Structure functions from the Compton amplitude

R Horsley[1], Y. Nakamura[2], H. Perlt[3], P.E.L. Rakow[4], G. Schierholz[5], K. Somfleth[6], R. Young[6] and J. Zanotti[6] (*QCDSF/CSSM/UKQCD Collaboration*)

[1]University Edinburgh, [2]RIKEN, [3]Universität Leipzig, [4]University Liverpool, [5]DESY, [6]University Adelaide

LATTICE 2019, Wuhan, China, June 16 - June 22, 2019

Abstract: We have initiated a program to compute the Compton amplitude with the Feynman-Hellman method. This amplitude is related to the structure function via a Fredholm integral equation of the first kind. It is known that these types of equations are inherently ill–posed - they are, e.g., extremely sensitive to perturbations of the system. We discuss some methods which are candidates to handle these problems. Among them we investigate simple model-fitting, singular value decomposition, conjugate gradient for least squares and Bayesian approaches. Special attention is drawn to the physical region of the ω parameter, where we have to take the principal value.

Introduction

We have initiated a program to compute particle distribution functions (PDF) with the Feynman-Hellmann method. *Chambers et al. Phys.Rev.Lett. 118 (2017) no.24, 242001.*

Starting point is the relation (unpolarized case)

$$T_{\mu\nu}(p,q) = \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\langle p|T J_\mu(x) J_\nu(0)|p\rangle\,,$$

p – momentum of nucleon states, q – momentum of the virtual photon, $J_\mu(x)$– electromagnetic current

The relation to the structure functions F_1 and F_2 is given by

$$\begin{aligned} T_{\mu\nu}(p,q) &= \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) 4\omega \int_0^1 dx\, \frac{\omega x}{1-(\omega x)^2}\, F_1(x,q^2) \\ &+ \left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right) \frac{8\omega}{2p\cdot q} \\ &\times \int_0^1 dx\, \frac{1}{1-(\omega x)^2}\, F_2(x,q^2)\,. \end{aligned}$$

($\omega = 2p\cdot q/q^2$). To simplify the computation we chose $\mu=\nu=3$ and $p_3 = q_3 = q_4 = 0$:

$$T_{33}(p,q) = 4\omega \int_0^1 dx\, \frac{\omega x}{1-(\omega x)^2} F_1(x,q^2) \quad (1)$$

For $\omega > 1$ one has to take the principal value.

Introduction

Relation (1) is the key relation for our considerations. From this equation it is clear that we need an efficient method to compute the lhs – the matrix element T_{33}.

We apply the Feynman-Hellmann method for the computation of the Compton amplitude. We change the Lagrangian

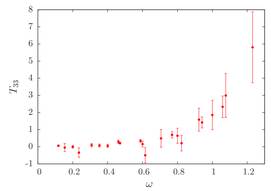
$$\mathcal{L}(x) \to \mathcal{L}(x) + \lambda \mathcal{J}_3(x)\,, \quad \mathcal{J}_3(x) = Z_V \cos(\vec{q}\cdot\vec{x})\, e_f\, \bar{\psi}_f(x)\gamma_3\psi_f(x)\,,$$

where ψ_f is the quark field of flavor $f = u, d, s, \cdots$ to which the photon is attached, and e_f is its electric charge.

Taking the second derivative of the nucleon two-point function $\langle N(\vec{p},t)\bar{N}(\vec{p},0)\rangle_\lambda \simeq C_\lambda\, \mathrm{e}^{-E_\lambda(p,q)\,t}$ with respect to λ on both sides, we obtain

$$-2E_\lambda(p,q)\frac{\partial^2}{\partial\lambda^2}E_\lambda(p,q)\,\big|_{\lambda=0} = T_{33}(p,q)\,.$$

A recent result for T_{33} as function of ω for $q^2 = (3.55,\ 4.64,\ 5.46)$ GeV^2 is given by



Fredholm Integral Equation of First Kind

To invert (1) for $F_1(x)$ one has to exploit methods for solutions of Fredholm integral equations of the first kind

However, it is known that problems of this class are ill–posed

- the solutions are not unique
- the solutions are not continuous functions of the data (T_{33}) – small perturbations of the data can cause arbitrary large perturbations of the solutions

It is clear that especially the second point is of great importance for us – how precise do we have to calculate T_{33} in order to get meaningful results for the particle distribution functions

There is a vast amount of literature which investigate this type of equations because they arise in many areas of science and engineering. They show, however, that there is no general solution: one has to inspect every special equation and chose the appropriate method (if any exists)

In view of particle distributions there is an interesting detailed investigation by *Karpie et al., JHEP 1904 (2019) 057.*

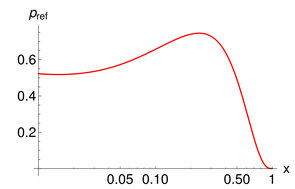
In our case we have a different kernel – so, we have to check various algorithms for their suitability.

The Model

In order to test various methods we will use a parton distribution function obtained from a global fit to experimental data *(Martin et al. Eur. Phys. J .C63:189-285,2009).*

We take as our reference function the combined valence and sea quark distribution (multiplied with x)

$$\begin{aligned} p_{ref}(x) &= 1.43\, x^{0.45}(1-x)^{3.04}\left(1 + 8.99\,x - 2.37\sqrt{x}\right) \\ &+ \frac{0.22\,(1-x)^{8.88}\left(1 + 17.86\,x - 2.9\sqrt{x}\right)}{x^{0.16}} \end{aligned}$$



We expect that this x–dependence is similar to our expected result from the FH-approach. This function is then used to generate the T_{33} data and to compare with the results of our tested inversion algorithms.

Generation of Data, PV

For $\omega > 1$ one should use the principal value to compute the integral (1).

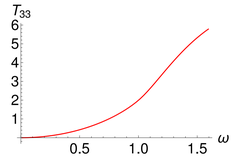
Writing the kernel as

$$\begin{aligned} \ker(x,\omega) &= \frac{4\,\omega^2}{1-x^2\,\omega^2} \equiv \left(\frac{1}{x-\frac{1}{\omega}}\right)\left(-\frac{4\omega}{1+x\omega}\right) \\ &= \ker_1(x,\omega)\,\ker_2(x,\omega) \end{aligned}$$

and denoting $\tilde{p}(x,\omega) = p(x)\,\ker_2(x,\omega)$ we compute T_{33} as ($p(x) = x\,F_1(x)$)

$$\begin{aligned} T_{33}(\omega) &= \tilde{p}(1/\omega,\omega)\,\log\left((1-1/\omega)\omega\right) \\ &+ \int_0^1 dx\, \frac{\tilde{p}(x,\omega) - \tilde{p}(1/\omega,\omega)}{x - 1/\omega} \end{aligned}$$

For $0 < \omega < 1.6$ we get for our example distribution



A moment fit of T_{33} as a polynomial in ω^{2k} up to ω^{14} gives

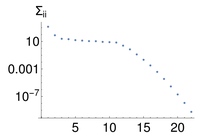
$$\begin{aligned} T_{33}(\omega) &= 1.41(6)\omega^2 + 2.3(4)\omega^4 - 6.3(1)\omega^6 + 8.3(2)\omega^8 \\ &- 4.9(8)\omega^{10} + 1.3(3)\omega^{12} - 0.142(3)\omega^{14} \end{aligned}$$

SVD, CGLS

A standard way of solving (1) is to discretize both x and ω and obtain the linear equation system to be solved for $p(x_i) = p_i$

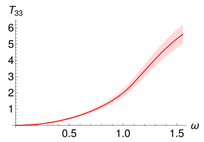
$$T_{33,i} = \sum_{j=1}^{M} K_{ij}\, p_j\,,\ i = 1,\cdots,N\,. \quad (2)$$

In general $M \neq N$ – and a singular value decomposition (SVD) approach could be attempted. In our case the singular values are distributed as



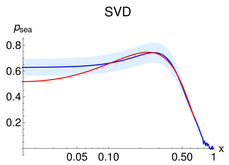
We observe a typical exponential decrease – but also a pronounced plateau, which indicates a possible meaningful application of the pseudoinverse to solve (2).

In order to investigate the sensitivity of the results on perturbations of the data we give the T_{33} a certain band of variation ($\pm 10\%$)

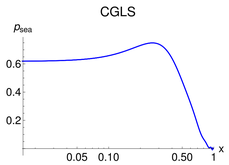


SVD, CGLS

The result for the inversion of the SVD is given ($\omega < 1.3$): red curve: input from above, the blue region indicates the variation with the T_{33}



Another method is the **conjugate gradient for least squares** (CGLS) used in many fields. It reduces iteratively the residuum of the discretized Fredholm equation. We get



The result is much more stable with respect to the variation of the data – the final band of variation for the pdf is nearly zero.

Bayes, Summary

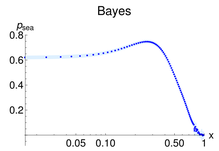
Karpie et al. proposed a variant of the Bayes method. Let $p_n = p(x_n)$ the wanted discrete PDF. Then

$$\frac{\delta\mathcal{P}(p|\mathcal{T}_{33})}{\delta p}\Big|_{p=p_{Bayes}} = 0$$

with

$$\mathcal{P}(p|\mathcal{T}_{33}) = \frac{\mathcal{P}(\mathcal{T}_{33}|p)\,\mathcal{P}(p)}{\mathcal{P}(\mathcal{T}_{33})}$$

- $\mathcal{P}(\mathcal{T}_{33}|p)$ – likelihood probability (from T_{33} data)
- $\mathcal{P}(p)$ – prior probability (Tikhonov regularization; fit to the data)
- $\mathcal{P}(\mathcal{T}_{33})$ – evidence (indep. of p)



Summary

- The greatest sensitivity to the data shows the SVD method
- CGLS has the least sensitivity
- Results show a deviation from the (in our case known) pdf for small x
- For $\omega > 1$ there are small wiggles in the vicinity of $x = 1$

Printed at Universitätsrechenzentrum Leipzig