1. Why nucleon sigma terms?

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   - Charm sigma term

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Why nucleon sigma terms?

Nucleon couples to the external Higgs field. That coupling is mediated by the couplings \( g_q \) of the quark flavors \( q \) to the Higgs field.

Elementary fermions have a mass, which is proportional to the fermion-Higgs coupling \( g_f \). As a consequence \( m_f = g_f \frac{\partial m_f}{\partial g_f} \).

We can define their logarithmic derivative also for non-elementary particles, like the nucleon:

\[
M_{Nf} q_N = \sigma_{qN} := g_q \frac{\partial M_N}{\partial g_q} = m_q \frac{\partial M_N}{\partial m_q}
\]

For the nucleon \( \sum_q \sigma_{qN} < M_N \), because a large part is generated by the scale anomaly of QCD.
Sigma terms and the mass of the nucleon

The Hamiltonian of QCD can be decomposed like [1]

\[ H = \sum_q m_q \bar{q}q + O_{\text{rest}}, \quad \frac{\partial O_{\text{rest}}}{\partial m_q} = 0 \]

The Feynman-Hellmann theorem states

\[ \sigma_{qN} = m_q \frac{\partial M_N}{\partial m_q} = m_q \langle N | \bar{q}q | N \rangle, \quad \langle N | N \rangle = 1 \]

so that \( M_N = \sum_q \sigma_{qN} + \langle N | O_{\text{rest}} | N \rangle \).


Why nucleon sigma terms?

Sigma terms and the mass of the nucleon

What is $\langle N \mid O_{\text{rest}} \mid N \rangle$?

We look at the trace $\theta^\mu_{\mu}$ of the energy momentum tensor [1]:

$$\langle N \mid \theta^\mu_{\mu} \mid N \rangle = \left\langle N \mid \sum_q m_q \bar{q}q \mid N \rightangle + \left\langle N \mid \gamma_m \sum_q m_q \bar{q}q + \frac{\beta}{g} G^2 \mid N \rightangle = M_N$$

Note: $\gamma_m \sum_q m_q \bar{q}q + \frac{\beta}{g} G^2$ does not depend on $m$, because the $\gamma_m m$-term cancels with a contribution in the renormalized $G^2$-term.

We find $\langle N \mid O_{\text{rest}} \mid N \rangle = M_N^a$ and

$$M_N = \sum_q \sigma_{qN} + M_N^a$$

Is $M_N = \sum_q \sigma_{qN} + M^a_N$ the decomposition of the nucleon mass?

It is a decomposition in the following sense:

- It is a sum of positive contributions.
- Each contribution is a scale and scheme independent observable in QCD.
- The individual contributions have a clear physical meaning. (experiments!)
- The contributions are properties of the physical nucleon state, making no references to unphysical theories.

However, the nucleon states do also depend on the quark mass. Therefore

$$M_N = \langle N(m_q) \mid H(m_q) \mid N(m_q) \rangle,$$

$$M_{N}^{(\phi)} - M_{N}^{(\chi)} = \sigma_{qN} + O(m_{q}^{(\phi)^2}).$$
Sum of sigma terms determines how strongly the nucleon couples to WIMPs via the spin-independent channel.

Sigma terms are required for the interpretation of direct dark matter detection experiments. [1]

Why nucleon sigma terms?

What is known about the sigma terms?

- **Ellis et. al. 18**
- **Hobbs et. al. 17**
- **Hobbs et. al. 17**
- **Hobbs et. al. 17**
- **Duan et. al. 16**
- **RQCD 16**
- **ETM 16**
- **QCD 13**
- **MILC 12**

\[ \sigma_{\text{nn}}^N \text{[MeV]} \]

\[ \sigma_{\text{nn}}^N \text{[MeV]} \]

\[ \sigma_{\text{CN}} \text{[MeV]} \]

![Graph](image_url)

Lukas Varnhorst for BMW collaboration

Nucleon sigma terms
How to determine the sigma terms?

Up, down, strange, and charm sigma terms can be determined on the lattice.

For the top, bottom, and charm sigma terms an expansion in $1/m_q$ can be employed. The validity can be checked in case of $\sigma_{cN}$, where the $\mathcal{O}(m_q^{-2})$ effects are biggest.
Direct evaluation of matrix elements $\langle N \mid m_q \bar{q}q \mid N \rangle$ requires disconnected contributions:

We use the Feynman-Hellmann method.
Strategy for the light and strange sigma terms:

1. Determine $M_N(M_{\pi}^2, M_{K^*}^2)$ from fit to lattice data. Use Wilson fermions.

2. Calculate $J = \frac{\partial \log M_{\text{meson}}^2}{\partial \log m_{\text{quarks}}}$ to get light and strange sigma term. Use staggered fermions.

3. Use isospin splitting relationships to disentangle up and down sigma terms.

For details, see last lattice conference.
Lattice calculations

Light- and strange sigma terms

\begin{align*}
\beta &= 3.2 \\
\beta &= 3.2 \text{ QCD+QED} \\
\beta &= 3.3 \\
\beta &= 3.4 \\
\beta &= 3.5
\end{align*}

M_N [MeV] vs \( M^2_{\pi} \) [MeV^2]

\( \sigma_{uN} \), \( \sigma_{dN} \), \( \sigma_{sN} \) preliminary
The results for the mixing matrix are:

\[
\begin{align*}
\frac{m_{ud}}{M_{\pi}^2} \frac{\partial M_{\pi}^2}{\partial m_{ud}} &= 0.99(3)(4) \\
\frac{m_{ud}}{M_{K}^2} \frac{\partial M_{K}^2}{\partial m_{ud}} &= -0.08(4)(6) \\
\frac{m_s}{M_{\pi}^2} \frac{\partial M_{\pi}^2}{\partial m_s} &= 0.04(4)(2) \\
\frac{m_s}{M_{K}^2} \frac{\partial M_{K}^2}{\partial m_s} &= 1.03(3)(2)
\end{align*}
\]

The result for the strange to light quark mass ratio are

\[
\frac{m_s}{m_{ud}} = 27.293(33)(08)
\]

(FLAG result: \(\frac{m_s}{m_{ud}} = 27.30(34))\).
Strategy for the charm sigma term:

We measured the nucleon mass on 9 staggered, Symanzik improved, 4 times stout smeared ensembles:

\[ \frac{m_c^{(\phi)}}{m_c} \]

Three lattice spacings: \( \beta = 3.75, \beta = 3.7753, \) and \( \beta = 3.84. \)

Three charm masses: \( \frac{3}{4} m_c^{(\phi)}, m_c^{(\phi)}, \frac{5}{4} m_c^{(\phi)}. \)

We use finite differences to determine the charm sigma term.
Determination of the nucleon mass:

With staggered fermions the nucleon propagator behaves like:

$$C_O(t) = \sum_{i=0}^{N_{\text{states}}-1} a_i p_i^{t+1}(\exp(-m_i t) + (-1)^{t+1} \exp(-m_i(N_t - t)))$$

We need not only the mass, but a propagator with the parity partner removed. To that end we construct operators of the form

$$O' = \sum_{\tau=0}^{m} b_j \exp(H\tau)O \exp(-H\tau)$$

and apply a variational method.
We construct the matrix [1-3]

\[
M(t) = \begin{pmatrix}
C_O(t) & C_O(t+1) & \cdots & C_O(t+m) \\
C_O(t+1) & C_O(t+2) & \cdots & C_O(t+m+1) \\
\vdots & \vdots & \ddots & \vdots \\
C_O(t+m) & C_O(t+m+2) & \cdots & C_O(t+2m)
\end{pmatrix}.
\]

and solve the GEVP [4]

\[
M(t_0)\vec{v}_i(t_0, t_1) = M(t_1)\lambda_i(t_0, t_1)\vec{v}_i(t_0, t_1).
\]

We get the correlation functions of the \(O'\) via

\[
C'_i(t; t_0, t_1) = \vec{v}^\dagger_i(t_0, t_1)M(t)\vec{v}_i(t_0, t_1)
\]

We performed a study with mock data:

We performed a study with mock data:

This method is well suited for the extraction of ground states in the presence of many excited (oscillating) states. It is also applied for the extraction of $\Omega$ masses in “High precision determination of $\omega_0$” [Jana N. Guenther].

For the nucleon the “projected” correlation function behaves like

\[
\begin{align*}
t_{\text{start}} &= 6 \\
t_{\text{stop}} &= 14 \\
am &= 0.6316 \pm 0.0306 \\
\chi^2/n_{\text{dof}} &= 6.19/6
\end{align*}
\]
As an alternative we used

\[ D_0(t) = C_0(t) + e^{\bar{m}} C_0(t + 1) \]

We defined the effective mass

\[ m_{\text{eff}}(t) = -\log \left( \frac{D_0(t)}{D_0(t + 1)} \right) \]

and its average between the times \( t_a \) and \( t_b \). We then minimized the deviation of the effective mass from the averaged effective mass.

The results agree very well with the GEVP based method.
**Finite difference approximation:** For each lattice spacing we used finite differences to estimate the charm sigma term. We used two differences:

\[
\begin{align*}
\Delta^+ M_N &= M_N(m_c = \frac{5}{4}m_c^{\text{central}}) - M_N(m_c = m_c^{\text{central}}) \\
\Delta^- M_N &= M_N(m_c = m_c^{\text{central}}) - M_N(m_c = \frac{3}{4}m_c^{\text{central}})
\end{align*}
\]

We combined them in two ways:

- The standard finite difference formula (error: \(O((\delta m_c/m_c)^2) = O(1/16))\)
  \[
  \sigma_{cN} = m_c \frac{\partial M_N}{\partial m_c} = 2 \frac{\Delta^+ M_N + \Delta^- M_N}{M_N^{(\phi)}}.
  \]

- Based on the HQ behaviour of sigma terms (error: \(O((\sigma_{cN}/M_N^{(\phi)})^3) = O(3 \times 10^{-4}))\)
  \[
  \sigma_{cN} = \frac{1}{\log \frac{5}{4} \log \frac{4}{3} \log \frac{5}{3}} \left( \log^2 \frac{4}{3} \Delta^+ M_N + \log^2 \frac{5}{4} \Delta^- M_N \right)
  \]
For the continuum extrapolation we used:

1. A constant fit in $a^2$ with only the two finest lattice spacings included.
2. A linear fit in $a^2$ with all lattice spacings included.
3. A quadratic fit in $a^2$ with all lattice spacings included.
Sigma terms in effective $N_f < 6$ theories: [1,2]

We use the histogram method to determine our systematic uncertainties. All uncertainties from the different parts are combined. We will provide a small program, that allows to calculate fully correlated statistical and systematic errors for arbitrary linear combinations of sigma terms.