Developments in the position-space approach to the HLbL contribution to the muon $g - 2$ on the lattice

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in Collaboration with
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Anomalous Magnetic Moment of the Muon

\[ \mu = g \frac{e}{2m} S \]

anomalous magnetic moment: \( a_\mu = \frac{g-2}{2} \)

\[ \approx 3 \text{ to } 4 \text{ standard deviations} \]

tension between \( a_\mu^{\text{exp}} \) and \( a_\mu^{\text{theo}} \)

→ new physics?
Anomalous Magnetic Moment of the Muon

**gyromagnetic moment:**  \[ \mu = g \frac{e}{2m} S \]

**anomalous magnetic moment:**  \[ a_\mu = \frac{g-2}{2} \]

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tension between \( a_\mu^{\text{exp}} \) and \( a_\mu^{\text{theo}} \)

\[ \rightarrow \text{new physics?} \]

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### reduce uncertainties

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<th>theory for HLbL</th>
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<td>reduce model uncertainties</td>
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<td>for dominant contribution (( \pi^0, \eta, \eta' ; \pi\pi ))</td>
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<td>using experimental input</td>
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<td>Colangelo et al. ’14,…,’17</td>
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<td>Blum et al. (’05,…)’15,…,’17</td>
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<td>Pion TFF: Gérardin et al. ’16, ’19</td>
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Euclidean position-space approach to $a_{\mu}^{\text{HLbL}}$

![Diagram](image)

**master formula**

\[
a_{\mu}^{\text{HLbL}} = \frac{me^6}{3} \int d^4 y \left[ \int d^4 x \, \bar{\mathcal{L}}_{[\rho, \sigma]; \mu \nu \lambda}(x, y) \right. \\
\left. i \hat{\Pi}_{\rho; \mu \nu \lambda \sigma}(x, y) \right].
\]

\[
i \hat{\Pi}_{\rho; \mu \nu \lambda \sigma}(x, y) = - \int d^4 z \, z_\rho \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle.
\]

- $\bar{\mathcal{L}}_{[\rho, \sigma]; \mu \nu \lambda}(x, y)$ computed in the continuum & infinite-volume
- no power-law finite-volume effects from the photons
- manifest Lorentz covariance
Outline

1. Tests of the QED Kernel
2. Lattice QCD
3. Conclusion
1 Tests of the QED Kernel

2 Lattice QCD

3 Conclusion
Tests in Continuum, Infinite Volume

**master formula**

\[
\alpha_{\mu}^{\text{HLbL}} = \frac{m e^6}{3} 8 \pi^3 \int_0^\infty d|y| |y|^3 \left[ \int_0^\infty d|x| |x|^3 \int_0^\pi d\beta \sin^2 \beta \tilde{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y) i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x, y) \right].
\]

\[
i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x, y) = -\int d^4 z z_{\rho} \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle.
\]

- \( \int d|x|, \int d|y| \) and \( \int d\beta \) evaluated numerically
- \( \int d^4 z \) evaluated (semi-)analytically
Contribution of the $\pi^0$ to $a_{\mu}^{HLbL}$ (VMD Model)

- dashed line = result from momentum-space integration
  - we reproduce the known result
  - contribution is perhaps surprisingly long-range
  - integrand peaked at short distances
Lepton loop integrand contribution to $a_{\mu}^{\text{HLbL}}$

- we reproduce the known result
- contribution is long-range
- integrand sharply peaked at short distances

The QED kernel is correct
- we reproduce the $\pi^0$-pole in VMD model
- we reproduce the lepton loop
What next?

**achievements**

- method for $a_{\mu}^{\text{HLbL}}$ on the lattice
- verified the QED kernel
- learned about the integrand

**challenges in the view of lattice computations**

- contributions are quite long range
- integrand peaked at small distances
What next?

**achievements**
- method for \( a_{\mu}^{\text{HLbL}} \) on the lattice
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**challenges in the view of lattice computations**
- contributions are quite long range
- integrand peaked at small distances

**a way to improve**
- do subtractions on the kernel (first proposed by Blum et al. ’17)
- exploit \( \int_x i\hat{\Pi}(x, y) = \int_y i\hat{\Pi}(x, y) = 0 \)
- example:
  \[
  \mathcal{L}^{(0)} = \tilde{\mathcal{L}}_{[\rho, \sigma]; \mu
\nu \lambda}(x, y)
  \]
  \[
  \mathcal{L}^{(1)} = \tilde{\mathcal{L}}_{[\rho, \sigma]; \mu\nu \lambda}(x, y) - \frac{1}{2} \tilde{\mathcal{L}}_{[\rho, \sigma]; \mu\nu \lambda}(x, x) - \frac{1}{2} \tilde{\mathcal{L}}_{[\rho, \sigma]; \mu\nu \lambda}(y, y)
  \]
  \[
  a_{\mu}^{\text{HLbL}} = \frac{me^6}{3} \int_{x, y} \mathcal{L}^{(0)}(x, y)i\hat{\Pi}(x, y) = \frac{me^6}{3} \int_{x, y} \mathcal{L}^{(1)}(x, y)i\hat{\Pi}(x, y)
  \]
Subtractions

master formula

\[ a_{\mu}^{\text{HLbL}} = \frac{me^6}{3} 8\pi^3 \int_0^\infty d|y||y|^3 \left[ \int_0^\infty d|x||x|^3 \int_0^\pi d\beta \sin^2 \beta \bar{L}_{[\rho,\sigma];\mu\nu\lambda}(x, y) i\bar{\Pi}_{\rho;\mu\nu\lambda\sigma}(x, y) \right]. \]

subtractions on the kernel

- we try (short notation):
  \( L^{(0)} = \bar{L}(x, y) \) (standard kernel)
  \( L^{(1)} = \bar{L}(x, y) - \frac{1}{2} \bar{L}(x, x) - \frac{1}{2} \bar{L}(y, y) \)
  \( L^{(2)} = \bar{L}(x, y) - \bar{L}(0, y) - \bar{L}(x, 0) \)
  \( L^{(3)} = \bar{L}(x, y) - \bar{L}(0, y) - \bar{L}(x, x) + \bar{L}(0, x) \)

- \( L^{(0)}(0, 0) = 0 \)
- \( L^{(1)}(x, x) = 0 \)
- \( L^{(2)}(0, y) = L^{(2)}(x, 0) = 0 \)
- \( L^{(3)}(x, x) = L^{(3)}(0, y) = 0 \)

- with all kernels \( L^{(0,1,2,3)} \) we can reproduce the known result
- we expect \( L^{(2,3)} \) to be advantageous on the Lattice

\[ y \text{ integrand lepton loop } m_l = m_\mu \]
Charged Pion Loop

charged pion loop from scalar QED, $\mathcal{L} = \left( \partial_\mu + ieA_\mu \right) \phi^* \left( \partial_\mu - ieA_\mu \right) \phi$

(note that we did not apply form factors)

the $\pi^+$-loop contribution is successfully reproduced
1 Tests of the QED Kernel

2 Lattice QCD

3 Conclusion
### Lattice Setup

#### CLS $N_f = 2 + 1$ ensembles

<table>
<thead>
<tr>
<th>CLS</th>
<th>$L^3 \times T$</th>
<th>$a$ [fm]</th>
<th>$m_\pi$ [MeV]</th>
<th>$m_\pi L$</th>
<th>$L$ [fm]</th>
<th>#confs</th>
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<tr>
<td>H105</td>
<td>$32^3 \times 96$</td>
<td>0.086</td>
<td>285</td>
<td>3.9</td>
<td>2.7</td>
<td>1000</td>
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<tr>
<td>N101</td>
<td>$48^3 \times 128$</td>
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<td>285</td>
<td>5.9</td>
<td>4.1</td>
<td>400</td>
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<tr>
<td>N203</td>
<td>$48^3 \times 128$</td>
<td>0.064</td>
<td>340</td>
<td>5.4</td>
<td>3.1</td>
<td>750</td>
</tr>
<tr>
<td>N200</td>
<td>$48^3 \times 128$</td>
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<td>800</td>
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<tr>
<td>D200</td>
<td>$64^3 \times 128$</td>
<td></td>
<td>200</td>
<td>4.2</td>
<td>4.2</td>
<td>1100</td>
</tr>
</tbody>
</table>

- $O(a)$ improved Wilson fermions
$N_f = 2 + 1$ CLS Ensembles

![Graph showing CLS ensembles with markers for different values of $m_\pi$ and $a^2$.]
Lattice Setup

### method 1 (presented so far)

\[
\int_{y,x,z} \mathcal{L}(x, y)(-z_\rho)\left[\Pi^{(1)}(x, y, z) + \Pi^{(1)}(y, x, z) + \Pi^{(1)}(x - y, -y, z - y)\right]
\]

**computational cost**
- \((1+N)\) forward propagators
- \(6(1+N)\) sequential prop.

### method 2

\[
\int_{y,x} \left(\mathcal{L}(x, y) + \mathcal{L}(y, x) - \mathcal{L}(x, x - y)\right)i\hat{\Pi}^{(1)}(x, y) + \int_{z} \mathcal{L}(x, x - y)x_\rho \Pi^{(1)}(x, y, z)
\]

**computational cost**
- \((1+N)\) forward propagators

- If the 1-dim. integral over \(|y|\) is done with \(N\) evaluations of the integrand.
- we sum over \(x\) and \(z\) explicitly over the whole lattice
Comparison of the Method 1 and 2

$\pi^0$ pole contrib. to conn. contractions;
$m_{\pi^0} = 135\text{MeV}, \text{VMD FF}; \text{kernel } L^{(2)}$;
$a_\mu = (34/9) \times 57 \times 10^{-11}$.

- method 2: reduced computational cost but longer range
Integrand of $a_{\mu}^{cH\ell bL}$ with $\mathcal{L}^{(2)}$, $m_\pi = 340$ MeV, $a = 0.064$ fm

- fully connected contribution only
- we already observe a good signal
- integrand non-zero up to 2 fm

$a_{\mu}^{cH\ell bL} = 82(9) \times 10^{-11}$
Discretisation Effects, $m_\pi = 285$ MeV

Discretisation effects seem to be small (we are increasing statistics)
Finite Size Effects, $a = 0.086$ fm

- $m_\pi L = 4.0$
- $m_\pi L = 6.0$

- Need more statistics to test for finite-size effects
Integrand of $a^{cHLbL}_\mu$ with $\mathcal{L}^{(2)}$, $m_\pi = 340$ MeV, $a = 0.064$ fm

- observe effects not covered by the $\pi^0$-exchange prediction at short distances
- more statistics needed to test for agreement with prediction at long distances
- $\pi^0$-exchange prediction of the finite-size effects might be valuable to improve the accuracy of $a^{HLbL}_\mu$
the results show an upward trend for decreasing pion mass

currently collecting more statistics in long distance regime
Pion Mass Dependence of $a_{\mu}^{cHLbL}$

- effect of the mass for small distances is small
- deviation more pronounced for larger distances

$\begin{align*}
\text{Kernel } & L(2), \text{ method 2; } \pi^0 \text{ contribution:} \\
\text{VMD TFF, } m_V = 775.5 \text{ MeV, } F_{\gamma\gamma(0,0)} = 0.274 \text{ GeV}^{-1}
\end{align*}$
1 Tests of the QED Kernel

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Conclusions

- Explicit formula for $a_{\mu}^{HLbL}$
  - QED kernel function multiplying the position-space QCD correlation function

- Tests
  - QED kernel: reproduce known results for $\pi^0$ pole, $\pi^+$-loop and lepton-loop in the continuum for the standard kernel $L^{(0)}$ and subtracted kernels $L^{(1,2,3)}$

- Lattice QCD
  - Preliminary results for the fully connected contribution (in QED$_\infty$)
  - The discretisation effects seem to be small
  - VMD results suggest to further investigate finite-size effects

- Future
  - Collect more statistics
  - Study finite-size effects in VMD to complement lattice computation
  - Perform chiral and continuum extrapolations
  - Implement disconnected contribution