Structure and transitions of nucleon excitations via parity-expanded variational analysis

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- $m_\pi = 156$–$702\,\mathrm{MeV}$

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Step 2: Perform variational analysis

- Seek operators $\{\phi^{\alpha}_{\boldsymbol{p}}\}$ that couple strongly to a single energy eigenstate

$$\phi^{\alpha}_{\boldsymbol{p}} = \sum_{i} v^{\alpha}_{i}(\boldsymbol{p})\ \chi^{i}_{\boldsymbol{p}} + \sum_{i'} v^{\alpha}_{i'}(\boldsymbol{p})\ \chi^{i'}_{\boldsymbol{p}}$$

$$\overline{\phi}^{\alpha}_{\boldsymbol{p}} = \sum_{i} u^{\alpha}_{i}(\boldsymbol{p})\ \overline{\chi}^{i}_{\boldsymbol{p}} + \sum_{i'} u^{\alpha}_{i'}(\boldsymbol{p})\ \overline{\chi}^{i'}_{\boldsymbol{p}}$$

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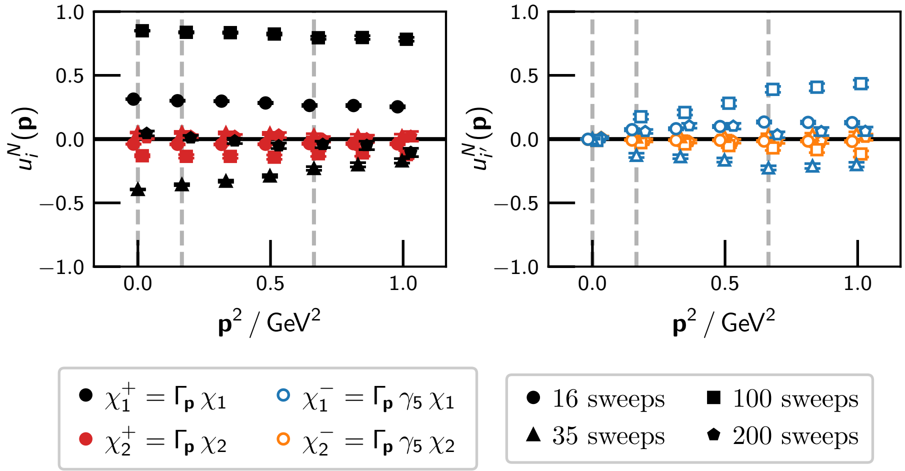
- Coefficients can be found by solving generalised eigenvalue problem

$$\boldsymbol{v}^{\alpha}(\boldsymbol{p})\ G(\boldsymbol{p};\ t_0) = \exp(-E^{\alpha}(\boldsymbol{p})\,\Delta t)\ \boldsymbol{v}^{\alpha}(\boldsymbol{p})\ G(\boldsymbol{p};\ t_0 + \Delta t)$$

$$G(\boldsymbol{p};\ t_0)\ \boldsymbol{u}^{\alpha}(\boldsymbol{p}) = \exp(-E^{\alpha}(\boldsymbol{p})\,\Delta t)\ G(\boldsymbol{p};\ t_0 + \Delta t)\ \boldsymbol{u}^{\alpha}(\boldsymbol{p})$$

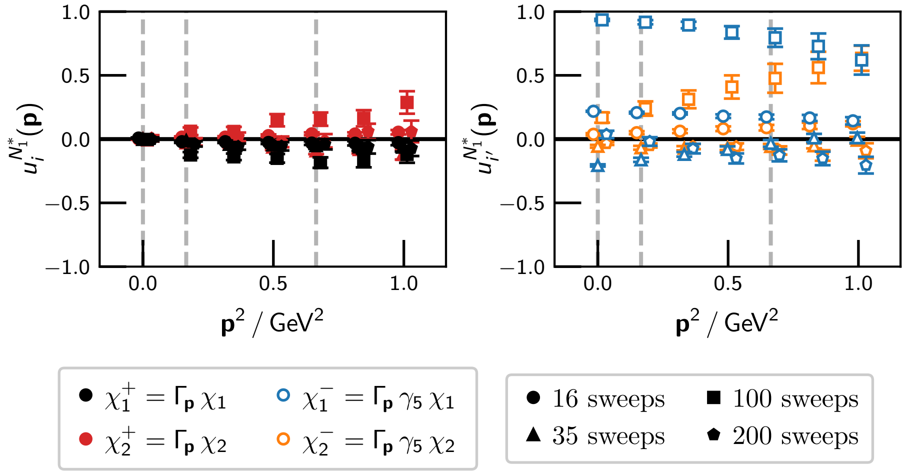
Eigenvector components

Ground state



Eigenvector components

First negative parity excitation



More information

"Parity-expanded variational analysis for nonzero momentum"
F. M. Stokes, W. Kamleh, D. B. Leinweber, M. S. Mahbub,
B. J. Menadue, B. J. Owen
Phys. Rev. D **92** (2015) 11, 114506
doi:10.1103/PhysRevD.92.114506
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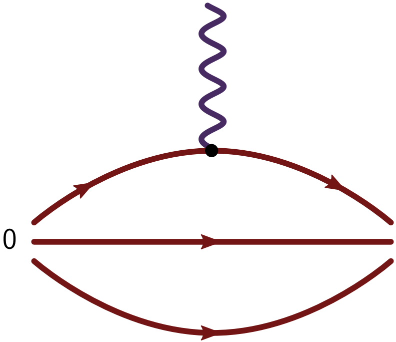

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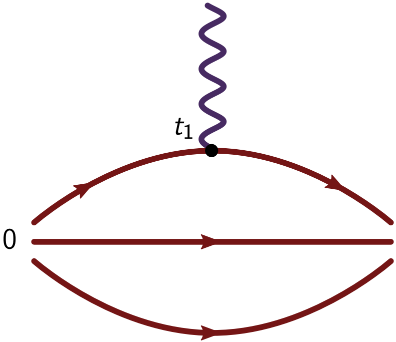

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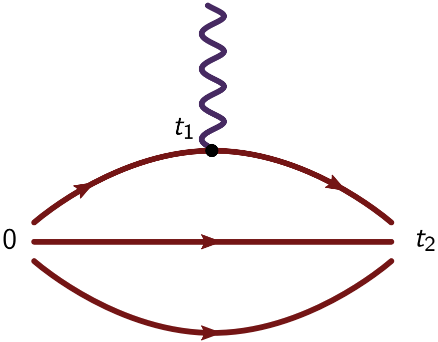

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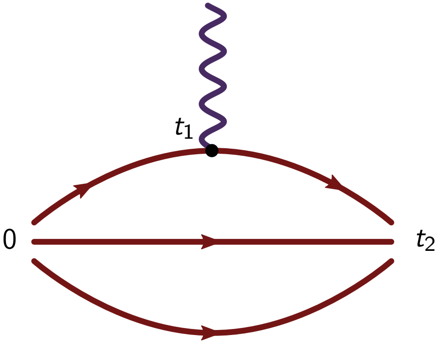

Matrix element

$$\langle B\,;p'\,;s'|\,j^\mu|\,B\,;p\,;s\rangle \propto$$
$$\overline{u}_B\left(\gamma^\mu F_1(Q^2) - \frac{\sigma^{\mu\nu}q_\nu}{2m_B}F_2(Q^2)\right)u_B$$

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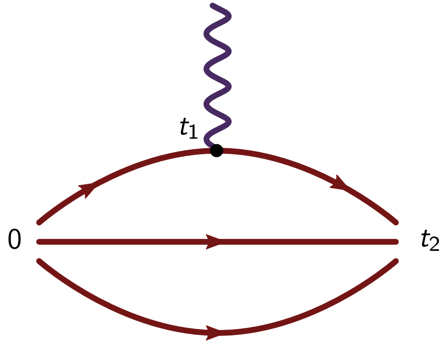

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Sachs form factors

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{2m_B}F_2(Q^2)$$
$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

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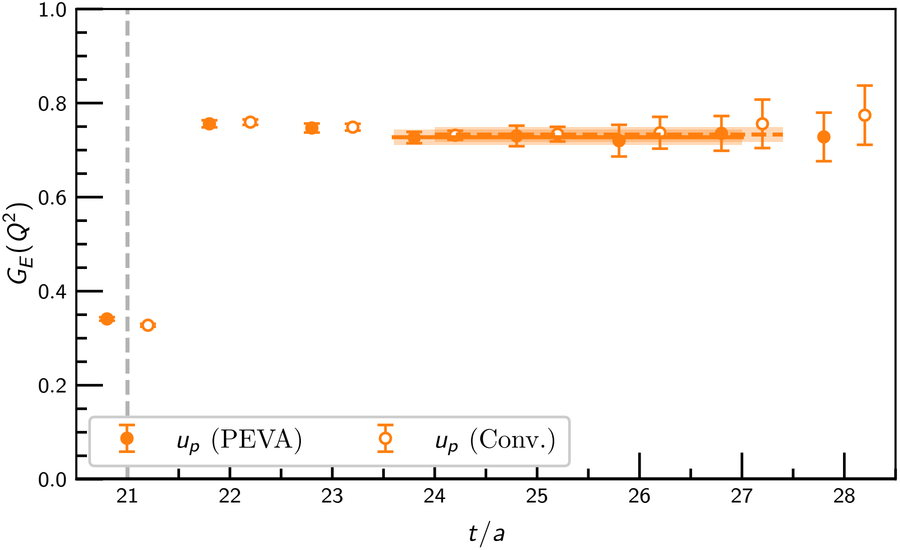
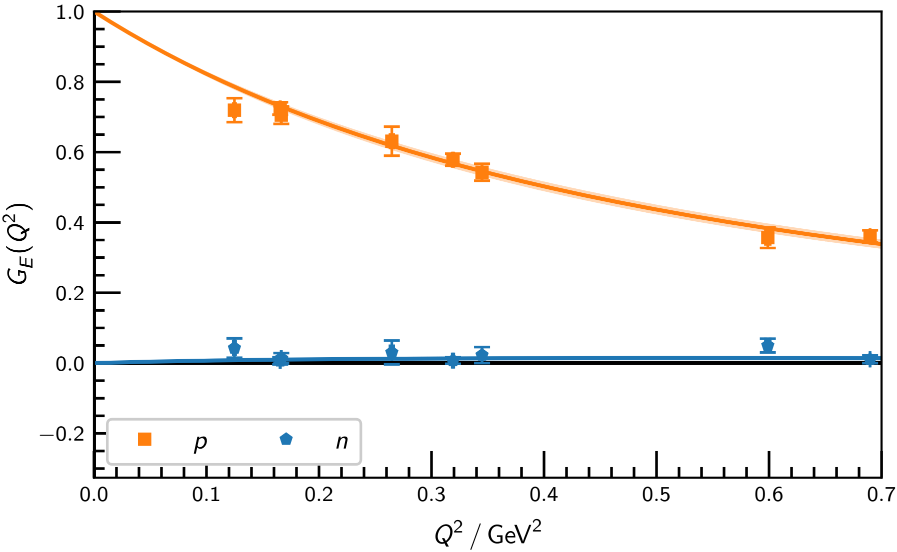
Ground state

Fits to $G_E(Q^2 = 0.166(4))$ ($m_\pi = 156\,\mathrm{MeV}$)



Ground state

Momentum-dependence of $G_E(Q^2)$ ($m_\pi = 156\,\text{MeV}$)



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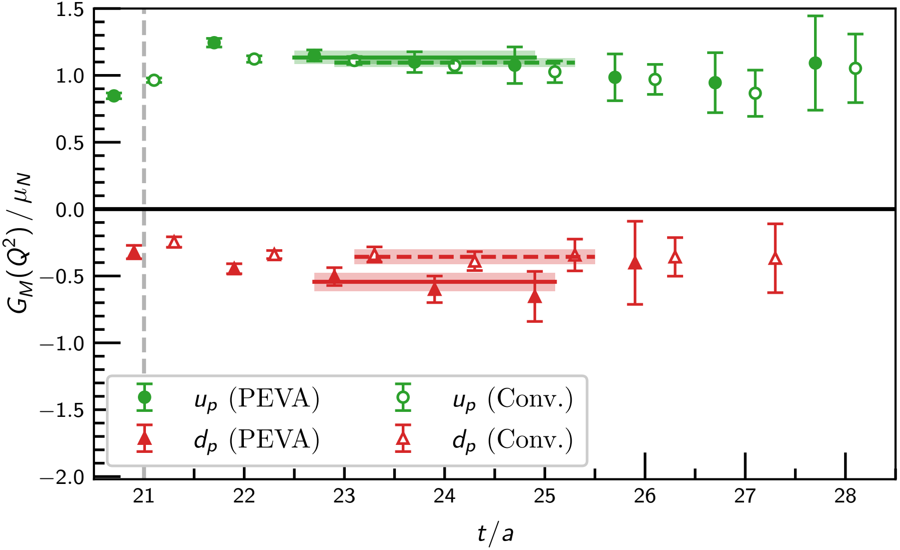
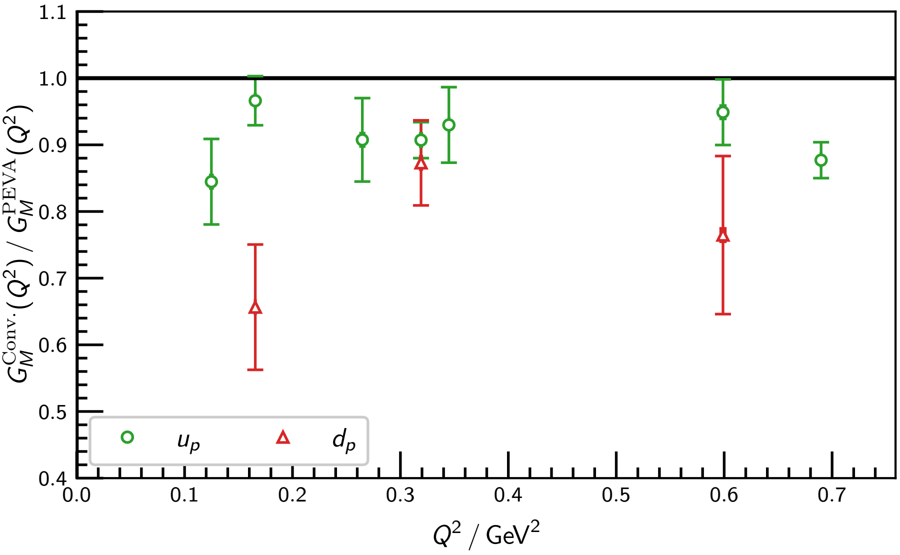
Fits to $G_M(Q^2 = 0.166(4))$ ($m_\pi = 156\,\text{MeV}$)



Ground state

Ratios of conventional $G_M(Q^2)$ plateaus to PEVA ($m_\pi = 156\,\mathrm{MeV}$)



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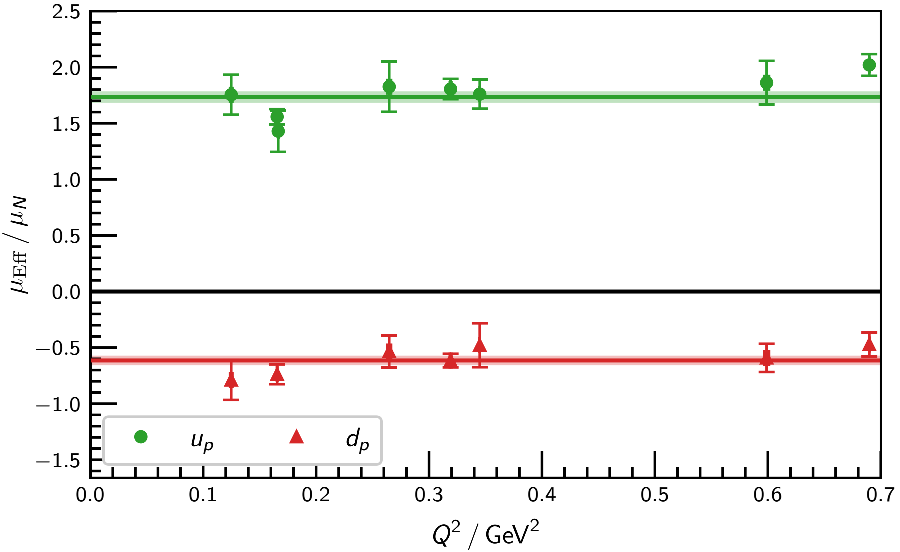
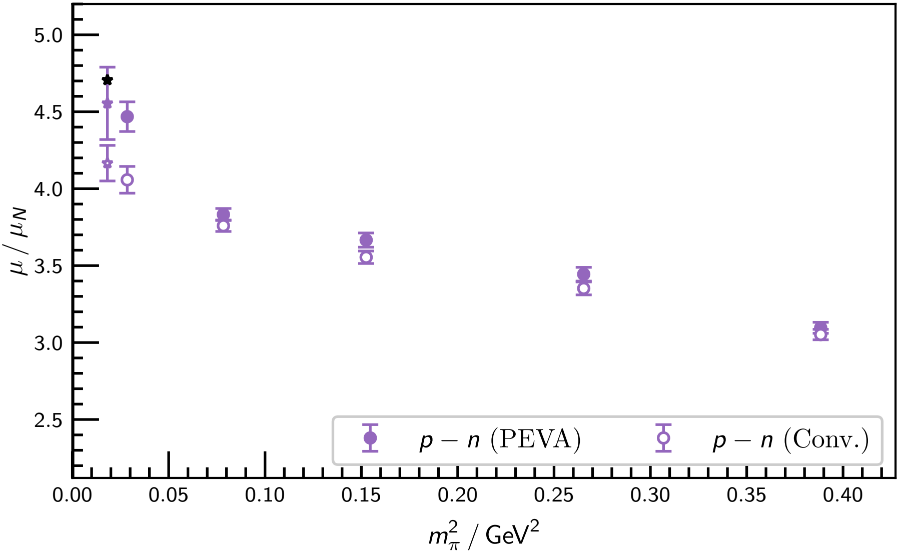
Magnetic moment estimate ($m_\pi = 156\,\mathrm{MeV}$)



Ground state

Pion-mass dependence of magnetic moment

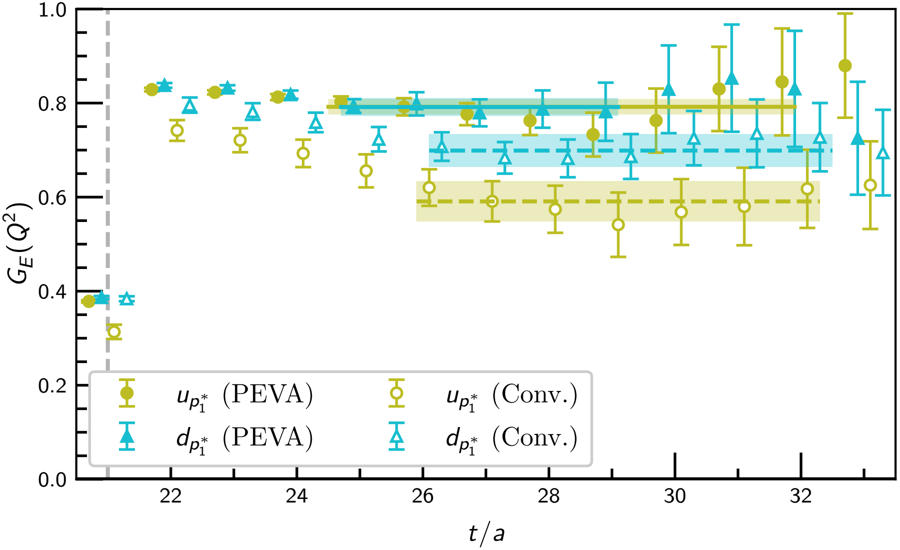


More information

“Opposite-Parity Contaminations in Lattice Nucleon Form Factors”
F. M. Stokes, W. Kamleh, D. B. Leinweber
Phys. Rev. D **99** (2019) 7, 074506
doi:10.1103/PhysRevD.99.074506
arXiv:1809.11002 (hep-lat).

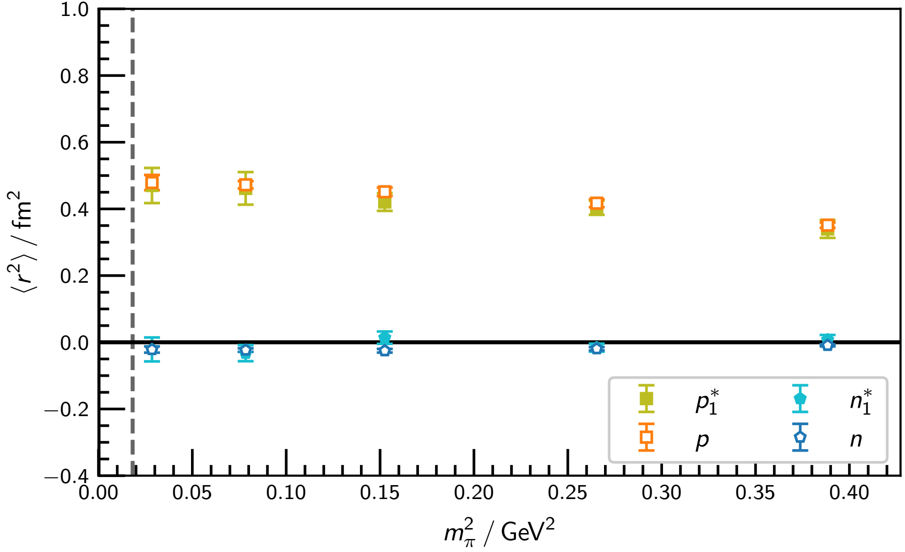
First negative-parity excitation

Fits to $G_E(Q^2 = 0.142(4))$ ($m_\pi = 702\,\mathrm{MeV}$)



First negative-parity excitation

Pion-mass dependence of charge radius



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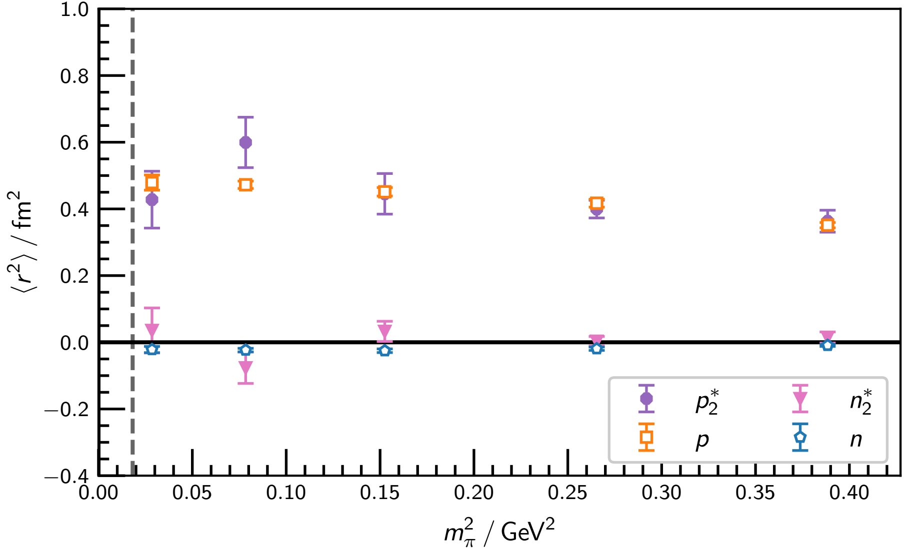

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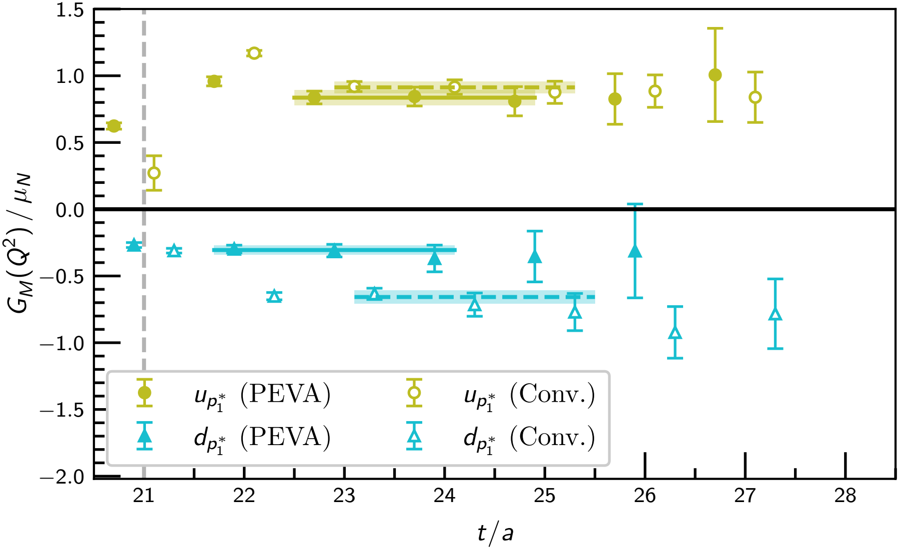

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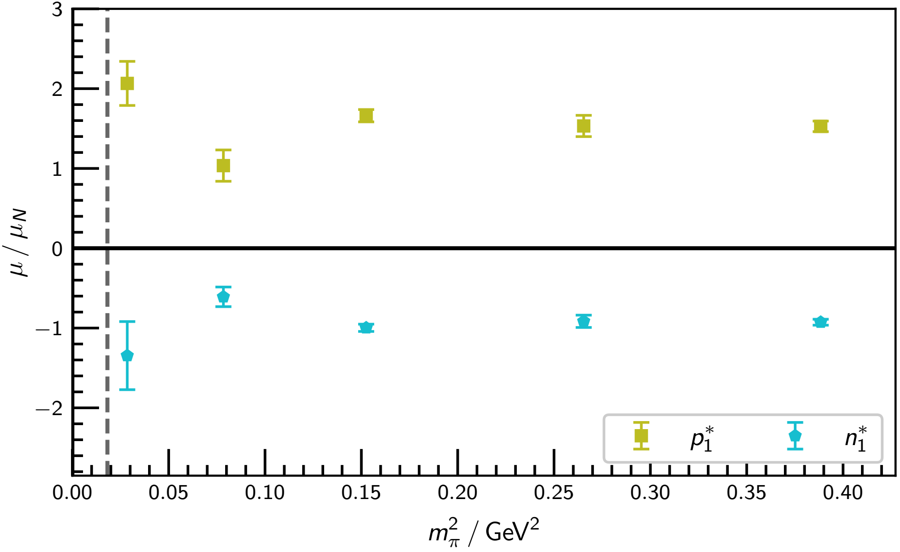

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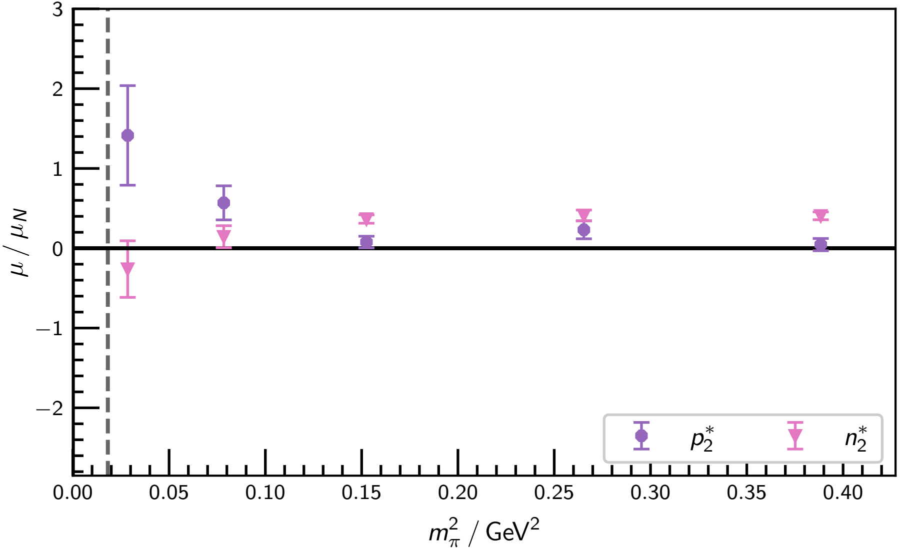

Comparison to constituent quark model

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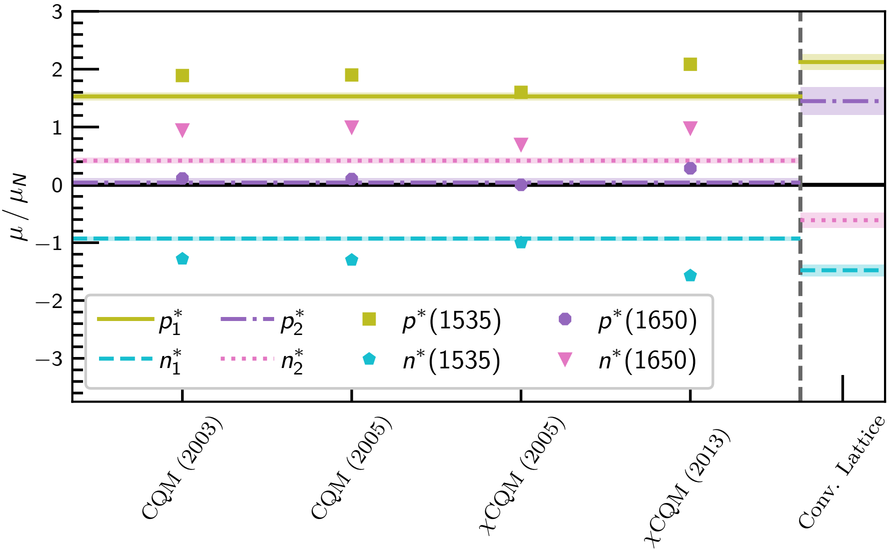

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- W.-T. Chiang, S. N. Yang, M. Vanderhaeghen and D. Drechsel, Nucl. Phys. A **723** (2003), doi:10.1016/S0375-9474(03)01160-6.
- J. Liu, J. He and Y. B. Dong, Phys. Rev. D **71** (2005), doi:10.1103/PhysRevD.71.094004.
- N. Sharma, A. Martinez Torres, K. P. Khemchandani and H. Dahiya, Eur. Phys. J. A **49** (2013), doi:10.1140/epja/i2013-13011-2.

Step 3: Compute three point correlation function

Matrix element

$$\langle \beta^- \, ; p' \, ; s' | \, j^\mu \, | \, \alpha^+ \, ; p \, ; s \rangle \propto$$
$$\overline{u}_\beta \left(\left(\delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \gamma^\nu \gamma^5 F_1^*(Q^2) - \frac{\sigma^{\mu\nu} q_\nu}{m_\beta - m_\alpha} \gamma^5 F_2^*(Q^2) \right) u_\alpha$$

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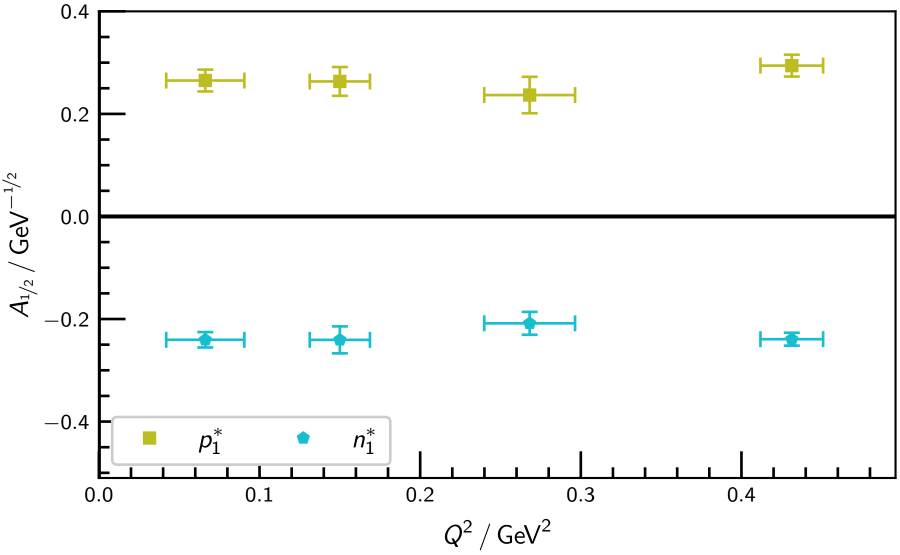
Helicity amplitudes

$$A_{1/2}(Q^2) = 2b_- \left(F_1^*(Q^2) + F_2^*(Q^2) \right)$$
$$S_{1/2}(Q^2) = -\sqrt{2} b_- \frac{(m_\beta - m_\alpha) |\vec{q}|}{Q^2} \left(F_1^*(Q^2) - \frac{Q^2}{m_\beta - m_\alpha} F_2^*(Q^2) \right)$$
$$b_- = \sqrt{\frac{Q^2 + (m_\beta + m_\alpha)^2}{8 m_\alpha (m_\beta^2 - m_\alpha^2)}}$$

Transition to first negative parity excitation

Transverse helicity amplitude at $m_\pi = 702\,\text{MeV}$



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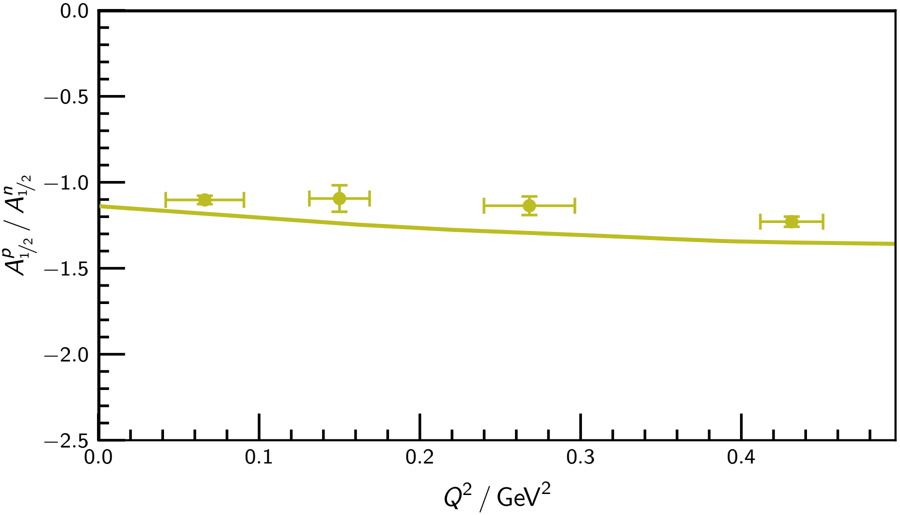
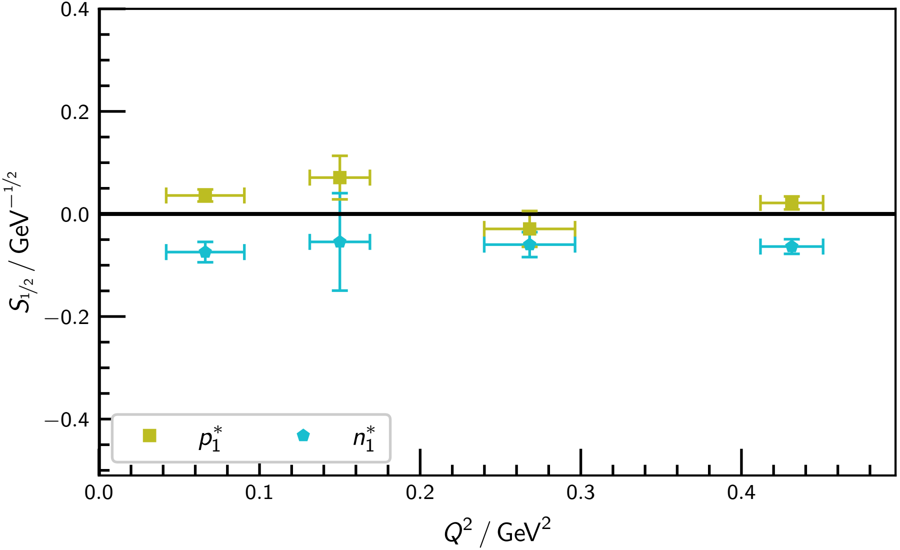
Transverse helicity amplitude ratio at $m_\pi = 702\,\mathrm{MeV}$



S. Capstick, B. D. Keister, Phys. Rev. D **51** (1995), doi:10.1103/PhysRevD.51.3598.

Transition to first negative parity excitation

Longitudinal helicity amplitude at $m_\pi = 702\,\mathrm{MeV}$



Transition to second negative parity excitation

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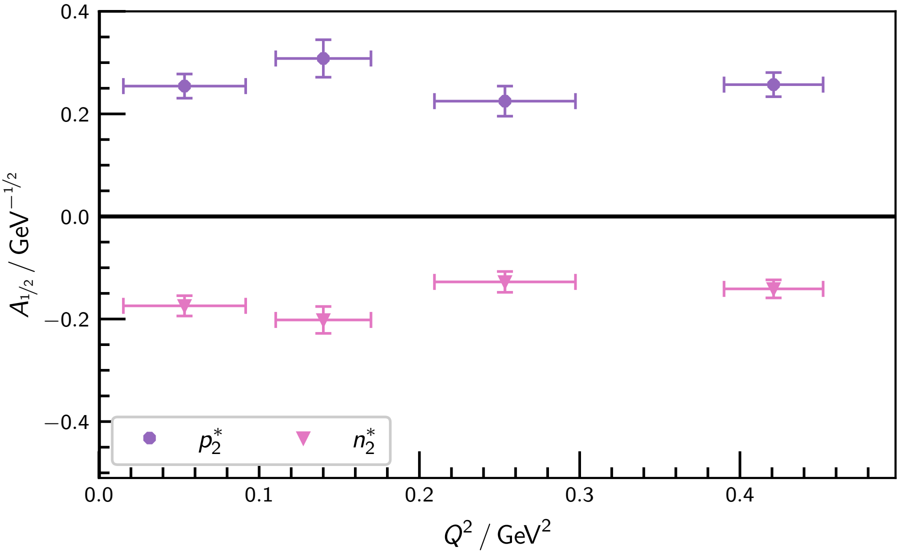

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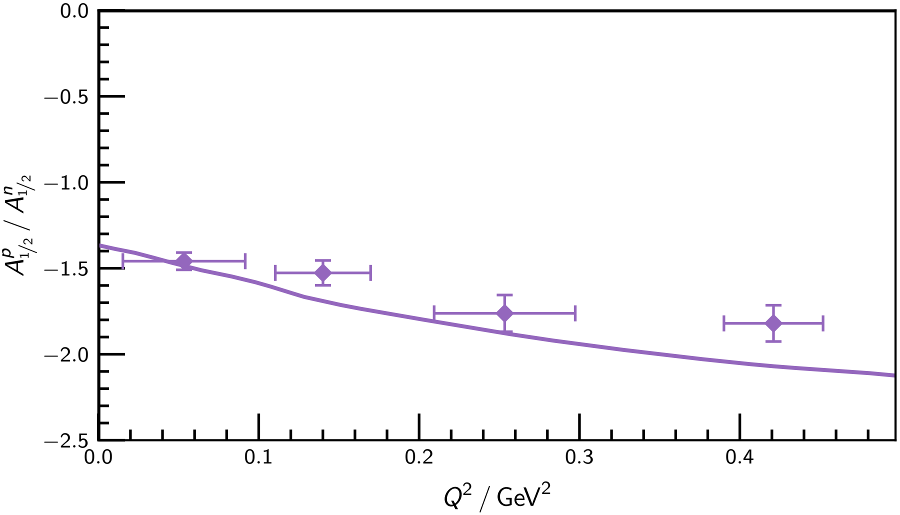

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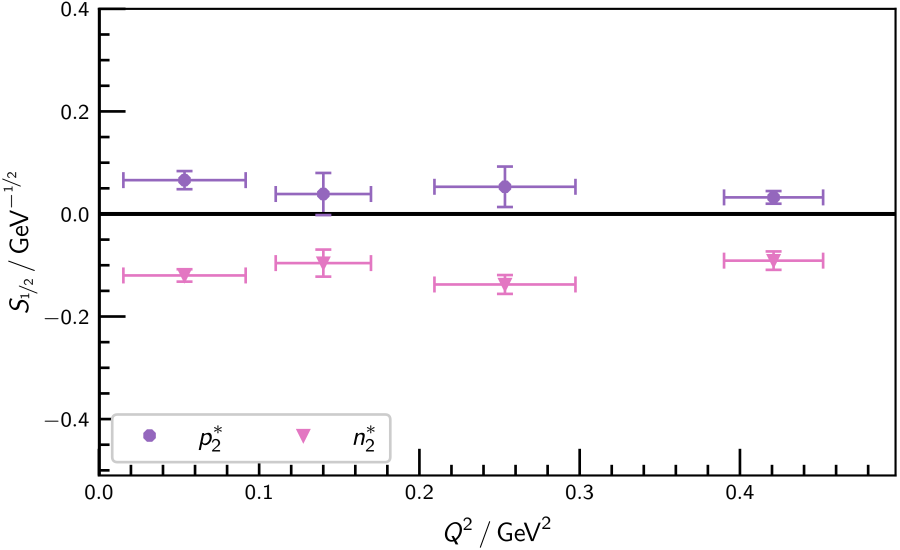
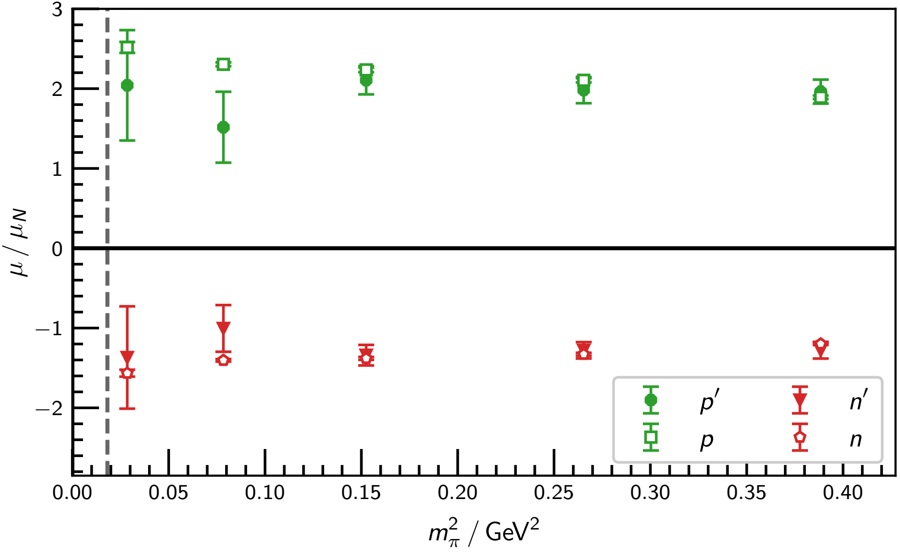


First positive-parity excitation

Pion-mass dependence of magnetic moment



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- Two low-lying localised $\frac{1}{2}^-$ nucleon excitations on the lattice

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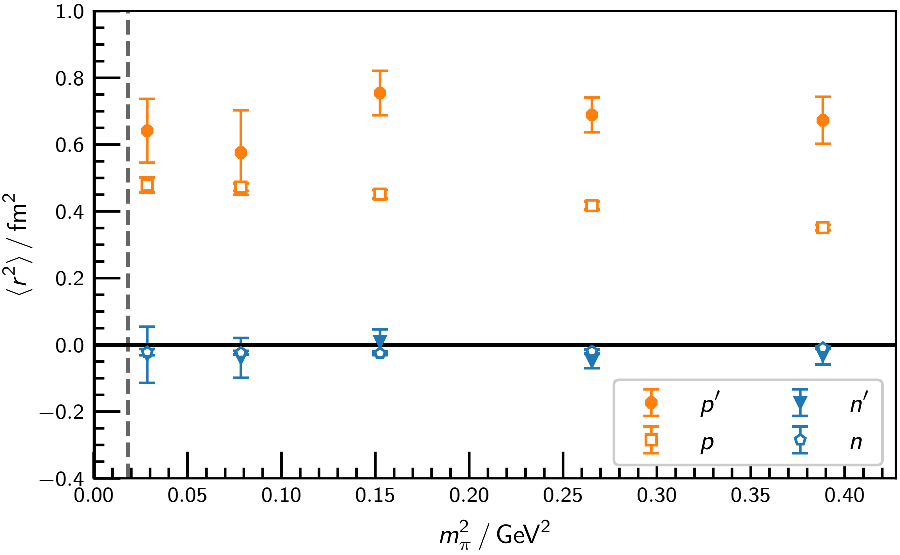
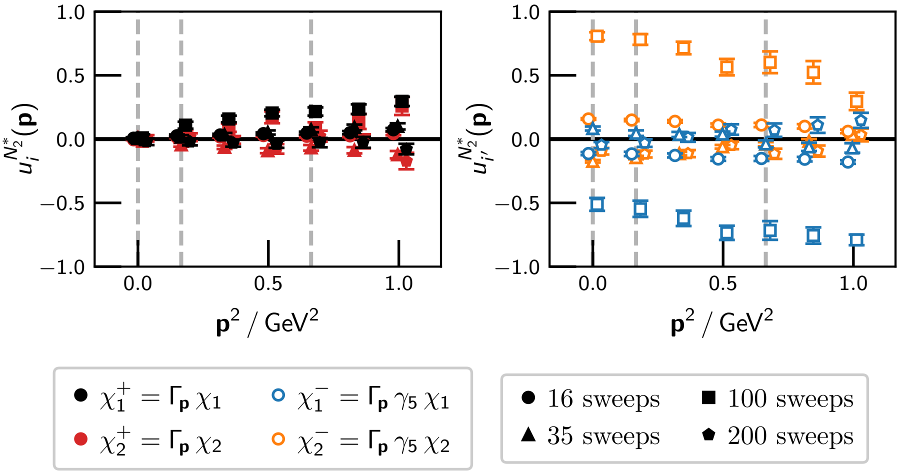
First positive-parity excitation

Pion-mass dependence of charge radius



Eigenvector components

Second negative parity excitation



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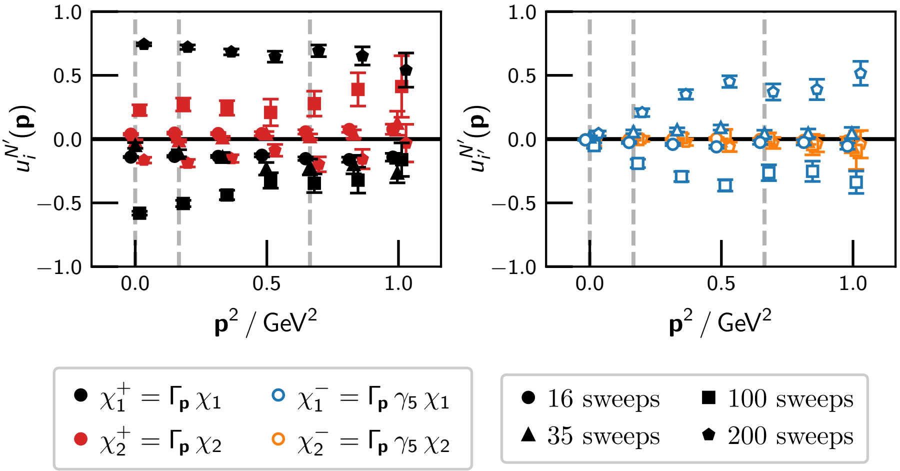
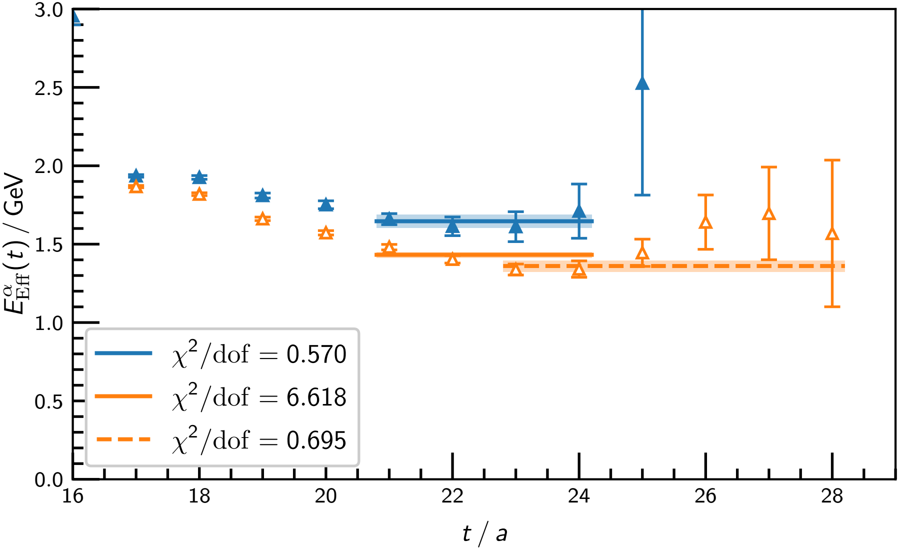


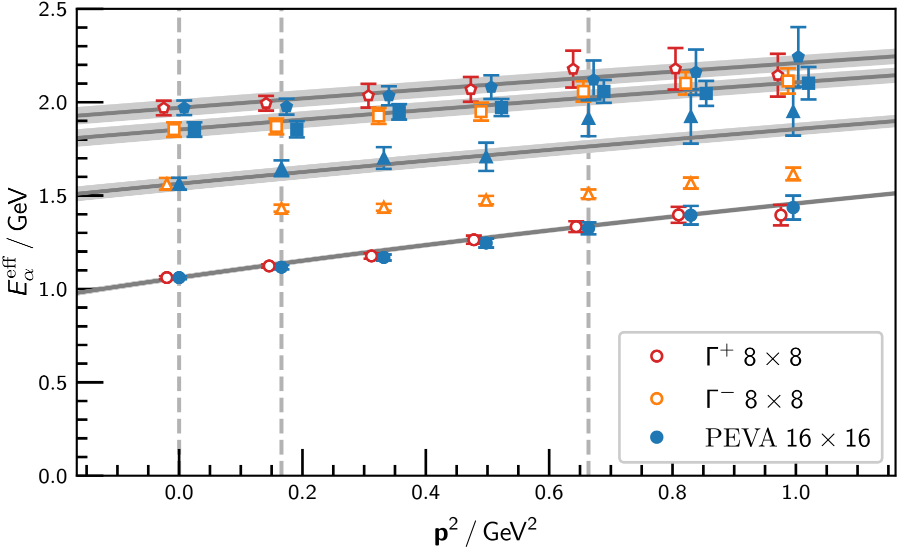
Effective energy

First negative parity excitation - $\boldsymbol{p}^2 \simeq 0.166\,\mathrm{GeV}^2$



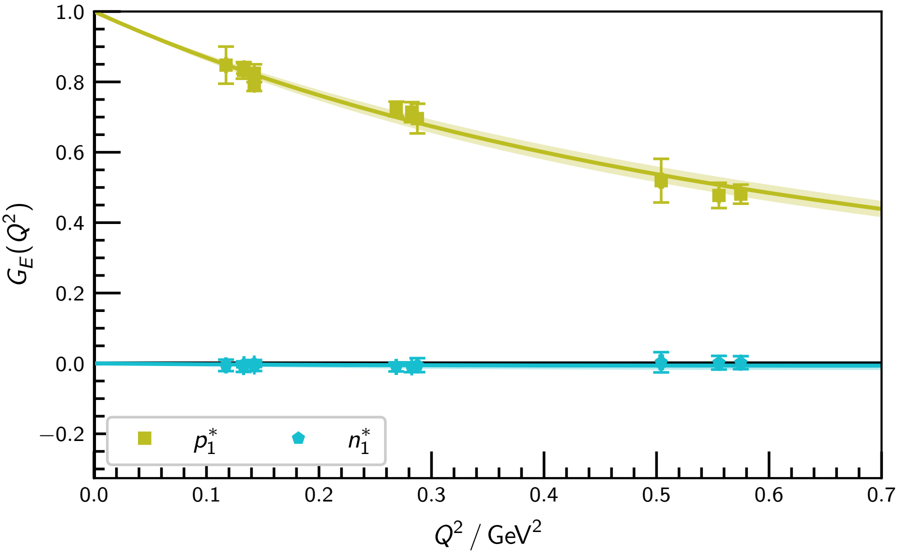
Effective energy

Nucleon spectrum



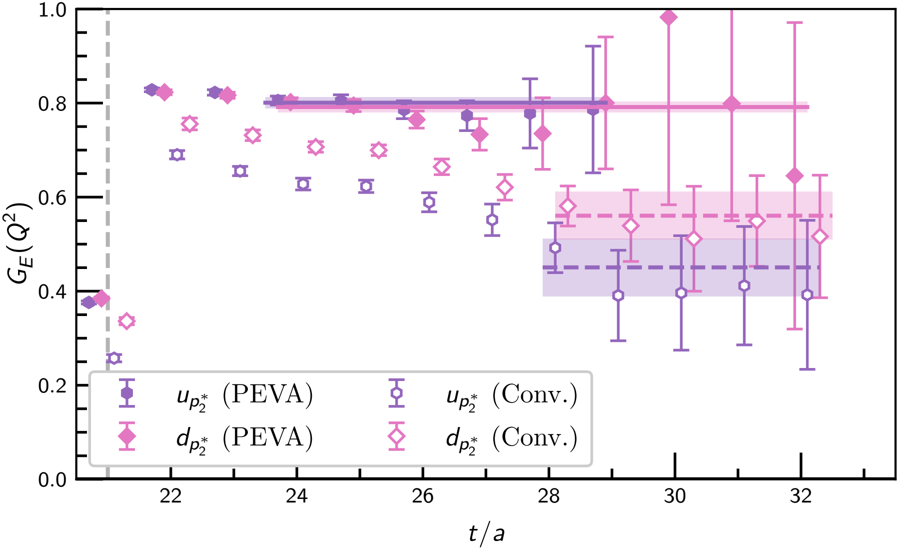
First negative-parity excitation

Momentum-dependence of $G_E(Q^2)$ ($m_\pi = 702\,\mathrm{MeV}$)



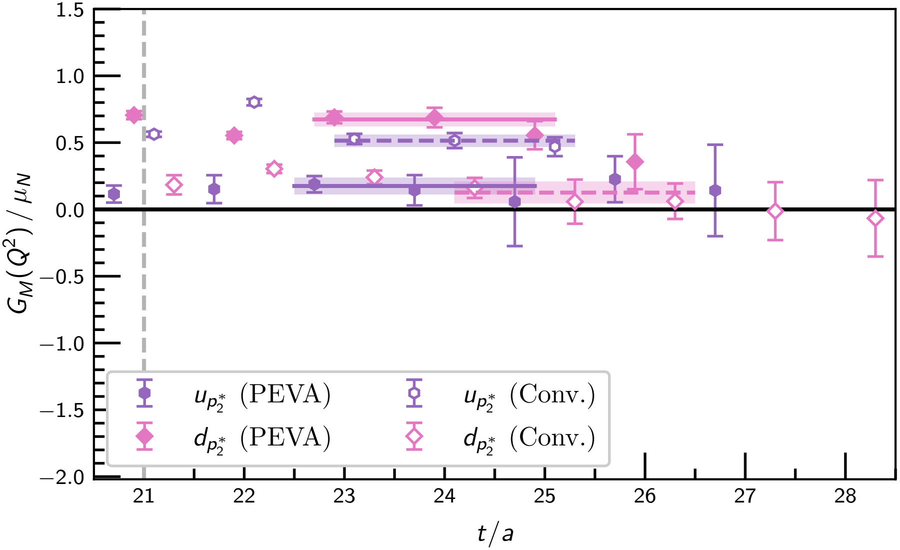
Second negative-parity excitation

Fits to $G_E(Q^2 = 0.142(4))$ ($m_\pi = 702\,\mathrm{MeV}$)



Second negative-parity excitation

Fits to $G_M(Q^2 = 0.142(4))$ ($m_\pi = 702\,\text{MeV}$)



Parity projection

$$\mathcal{G}_{ij}(\boldsymbol{p};\ t) = \sum_{B^\pm} \mathrm{e}^{-E_{B^\pm}(\boldsymbol{p})\,t}\, \lambda_i^{B^\pm}\, \overline{\lambda}_j^{B^\pm}\, \frac{-i\,\gamma\cdot p \pm m_{B^\pm}}{2E_{B^\pm}(\boldsymbol{p})}$$

- Introduce $\Gamma_\pm = (\gamma_4 \pm \mathbb{I})/2$ and define $G_{ij}(\Gamma_\pm;\boldsymbol{p};\ t) \equiv \mathrm{Tr}\left(\Gamma_\pm\,\mathcal{G}_{ij}(\boldsymbol{p};\ t)\right)$

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Zero momentum

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