Strong coupling constant and heavy quark masses in (2+1)-flavor QCD

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CMSE

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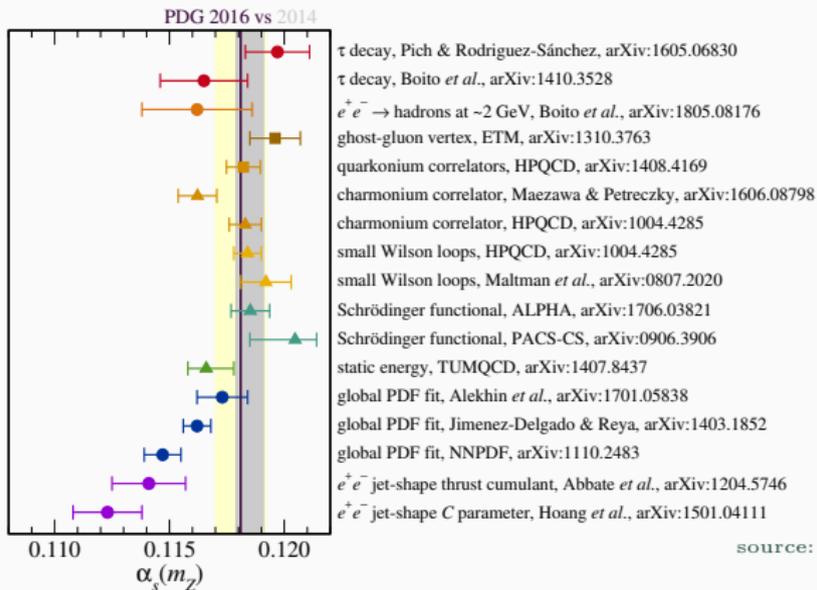**YM, PP: PR D94 (2016) ⇒ PP, JHW: arXiv:1901.06424**
TUMQCD: PR D90 (2014) ⇒ *in preparation*

Outline

Lattice determinations of α_s in context

PDG 2016 vs 2014

τ decay, Pich & Rodriguez-Sánchez, arXiv:1605.06830
τ decay, Boito et al., arXiv:1410.3528
$e^+e^- \to$ hadrons at ~2 GeV, Boito et al., arXiv:1805.08176
ghost-gluon vertex, ETM, arXiv:1310.3763
quarkonium correlators, HPQCD, arXiv:1408.4169
charmonium correlator, Maezawa & Petreczky, arXiv:1606.08798
charmonium correlator, HPQCD, arXiv:1004.4285
small Wilson loops, HPQCD, arXiv:1004.4285
small Wilson loops, Maltman et al., arXiv:0807.2020
Schrödinger functional, ALPHA, arXiv:1706.03821
Schrödinger functional, PACS-CS, arXiv:0906.3906
static energy, TUMQCD, arXiv:1407.8437
global PDF fit, Alekhin et al., arXiv:1701.05838
global PDF fit, Jimenez-Delgado & Reya, arXiv:1403.1852
global PDF fit, NNPDF, arXiv:1110.2483
e^+e^- jet-shape thrust cumulant, Abbate et al., arXiv:1204.5746
e^+e^- jet-shape C parameter, Hoang et al., arXiv:1501.04111

source: A. S. Kronfeld

- PDG has increased the global error of α_s since 2014
- Lattice QCD (HPQCD) dominates the global average and error
- Spread hints at underestimated systematic uncertainties?

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Lattice setup and heavy quark parameters

$\beta = 10/g_0^2$	$\frac{m_\ell}{m_s}$	$N_\sigma^3 \times N_\tau$	a^{-1} GeV	L_σ fm	am_{c0}	am_{b0}
6.740	0.05	48^4	1.81	5.2	0.5633(10)	
6.880	0.05	48^4	2.07	4.6	0.4800(10)	
7.030	0.05	48^4	2.39	4.0	0.4047(9)	
7.150	0.05	$48^3 \times 64$	2.67	3.5	0.3547(9)	
7.280	0.05	$48^3 \times 64$	3.01	3.1	0.3086(13)	
7.373	0.05	$48^3 \times 64$	3.28	2.9	0.2793(5)	
7.596	0.05	64^4	4.00	3.2	0.2220(2)	1.019(8)
7.825	0.05	64^4	4.89	2.6	0.1775(3)	0.7985(5)
7.030	0.20	48^4	2.39	4.0	0.4047(9)	
7.825	0.20	64^4	4.89	2.6	0.1775(3)	0.7985(5)
8.000	0.20	64^4	5.58	2.3	0.1495(6)	0.6710(6)
8.200	0.20	64^4	6.62	1.9	0.1227(3)	0.5519(6)
8.400	0.20	64^4	7.85	1.6	0.1019(27)	0.4578(6)

- Pseudoscalar meson operator $j_5(x) = \bar{\psi}(x)\gamma_5\psi(x)$
- RGI pseudoscalar meson correlator

$$G(\tau) = a^8 m_{h0}^2 \sum_{\boldsymbol{x}} \langle j_5(\boldsymbol{x},\tau)j_5(0,0)\rangle_U \quad \lim_{\tau\to 0} \quad \left(\frac{a}{\tau}\right)^4$$

- HISQ valence quarks, m_c and m_b tuned using η_c and η_b masses
- Meson correlators with $am_{h0} = 1, 1.5, 2, 3, 4$ am_{c0}, and am_{b0}

Quarkonium moments with HISQ action

- Time moments are finite for $n \geq 4$ defined on the lattice as

$$G_n = \sum_{\tau/a=1}^{N_\tau/2} \left(\frac{\tau}{a} \right)^n [G(\tau) + G(aN_\tau - \tau)]$$

- Use random color wall sources – statistical errors become irrelevant
- Fluctuations and mass dependence reduced in ratios $G_n^{\frac{1}{n-4}} / G_{n+2}^{\frac{1}{n-2}}$
- Artifacts $\sim \alpha_s^0 (am_{h0})^n$ cancel in reduced moments $R_n = \left(\frac{G_n^{QCD}}{G_n^0} \right)^{\frac{1}{n-4}}$
- Artifacts $\sim \alpha_s^m (am_{h0})^n$ persist in R_n, no artifacts $\sim (a\Lambda_{\mathrm{QCD}})^n$ relevant
- Artifacts are worse in lower moments ($\tau \sim a$) and for larger masses
- Finite size effects are worse in higher moments ($\tau \sim aN_\tau$) and for free theory moments G_n^0 ("quark-antiquark" scattering states, not hadrons)

Approach to the continuum limit
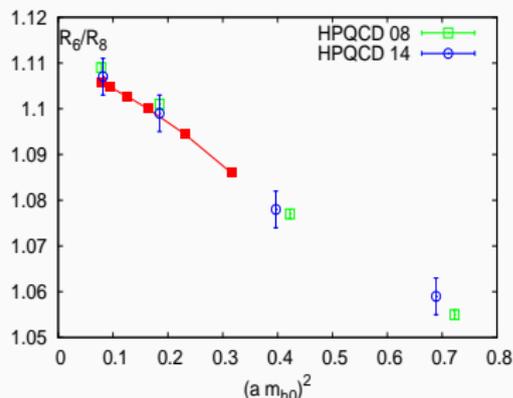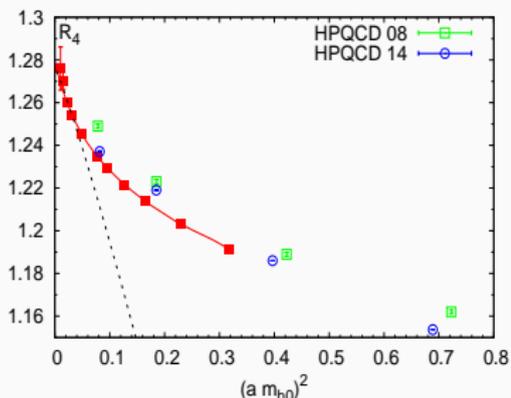
- Unresolved logs $\Rightarrow R_4$ under- and R_6/R_8 or R_8/R_{10} overestimated
- **Boosted coupling** $\alpha_s^{\mathrm{lat}} = 10/(4\pi\beta u_0^4)$, where u_0 is the tadpole factor, i.e., an average link U defined via the plaquette, $u_0^4 = \langle \mathrm{Tr}\, U_\square \rangle /3$
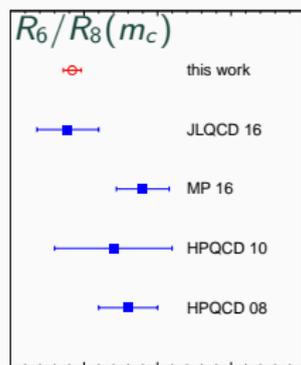- We extrapolate the reduced moments and ratios to the continuum using

$$R(\alpha_s^{\mathrm{lat}}, am_h) = \sum_{n=1}^{N} \sum_{j=1}^{J} c_{nj}\ (\alpha_s^{\mathrm{lat}})^n (am_h)^{2j}, \quad N \leq 3,\ J \leq 5$$

- Similar for larger m_h; control of continuum limit up to $m_h = 3m_c$
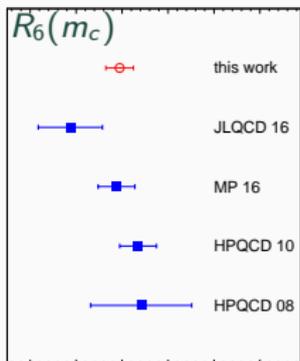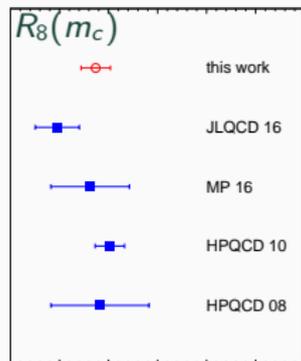
Continuum results at the charm scale
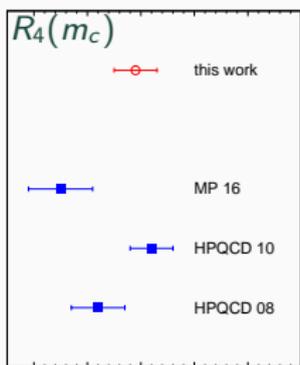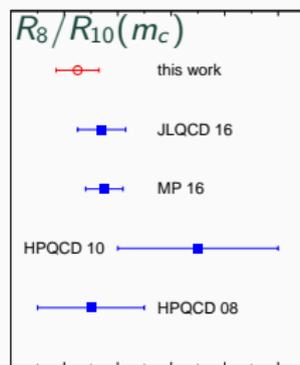
Outline
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Introduction
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Quarkonium moments
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Static energy
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Singlet free energy
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Summary
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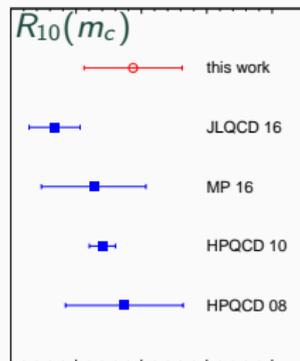
Reduced quarkonium moments in perturbation theory

- We compare to the known weak-coupling result[1] at order α_s^3

$$R_n = \begin{cases} r_4 & (n=4) \\ r_n \cdot \frac{m_{h0}}{m_h} & (n \geq 6) \end{cases} , \qquad r_n = 1 + \sum_{j=1}^{3} r_{nj}\left(m_h, \frac{\mu}{m_h}\right)\alpha_s^j(\mu)$$

- We estimate the uncertainty due to the truncation of the perturbative series with an α_s^4 term, whose coefficient is varied in the range $\pm 5 r_{n3}$
- **Nonperturbative physics enters only via QCD condensates**
⇒ Leading nonperturbative contribution due to the gluon condensate[2]
- We determine $\alpha_s(m_h)$ from the nonlinear equations

$$R_4(\alpha_s(m_h)) = 1 + \sum_{j=1}^{3} r_{4,j}(m_h, 1)\alpha_s^j(m_h) + \frac{1}{m_h^4}\frac{11}{4}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle, \quad \text{etc. },$$

using the gluon condensate $\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle = -0.006(12)\,\text{GeV}^4$ from τ decays[3]

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[1] Sturm, JHEP 0809 (2008) 075
  Kiyo et al., Nulc. Phys. B 823, 269 (2009)
  Maier et al., Nucl. Phys. B 824, 1 (2010)
[2] Broadhurst et al., Phys. Lett. B **329**, 103 (1994)
[3] Geshkenbein et al., Phys. Rev. D **64**, 093009 (2001)

α_s at the heavy quark scale m_h

$\frac{m_h}{m_c}$	R_4	R_6/R_8	R_8/R_{10}	av.	$\Lambda_{\mathrm{QCD}}^{N_f=3}$ MeV
1.0	0.3815(55)(30)(22)	0.3837(25)(180)(40)	0.3550(63)(140)(88)	0.3788(65)	315(9)
1.5	0.3119(28)(4)(4)	0.3073(42)(63)(7)	0.2954(75)(60)(17)	0.3099(48)	311(10)
2.0	0.2651(28)(7)(1)	0.2689(26)(35)(2)	0.2587(37)(34)(6)	0.2649(29)	285(8)
3.0	0.2155(83)(3)(1)	0.2338(35)(19)(1)	0.2215(367)(17)(1)	0.2303(150)	284(48)

- Three errors of $\alpha_s(m_h)$ due to the continuum-extrapolated lattice data, the truncation of the perturbative series, and the gluon condensate
- The latter two shrink at the expense of the lattice error for $m_h > m_c$
- All three errors generally increase for the ratios, and α_s from R_8/R_{10} is usually lower than α_s from R_4 or R_6/R_8 for no apparent reason
- Weighted average of the three observables at each scale, and determine the minimal uncertainty such that it has overlapping errors with each
- Consistency of three $\alpha_s(m_h)$ is powerful check for the continuum limit

At $\mu = m_c$: $\alpha_s(M_Z, N_f = 5) = 0.1166(7)$ vs $\alpha_s(M_Z, N_f = 5) = 0.1183(7)$[4]

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[4]McNeile et al., Phys.Rev. D82 (2010) 034512

Heavy quark masses m_h from higher moments

| $\frac{m_h}{m_c}$	R_6	R_8	R_{10}
1.0	1.2740(25)(17)(11)(61)	1.2783(28)(23)(00)(43)	1.2700(72)(46)(13)(33)
1.5	1.7147(83)(11)(03)(60)	1.7204(42)(14)(00)(40)	1.7192(35)(29)(04)(30)
2.0	2.1412(134)(07)(01)(44)	2.1512(71)(10)(00)(29)	2.1531(74)(19)(02)(21)
3.0	2.9788(175)(06)(00)(319)	2.9940(156)(08)(00)(201)	3.0016(170)(16)(00)(143)
4.0	3.7770(284)(06)(00)(109)	3.7934(159)(08)(00)(68)	3.8025(152)(15)(00)(47)
$\frac{m_b}{m_c}$	4.1888(260)(05)(00)(111)	4.2045(280)(07)(00)(69)	4.2023(270)(14)(00)(47)

- Four errors of m_h due to the continuum-extrapolated lattice data, truncation of the perturbative series, the gluon condensate, and $\alpha_s(m_h)$
- The error due to the lattice scale r_1 is not included in the table
- Continuum extrapolation of R_6, R_8, and R_{10} is unproblematic for all m_h
- At each $m_h \leq 3m_c$ we obtain Λ_{QCD} from m_h and $\alpha_s(m_h)$, and take the unweighted average of $\Lambda_{\mathrm{QCD}}^{N_f=3}$, and use the spread as systematic error

$$\Lambda_{\mathrm{QCD}}^{N_f=3} = 301 \pm 16\,\mathrm{MeV}, \qquad \alpha_s(M_Z, N_f = 5) = 0.1161(12),$$
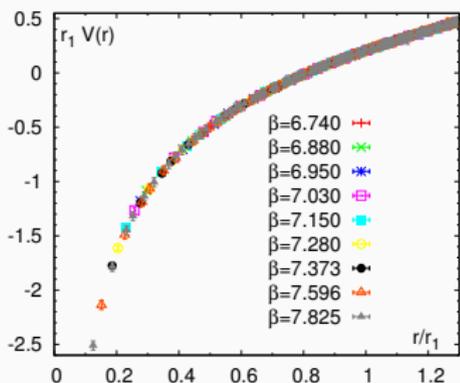
- For $m_h > 3m_c$: unweighted average of Λ_{QCD}, then use 4-loop running to obtain $\alpha_s(4m_c)$ and $\alpha_s(m_b)$, matching to 4 or 5 flavors at 1.5 or 4.7 GeV

$$m_c(m_c, N_f = 4) = 1.2672(84)\,\mathrm{GeV}, \quad m_b(m_b, N_f = 5) = 4.188(29)\,\mathrm{GeV}$$

Running of the $\overline{MS}$ mass renormalization factor $m_h(m_h)/(m_h/m_c)$
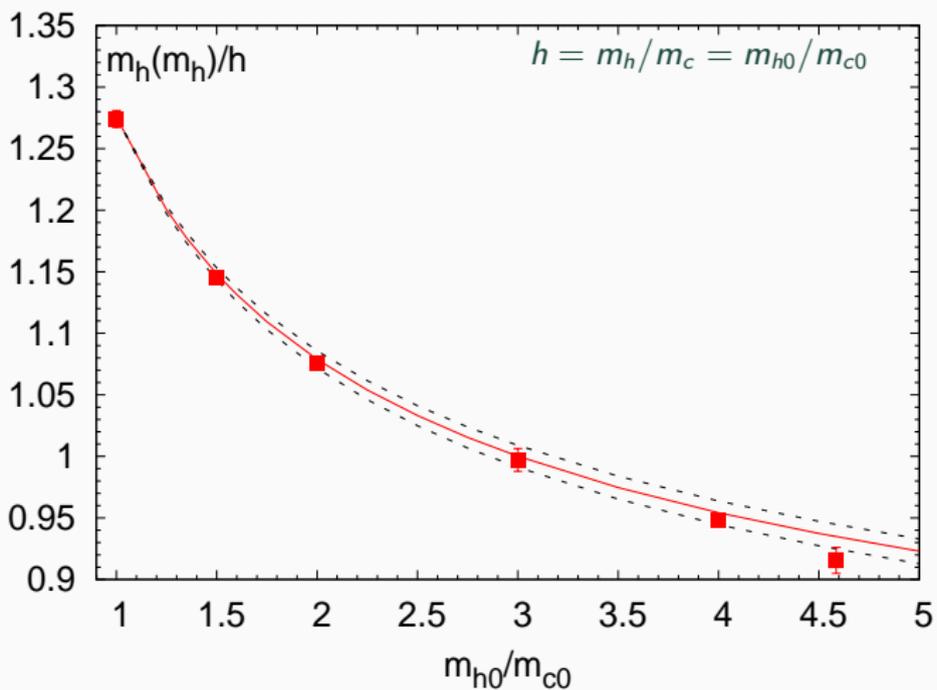
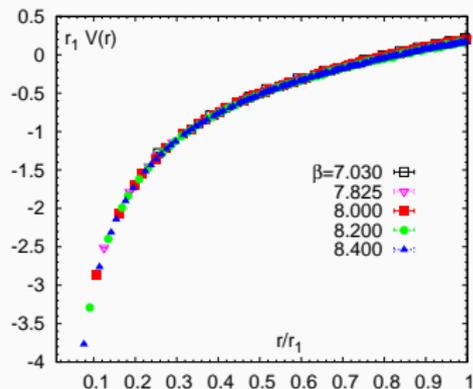# Static energy on the lattice: 2014 vs 2019

2014 edition[5], $a^{-1} \leq 4.9\,\mathrm{Gev}$

2019 edition[6], $a^{-1} \leq 7.9\,\mathrm{Gev}$

- Smallest distance $r = 0.04\,\mathrm{fm}$

- Perturbative errors dominant

- Very light pion $m_\pi = 160\,\mathrm{MeV}$

- Consistent with 2012 edition[7]

- Three extra fine lattice spacings at $T = 0$

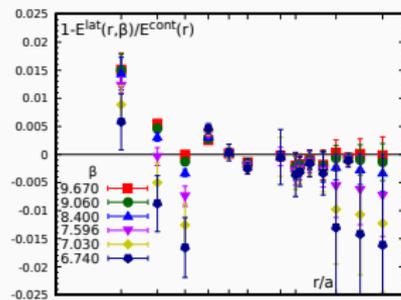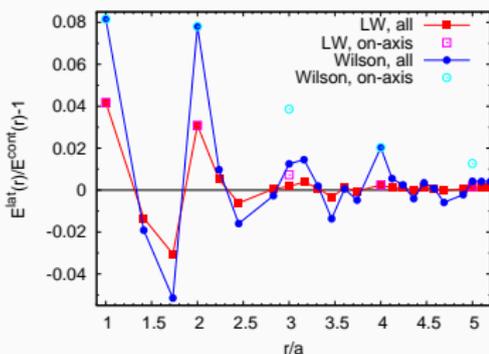- Include for shortest distances singlet free energies at $T > 0$ $\Rightarrow a^{-1} \lesssim 22\,\mathrm{GeV}$

[5] Bazavov et al., Phys. Rev. D90 (2014) 7, 074038
[6] Bazavov et al. [TUMQCD], *in preparation*
[7] Bazavov et al., Phys. Rev. D86 (2012) 114031

Lattice artifacts in the static quark-antiquark energy

- The static energy at short distances has percent-level lattice artifacts

- Improved gauge action (Lüscher–Weisz) – reduced symmetry breaking
- Tree-level improvement: $\frac{E^{\mathrm{lat}}(r)}{E^{\mathrm{cont}}(r)}$ for OGE without running coupling
- After tree-level correction – smaller cutoff effects with similar pattern[8]
- E on fine lattices as continuum estimate, correct for cutoff effects
- Alternatively use only data with $r/a \geq \sqrt{8}$ omitting $r/a = \sqrt{12}$

[8]Bazavov et al, Phys.Rev. D98 (2018) no.5, 054511

Static quark-antiquark energy in perturbation theory

- Static energy determined from large-time behavior of Wilson loops

$$E(r) = \Lambda_s - \frac{C_F \alpha_s}{r} \left(1 + \#\alpha_s + \#\alpha_s^2 + \#\alpha_s^3 + \#\alpha_s^3 \ln \alpha_s + \#\alpha_s^4 \ln^2 \alpha_s + \#\alpha_s^4 \ln \alpha_s + \ldots \right) \quad \text{@ N3LL}$$

- US contributions to the static energy can be understood in pNRQCD

$$E(r) = \Lambda_s - V_s(r, \mu_{us}) - i \frac{g^2}{N_c} V_A^2 \int\limits_0^\infty dt e^{-it(V_o - V_s)} \langle \mathrm{Tr}\, \boldsymbol{r} \cdot \boldsymbol{E}(t) \boldsymbol{r} \cdot \boldsymbol{E}(0) \rangle (\mu_{us}) + \ldots$$

  as to include the singlet potential and an ultrasoft contribution[9]

- The factorization of the ultrasoft contribution gives rise to the ultrasoft scale μ_{us}, the scale of transitions between singlet and octet

- Cancellation of intermediate scale[10]: $\ln \alpha_s = \ln \left(\frac{\mu_{us}}{1/r} \right) + \ln \left(\frac{\alpha_s/r}{\mu_{us}} \right)$

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[9]Brambilla et al., Nucl. Phys. B566 (2000) 275
[10]Brambilla et al., Phys. Rev. D60 (1999) 091502

Fitting lattice results of the static energy (2014)
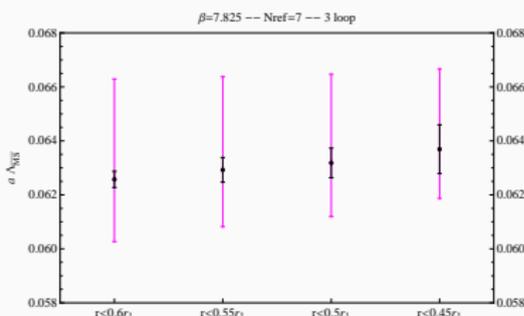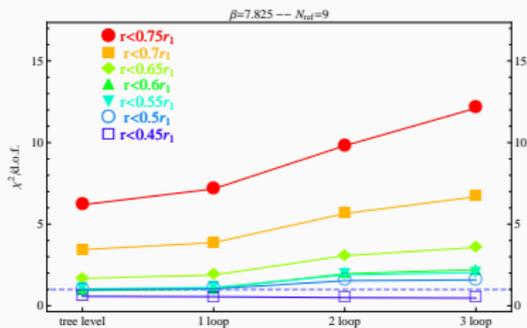
Different perturbative orders

- χ^2/dof reduces for higher orders at shorter distances
- ⇒ Weak-coupling suitable for static energy for $r \lesssim 0.15\,\mathrm{fm}$
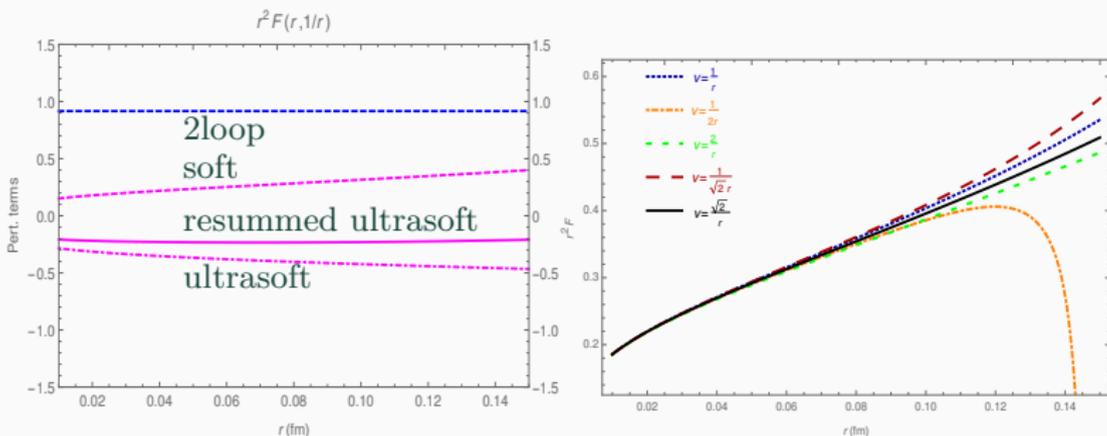- At shortest distances little sensititivy to perturbative order

When going to shorter distances

- Statistical errors increase
- Perturbative errors decrease

Perturbative errors estimated from

- scale variation $\nu = \frac{1}{\sqrt{2}r}$ to $\frac{\sqrt{2}}{r}$
- soft higher order term $\pm \# \frac{\alpha_s^4}{r}$

Perturbative uncertainty in the 2019 edition

- Ultrasoft logs are small – use *three-loop with leading US resummation*
- Soft scale variation generates the dominant uncertainty at three loop
- More conservative soft scale variation in 2019 edition: $\nu = \frac{1}{2r}$ to $\frac{2}{r}$
- Nonmonotonic soft scale dependence is minimal for $\nu \approx 1/(\sqrt{2}r)$
- Soft scale $\nu \approx 1/(2r)$ not suitable for $r \gtrsim 0.1\,\mathrm{fm}$

α_s from $T = 0$ in the 2019 edition
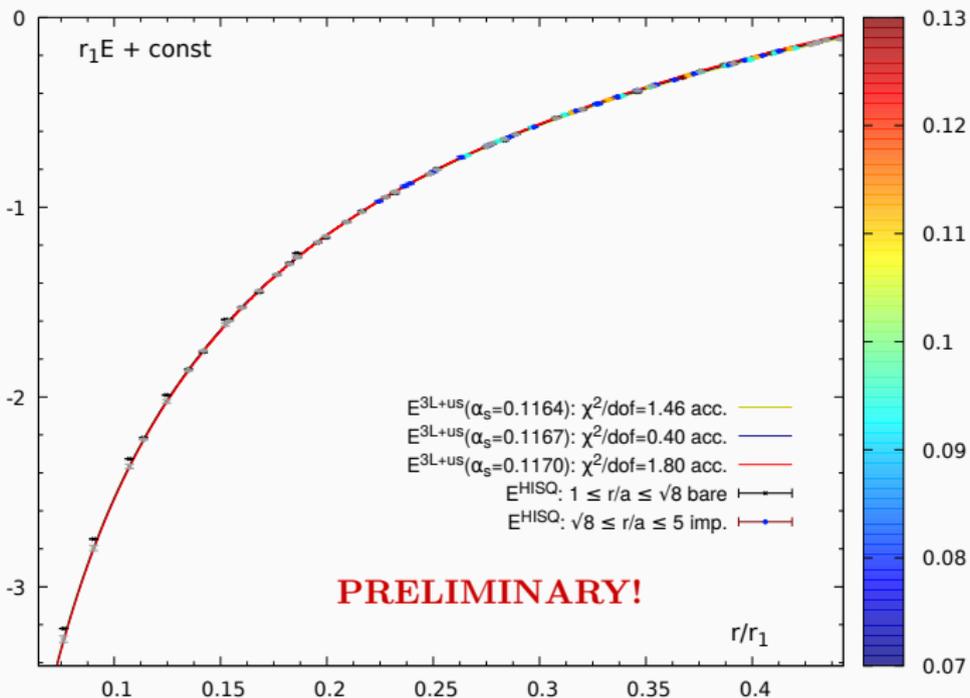
- Restrict lattice data to $r < 0.14\,\text{fm} \approx 0.45 r_1$
- Combined analysis of lattice data with $a \leq 0.06\,\text{fm}$, i.e., $a/r_1 \leq 0.2$
- Analysis for $r/a \geq \sqrt{8} \Rightarrow$ lattice artifacts are statistically irrelevant
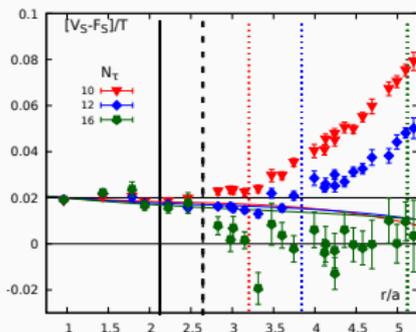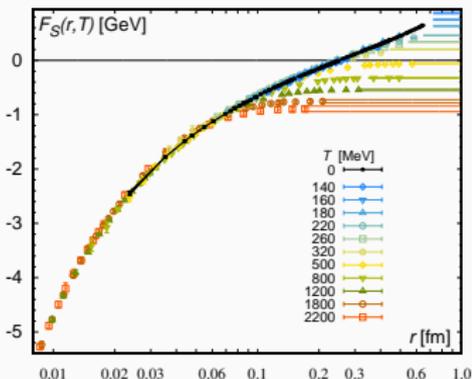
Systematic errors in the 2019 edition

$\min(r/a)$	$\max(r)\,\mathrm{fm}$	α_s	δ^{stat}	$\delta^{\mathrm{pert}}_{2014}$	$\delta^{\mathrm{pert}}_{2019}$
$\sqrt{8}$	0.097	0.1166	0.0007	$^{+0.0007}_{-0.0003}$	$^{+0.0015}_{-0.0005}$
$\sqrt{8}$	0.131	0.1167	0.0005	$^{+0.0008}_{-0.0003}$	$^{+0.0019}_{-0.0006}$
1	0.055	0.1164	0.0005	$^{+0.0003}_{-0.0001}$	$^{+0.0008}_{-0.0003}$
1	0.073	0.1166	0.0004	$^{+0.0004}_{-0.0001}$	$^{+0.0010}_{-0.0003}$
1	0.098	0.1167	0.0003	$^{+0.0005}_{-0.0002}$	$^{+0.0012}_{-0.0004}$
1	0.131	0.1167	0.0003	$^{+0.0007}_{-0.0003}$	$^{+0.0015}_{-0.0005}$

- Must keep $r \lesssim 0.1\,\mathrm{fm}$ to enable the full soft scale variation
- Central value α_s for soft scale $1/(\sqrt{2}r) \leq \nu \leq \sqrt{2}/r$ is very stable against variation of $\max(r)$
- Include $r/a < \sqrt{8}$ to reduce the impact of soft scale variation
- r_1 scale error: $\pm 1.7\,\mathrm{MeV}$ for Λ_{QCD}, ± 0.0001 for $\alpha_s(M_Z, N_f = 5)$

PRELIMINARY!

$$\Lambda^{N_f=3}_{\mathrm{QCD}} = 314^{+15}_{-8}\,\mathrm{MeV}, \qquad \alpha_s(M_Z, N_f = 5) = 0.11660^{+0.00106}_{-0.00056}$$

$T > 0$ data in the 2019 edition

- Singlet free energy for $T > 0$ with much finer lattice spacing[11]
- $T > 0$ effects exponentially suppressed for $\alpha_s/r \gg T$, i.e., $r/a \ll \alpha_s N_\tau$
- Nonconstant $T > 0$ effects are numerically small for $r/a \lesssim 0.3 N_\tau$

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[11]Bazavov et al, Phys.Rev. D98 (2018) no.5, 054511

α_s from $T > 0$
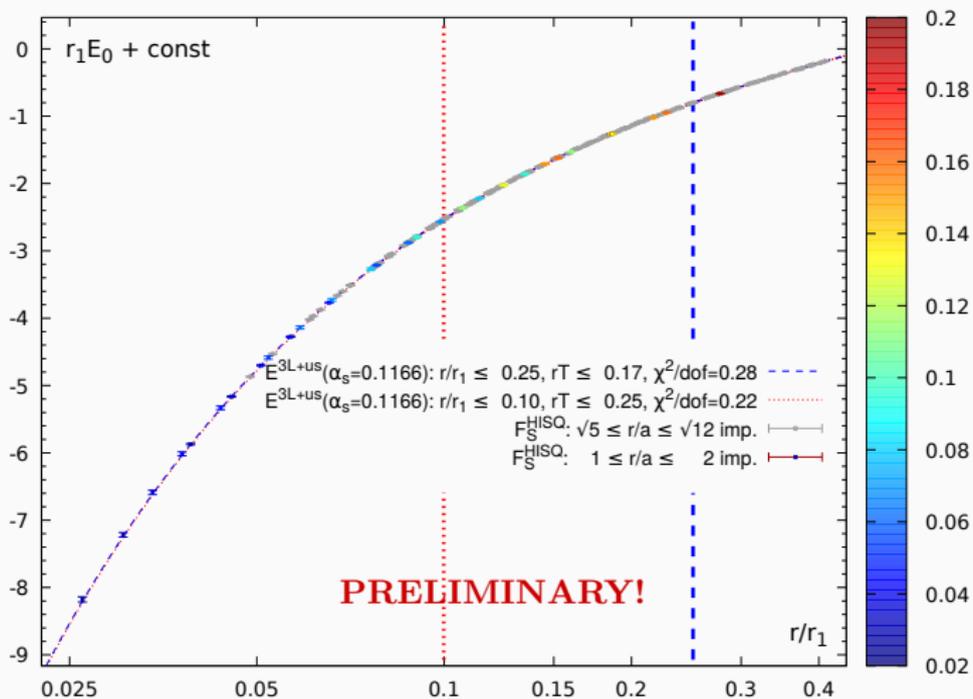
- Restrict $N_\tau = 12$ data to $r/a \leq 2$ or 3, i.e., $r \leq 0.17/T$ or $0.25/T$
- Cannot avoid having to correct for the lattice artifacts

$T = 0$ **vs** $T > 0$

| N_τ	$\max(r/a)$	$\max(r)\,\mathrm{fm}$	α_s	δ^{stat}	$\delta^{\mathrm{pert}}_{2014}$	$\delta^{\mathrm{pert}}_{2019}$
64	2	0.057	0.1165	0.0006	$+0.0003_{-0.0001}$	$+0.0008_{-0.0002}$
64	2	0.078	0.1166	0.0005	$+0.0004_{-0.0001}$	$+0.0010_{-0.0003}$
64	2	0.096	0.1166	0.0005	$+0.0004_{-0.0002}$	$+0.0011_{-0.0003}$
12	2	0.057	0.1165	0.0007	$+0.0002_{-0.0001}$	$+0.0006_{-0.0002}$
12	2	0.078	0.1166	0.0006	$+0.0003_{-0.0001}$	$+0.0008_{-0.0003}$
12	2	0.091	0.1167	0.0006	$+0.0003_{-0.0001}$	$+0.0008_{-0.0003}$
64	3	0.055	0.1164	0.0005	$+0.0003_{-0.0001}$	$+0.0008_{-0.0003}$
64	3	0.073	0.1166	0.0004	$+0.0004_{-0.0001}$	$+0.0010_{-0.0003}$
64	3	0.096	0.1167	0.0004	$+0.0005_{-0.0002}$	$+0.0011_{-0.0004}$
64	3	0.134	0.1167	0.0003	$+0.0006_{-0.0003}$	$+0.0013_{-0.0005}$
12	3	0.055	0.1167	0.0005	$+0.0002_{-0.0001}$	$+0.0006_{-0.0002}$
12	3	0.073	0.1168	0.0005	$+0.0003_{-0.0001}$	$+0.0007_{-0.0003}$
12	3	0.096	0.1168	0.0005	$+0.0003_{-0.0001}$	$+0.0009_{-0.0003}$
12	3	0.133	0.1168	0.0004	$+0.0005_{-0.0002}$	$+0.0011_{-0.0004}$

Complete agreement between α_s from $T = 0$ or $T > 0$

Summary

- We determine the strong coupling constant α_s and the charm and bottom quark masses using moments of PS quarkonium correlators, with 6 heavy quark masses, 11 lattice spacings and 2 sea quark masses
- We determine the strong coupling constant α_s from the static energy using 6 lattice spacings with more conservative perturbative errors and from the singlet free energy using 15 lattice spacings (and two N_τ)

Quarkonium	2016	2019
$\alpha_s(m_Z, N_f = 5)$	0.11622(84)	0.1161(12)
$\Lambda_{\rm QCD}(N_f = 3)$	308(12) MeV	301(16) MeV
Heavy quark scale	$\mu = m_c$	$\mu \leq 3m_c$
$m_c(m_c, N_f = 4)$	1.267(12) GeV	1.2672(84) GeV
$m_b(m_b, N_f = 5)$	4.184(89) GeV	4.188(29) GeV

Static energy	2014	2019 (**PRELIMINARY!**)
$\alpha_s(m_Z, N_f = 5)$	$0.1166^{+0.0012}_{-0.0008}$	$0.11660^{+0.00106}_{-0.00056}$
$\Lambda_{\rm QCD}(N_f = 3)$	315^{+18}_{-12} MeV	314^{+15}_{-08} MeV
Soft scale	$\nu = 1/\max(r) \gtrsim 2/r_1$	$\nu = 1/\max(r) \gtrsim 4/r_1$

Singlet free energy	past	2019 (**PRELIMINARY!**)
$\alpha_s(m_Z, N_f = 5)$	NA	$0.11638^{+0.00094}_{-0.00087}$
$\Lambda_{\rm QCD}(N_f = 3)$	NA	310^{+13}_{-12} MeV
Soft scale	NA	$\nu = 1/\max(r) \gtrsim 10/r_1$

Running of α_s at low scales
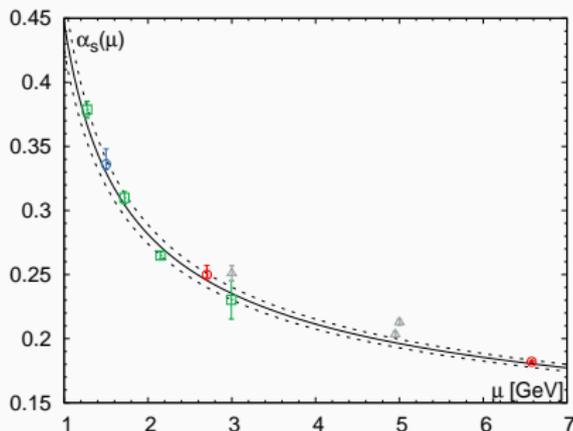
- 2014 HPQCD quarkonium correlators[12]
- 2019 quarkonium correlators[13]
- 2014 TUMQCD static energy[14]
- 2019 static energy and singlet free energy

[12] Chakraborty et al., Phys.Rev. D91 (2015) no.5, 054508
     McNeile et al., Phys.Rev. D82 (2010) 034512
     Allison et al., Phys.Rev. D78 (2008) 054513
[13] PP, JHW: arXiv:1901.06424
[14] Bazavov et al., Phys. Rev. D90 (2014) 7, 074038

Thank you!

n	r_{n1}	r_{n2}	r_{n3}
4	2.3333	-0.5690	1.8325
6	1.9352	4.7048	-1.6350
8	0.9940	3.4012	1.9655
10	0.5847	2.6607	3.8387

Table: The coefficients of the perturbative expansion of R_n

Continuum results for R_4 and ratios R_6/R_8, and R_8/R_{10}

m_h	R_4	R_6/R_8	R_8/R_{10}
$1.0m_c$	1.279(4)	1.1092(6)	1.0485(8)
$1.5m_c$	1.228(2)	1.0895(11)	1.0403(10)
$2.0m_c$	1.194(2)	1.0791(7)	1.0353(5)
$3.0m_c$	1.158(6)	1.0693(10)	1.0302(5)

Table: Continuum results for R_4, R_6/R_8 and R_8/R_{10} at different quark masses m_h

**Continuum results for R_6/m_{h0}, R_8/m_{h0}, and R_{10}/m_{h0}**

m_h	R_6/m_{h0}	R_8/m_{h0}	R_{10}/m_{h0}
$1.0m_c$	1.0195(20)	0.9174(20)	0.8787(50)
$1.5m_c$	0.7203(35)	0.6586(16)	0.6324(13)
$2.0m_c$	0.5584(35)	0.5156(17)	0.4972(17)
$3.0m_c$	0.3916(23)	0.3647(19)	0.3527(20)
$4.0m_c$	0.3055(23)	0.2859(12)	0.2771(23)
m_b	0.2733(17)	0.2567(17)	0.2499(16)

Table: Continuum results for R_n/m_{h0}, $n \geq 6$ for different quark masses, m_h

Wilson loops vs Wilson line correlators in Coulomb gauge
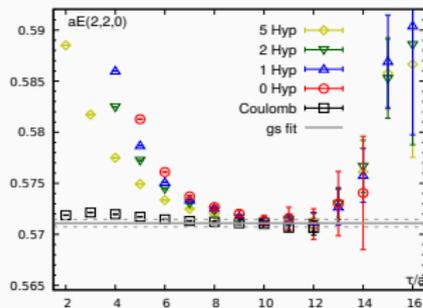
Wilson loops on the lattice

- \+ Explicit gauge invariance
- \– Cusp divergences due to corners
- \– Extra cusp divergences for off-axis separation
- \– Self-energy divergences due to spatial Wilson lines
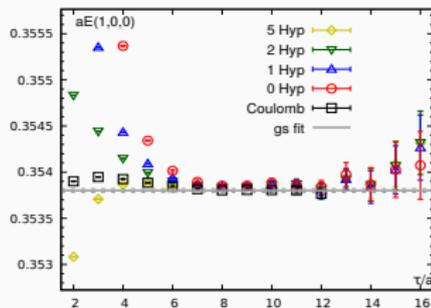
Wilson line correlator on the lattice

- \– Must fix some gauge, i.e. Coulomb gauge
- \+ No corners, no cusps
- \+ On- and off-axis separation have same divergence
- \+ No spatial Wilson lines

- Same ground state for both, but Wilson lines technically favorable
- Distortions at small distance and time for both operators

Quark mass dependence and topology

- Combine gauge ensembles with different light sea quark mass
- $\Rightarrow$ No statistically significant quark mass effects up to $r \approx 0.5r_1$
- Fine gauge ensembles with fully suppressed topological tunneling
- $\Rightarrow$ No statistically significant difference between static energy in different topological sectors up to $r \approx 0.5r_1$ observed[15]
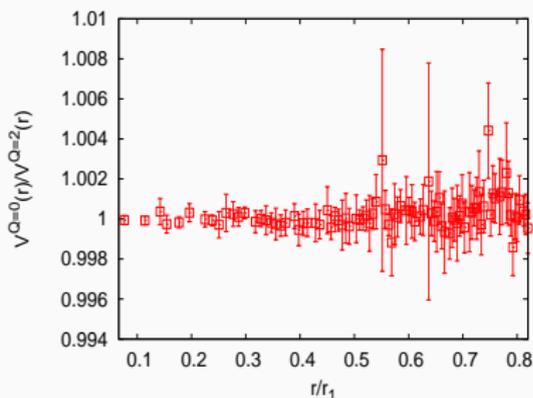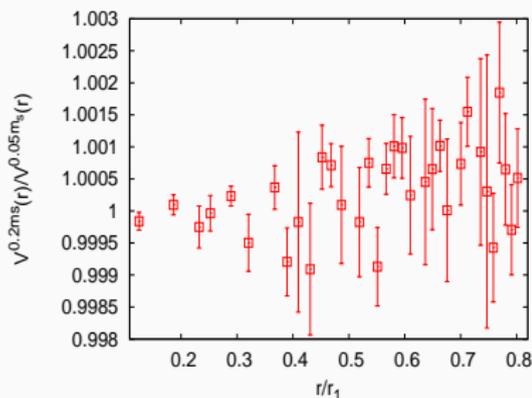
[15] Bazavov et al., arXiv:1811.11902

Nonperturbative improvement of the static energy
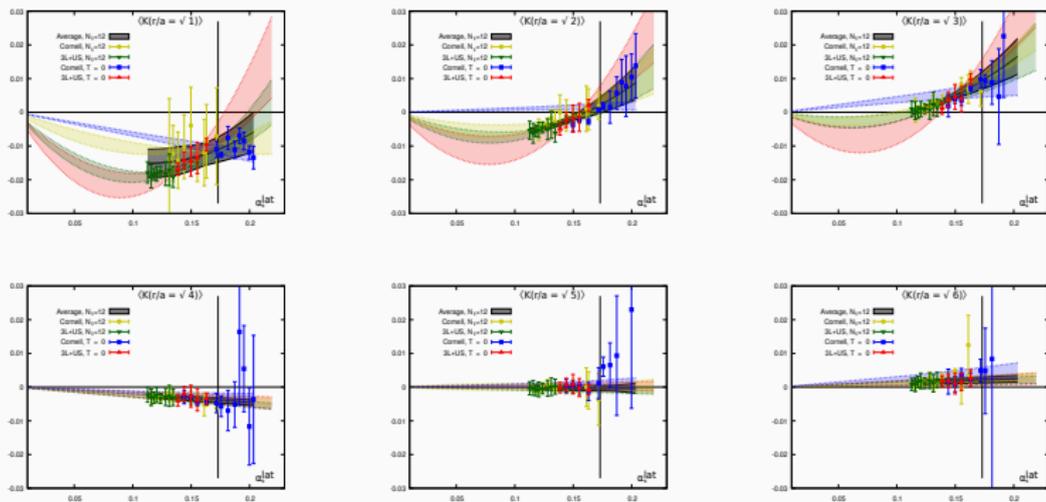
- Estimate continuum static energy using fine lattice using $r/a \geq \sqrt{5}$, determine corrections for coarser lattices
- → extrapolate the corrections in boosted coupling α_s^{lat} to finer lattices
- Estimate corrections to lattice static energy for $r/a \geq 1$ using *three-loop plus ultrasoft* result with Λ_{QCD} in a reasonable window