Renormalization of Bilinear and Four-Fermion Operators Through Temporal Moments

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JLQCD collaboration

June 17, 2019 LATTICE2019 @Wuhan

Outline

We propose a new method to calculate renormalization factors for lattice operators. It is based on the temporal moments of charmonium correlators.

1. Matching with charmonium temporal moments
2. Bilinear operators
3. Four-fermion operators
4. Summary

Matching Between Continuum and Lattice

Renormalization can be performed through a matching

$\overline{\mathrm{MS}}$ ← **lattice**

direct

perturbative

some "finite" physical quantities

e.g. X-space correlator

$\langle J(x)J(0)\rangle$

G. Martinelli et al. PLB 411 (1997)

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indirect

$\overline{\mathrm{MS}}$ ← **intermediate scheme** ← **lattice**

typically one-loop

nonperturbative

e.g. MOM (with gauge fixing)

G. Martinelli et al. NPB 445 (1995)

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direct — perturbative

some "finite" physical quantities

e.g. X-space correlator

$\langle J(x)J(0)\rangle$

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We use the charmonium temporal moments

$$\int dt\ t^n \langle J(t)J(0)\rangle$$

$\overline{\mathrm{MS}}$ ← **intermediate scheme** ← **lattice**

indirect — typically one-loop — nonperturbative

e.g. MOM (with gauge fixing)

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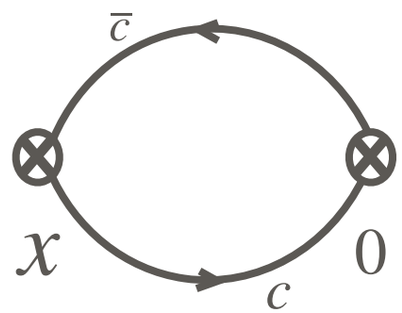
Moments of Charmonium Correlators

$\Pi(q^2)$



$$= \int d^4x \; e^{iqx} \langle 0 \,|\, \mathrm{T} J(t,x) J(0,0) \,|\, 0 \rangle$$

Moments: $$\left(\frac{\partial}{\partial q^2}\right)^k \Pi(q^2)\Big|_{q^2=0}$$

Matching condition (direct)

$$\left(\frac{\partial}{\partial q^2}\right)^k \Pi^{\overline{\mathrm{MS}}}(\mu; q^2)\Big|_{q^2=0} = \left(Z^{\overline{\mathrm{MS}}/\mathrm{lat}}(\mu, a)\right)^2 \left(\frac{\partial}{\partial q^2}\right)^k \Pi^{\mathrm{lat}}(a; q^2)\Big|_{q^2=0}$$

perturbative theory
known to $O(\alpha_s^3)$ A. Maier et al. NPB 824 (2010)

non-perturbative

- to determine $Z^{\overline{\mathrm{MS}}/\mathrm{lat}}(\mu, a)$ for the lattice current
- $k > 1$ to avoid extra divergence due to $x \to 0$

Bilinear Operators

Moments on the Lattice

Charmonium correlator

$$G^{PS}(t) = a^6 \sum_{\boldsymbol{x}} \left\langle 0 \left| j_5(\boldsymbol{x}, t) j_5(0,0) \right| 0 \right\rangle$$

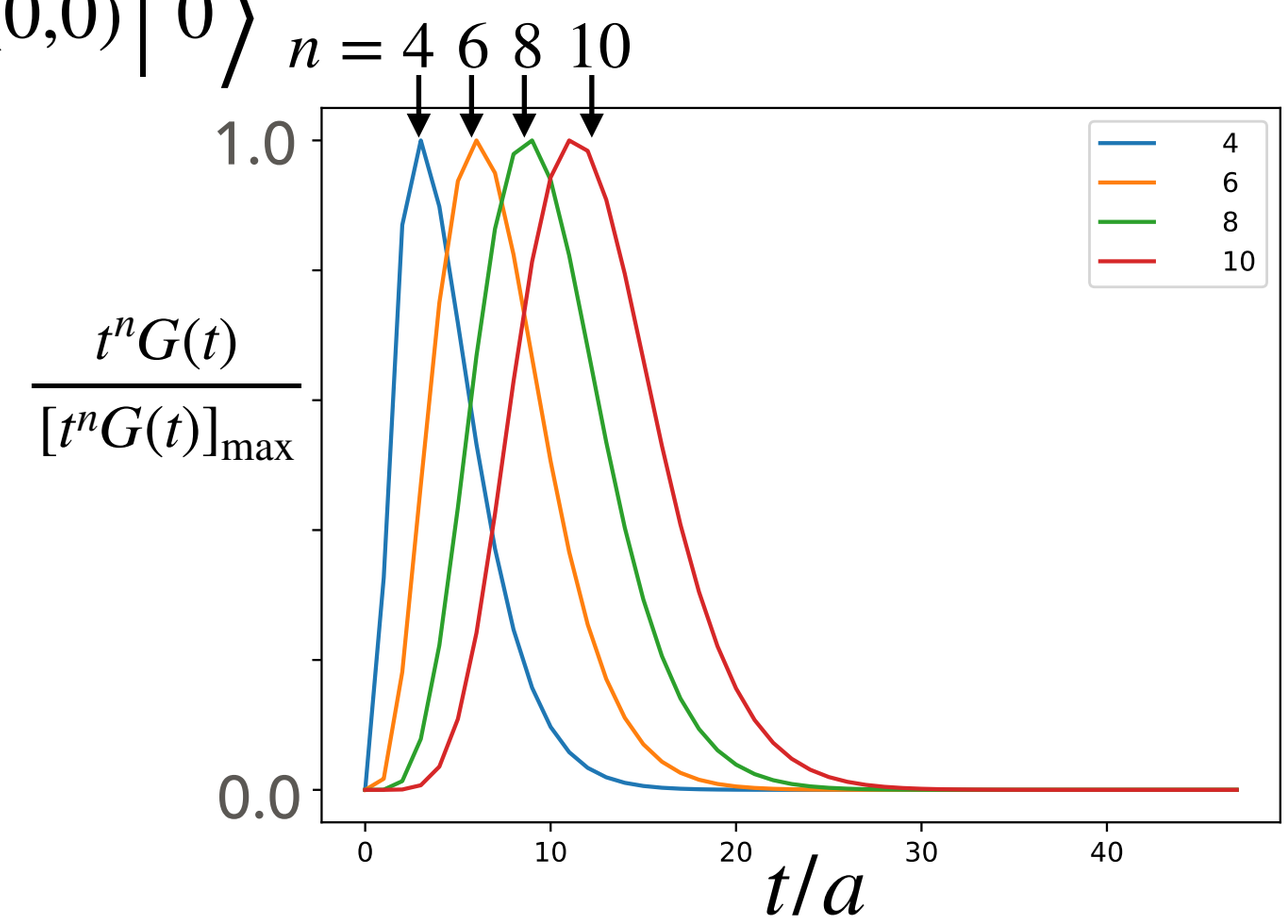
Temporal moments

$$G_n^{PS} = \sum_t \left(\frac{t}{a}\right)^n G^{PS}(t)$$

FT

$$\left(\frac{\partial}{\partial q^2}\right)^k \Pi(q^2)\Big|_{q^2=0}$$

$(2k = n)$



typical length scale $\sim 1/m_c$

Setup

- JLQCD ensemble

  Nf = 2+1 Möbius domain-wall fermion

β	a^{-1}[GeV]	$L^3 \times T(\times L_5)$	#meas	am_{ud}	am_s	am_c
4.17	2.453(4)	$32^3 \times 64\ (\times 12)$	100	0.007	0.04	0.44037
4.35	3.610(9)	$48^3 \times 96\ (\times 8)$	50	0.0042	0.025	0.27287
4.47	4.496(9)	$64^3 \times 96\ (\times 8)$	50	0.0030	0.015	0.210476

- inputs for perturbative expansion

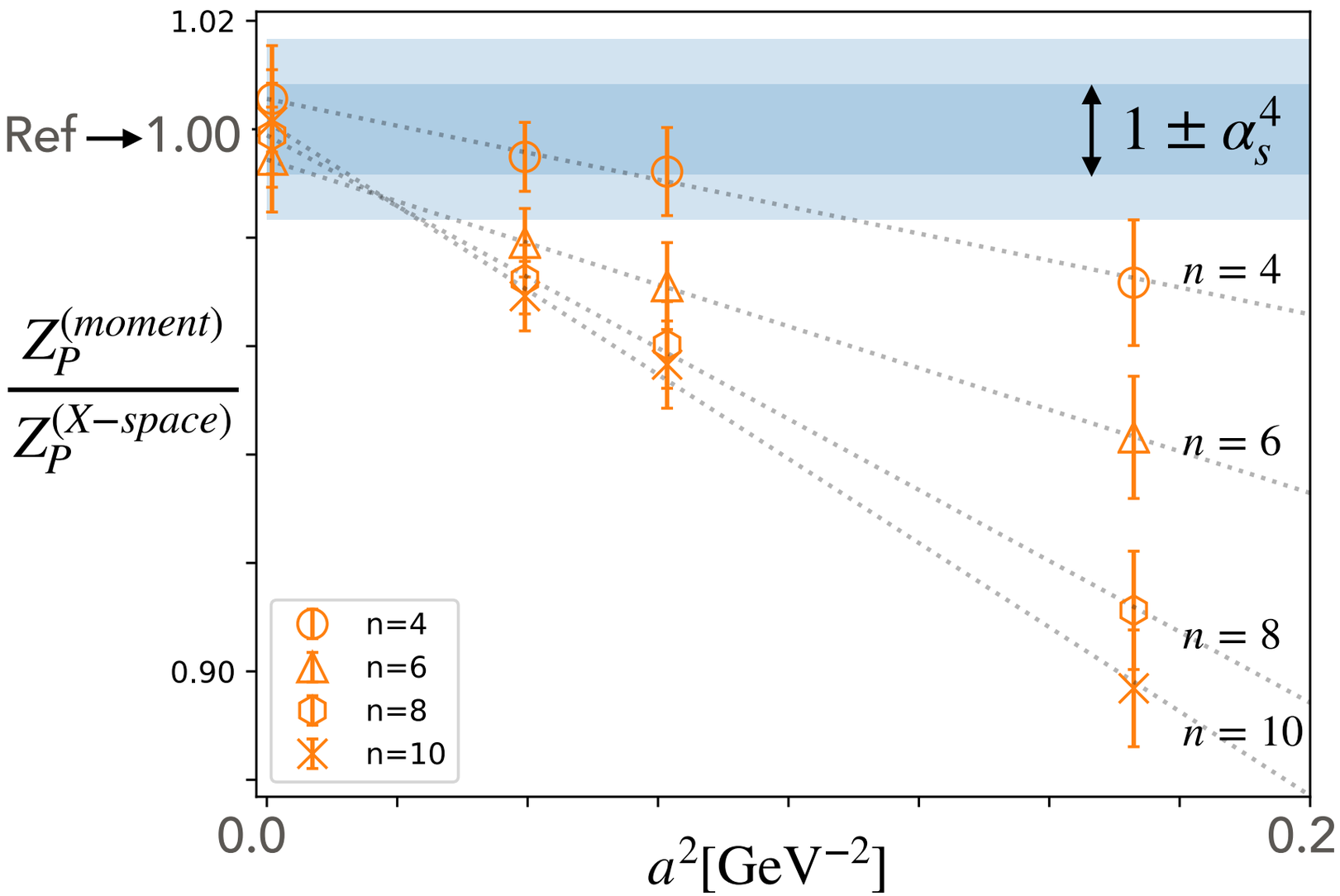
$$\alpha_s(M_z) = 0.1181 \qquad \alpha_s(2\ \text{GeV}) = 0.3022$$

$$m_c(m_c) = 1.275\ [\text{GeV}] \longrightarrow \overline{m}_c(2\ \text{GeV}) = 1.095\ [\text{GeV}]$$

Z_{moment} / $Z_{X-space}$

Taking the result of the X-space renormalization constant as a reference

M. Tomii et al. PRD 94, 054504



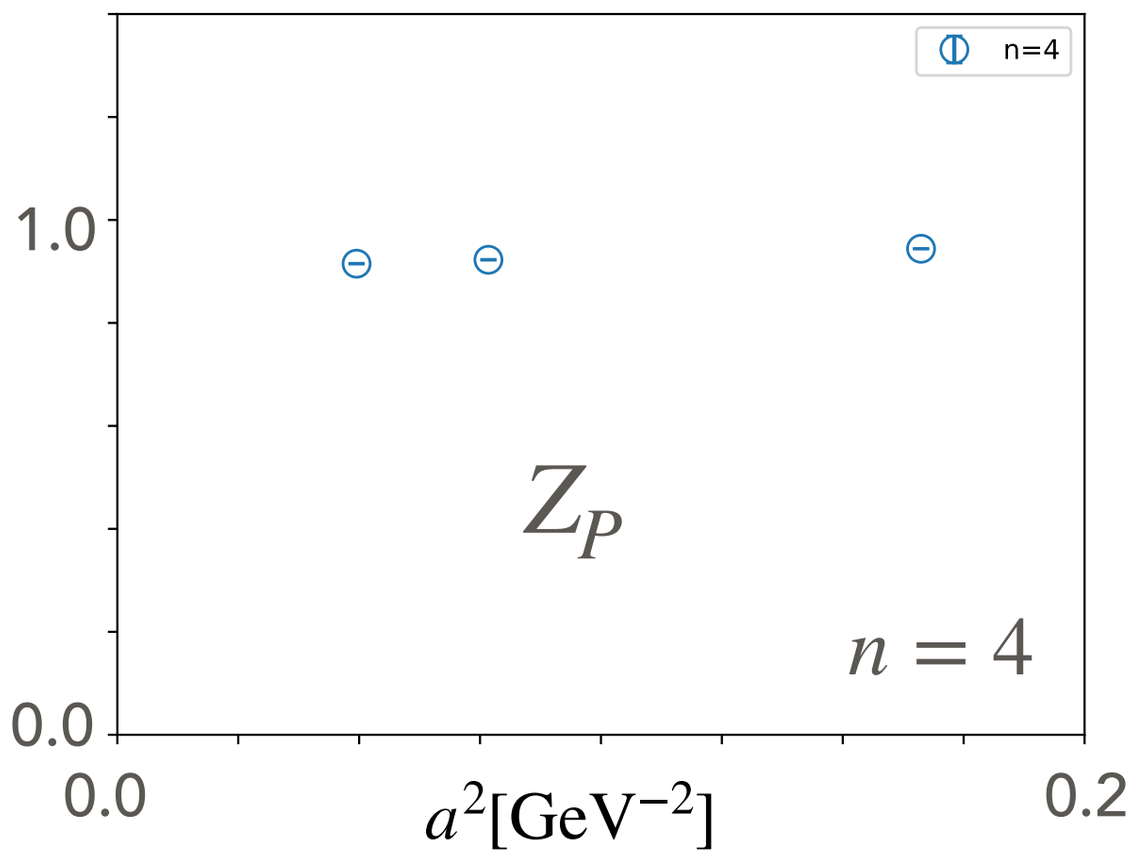
- Results are independent of the method, up to discretization effect and higher order effect

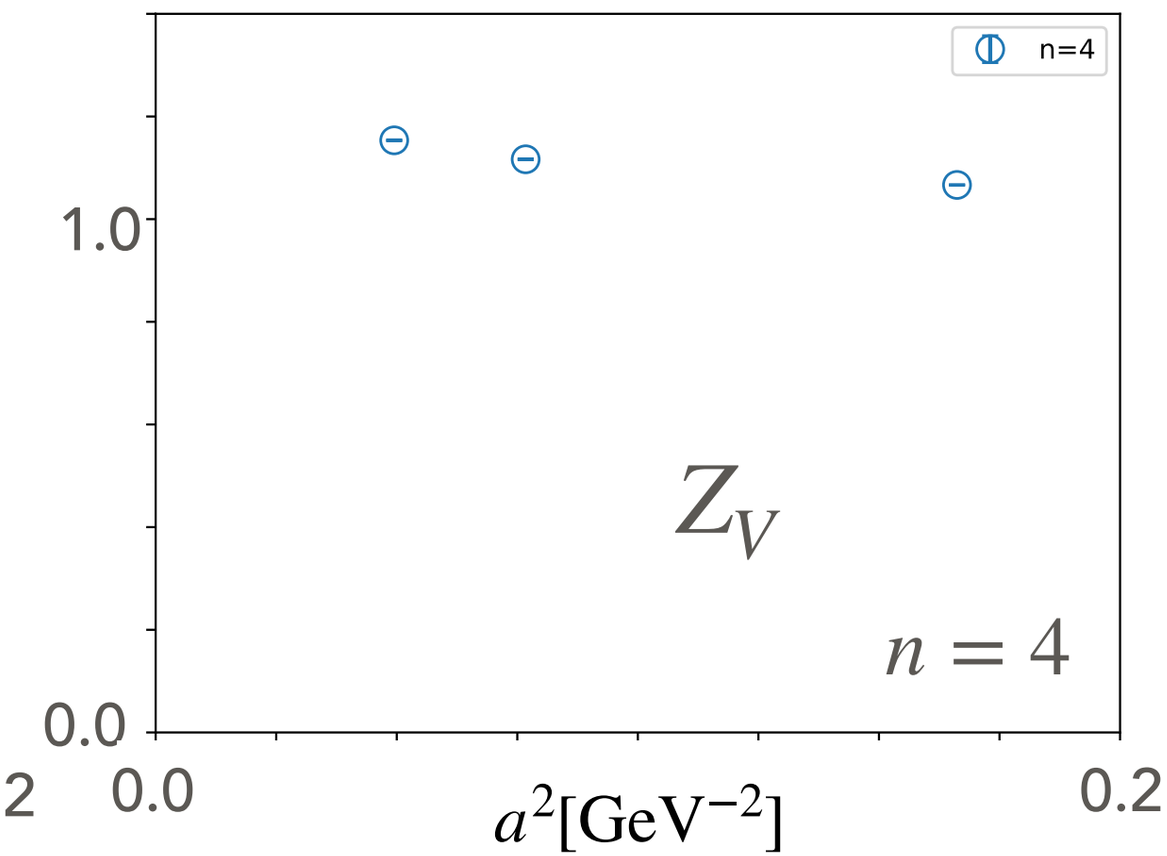
Moments as an Intermediate Scheme

Renormalization condition

$$\left(Z(\mu,a)\right)^2 \left(\frac{\partial}{\partial q^2}\right)^k \Pi^{\text{lat}}(a;q^2)\Big|_{q^2=0} = \left(\frac{\partial}{\partial q^2}\right)^k \Pi^{\text{tree}}(a;q^2)\Big|_{q^2=0}$$

Under this scheme, we obtain $(n = 2k)$





Four-Fermion Operators

Extension to Four-Fermion Operators

- effective weak Hamiltonian

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{cb} \left(C_1 O_1 + C_2 O_2 \right)$$

$$O_1 = \left(\bar{s}_i \gamma_\mu P_- c_j \right) \left(\bar{c}_j \gamma_\mu P_- b_i \right)$$

$$O_2 = \left(\bar{s}_i \gamma_\mu P_- c_i \right) \left(\bar{c}_j \gamma_\mu P_- b_j \right)$$

G_F :	Fermi constant
V_{cs}, V_{cb} :	CKM matrix elements
C_1, C_2 :	Wilson coefficient
P_- :	projection op.
i, j :	color indices

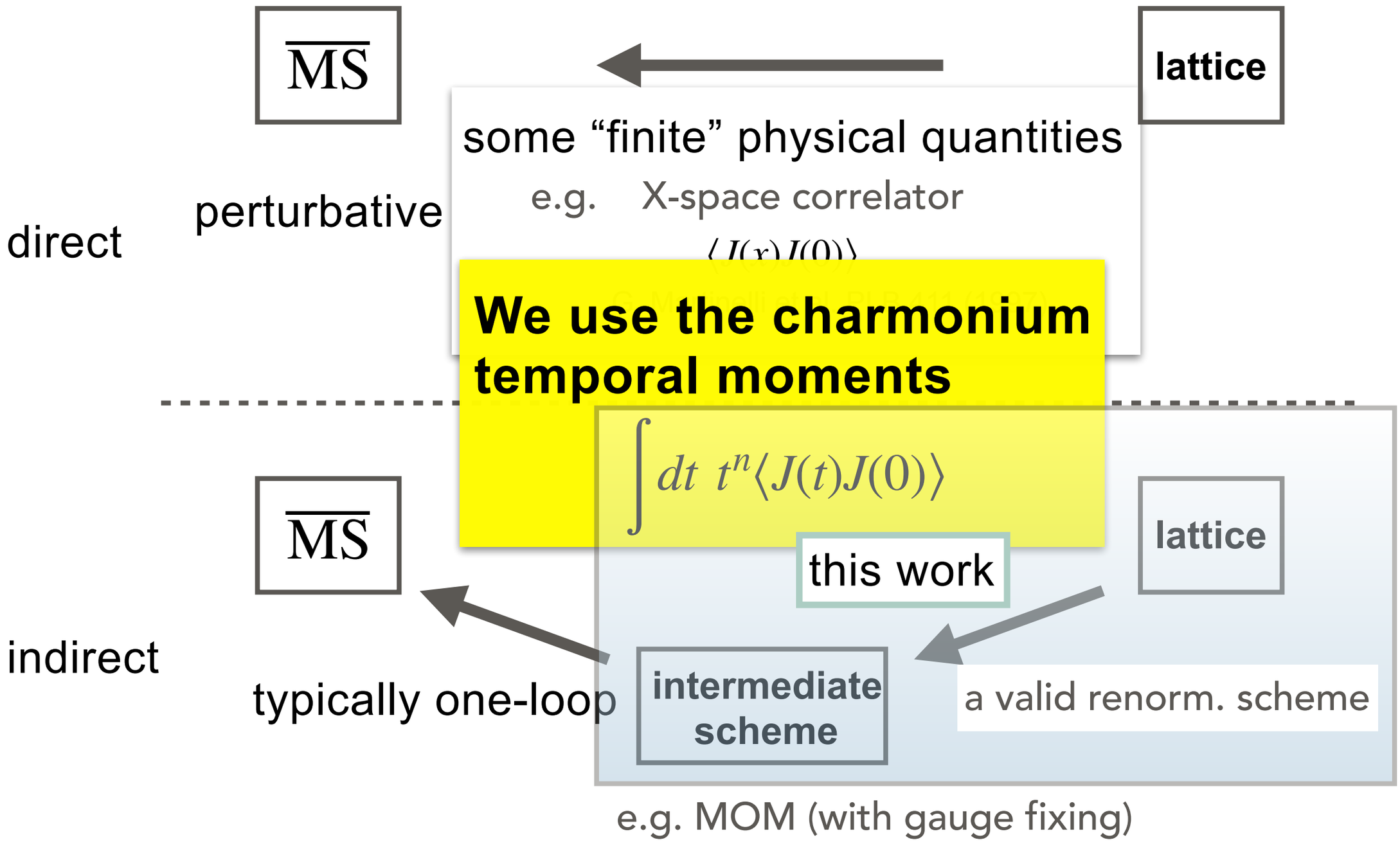
for an application, see Nakayama, Tue. 15:00

Renormalization through temporal moments

- Extension is straight-forward, but $\overline{\text{MS}}$ is NOT available.
- Use as an intermediate scheme

Matching Between Continuum and Lattice

Renormalization can be performed through a matching



G. Martinelli et al. NPB 445 (1995)

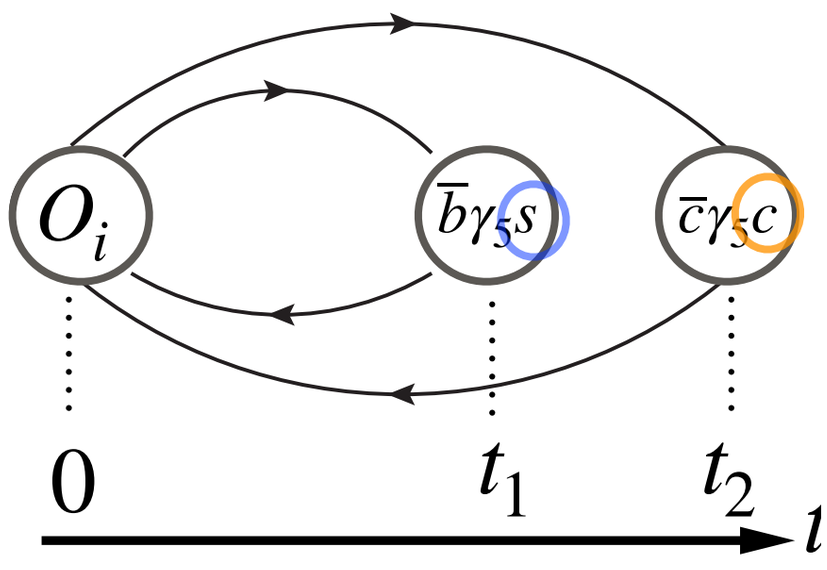
These will mix with each other under renormalization
The renormalization constants form a matrix

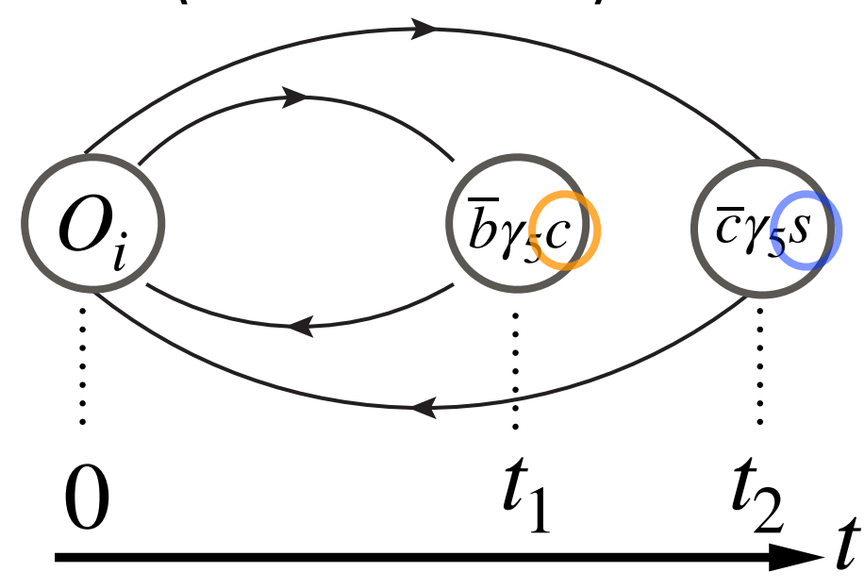
$$\begin{pmatrix} O_1^R \\ O_2^R \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} O_1 \\ O_2 \end{pmatrix}$$

We prepare two external states for each op.

$$A_i : \quad \left\langle (\bar{b}\gamma_5 s)\, O_1\, (\bar{c}\gamma_5 c) \right\rangle \quad \left\langle (\bar{b}\gamma_5 s)\, O_2\, (\bar{c}\gamma_5 c) \right\rangle$$

$$B_i : \quad \left\langle (\bar{b}\gamma_5 c)\, O_1\, (\bar{c}\gamma_5 s) \right\rangle \quad \left\langle (\bar{b}\gamma_5 c)\, O_2\, (\bar{c}\gamma_5 s) \right\rangle$$





temporal moments

$$A_i^{(n1,n2)} = (am_c^2) \sum_{t_1,t_2} \left(\frac{t_1}{a}\right)^{n_1} \left(\frac{t_2}{a}\right)^{n_2} a^{12} \sum_{\mathbf{x}_1,\mathbf{x}_2} \left\langle \left(\bar{b}\gamma_5 s\right)(t_1,\mathbf{x}_1) O_i(0,\mathbf{0}) \left(\bar{c}\gamma_5 c\right)(t_2,\mathbf{x}_2) \right\rangle$$

$$B_i^{(n1,n2)} = (am_c^2) \sum_{t_1,t_2} \left(\frac{t_1}{a}\right)^{n_1} \left(\frac{t_2}{a}\right)^{n_2} a^{12} \sum_{\mathbf{x}_1,\mathbf{x}_2} \left\langle \left(\bar{b}\gamma_5 c\right)(t_1,\mathbf{x}_1) O_i(0,\mathbf{0}) \left(\bar{c}\gamma_5 s\right)(t_2,\mathbf{x}_2) \right\rangle$$

n_1, n_2 should be odd, otherwise the moments vanish at tree level.

To avoid divergence, $n_1, n_2 > 2$. we take $n_1 = n_2 = 3$

Renormalization condition

$$\begin{cases} A_i^{(3,3)}\Big|_{\text{renorm.}} = A_i^{(3,3)}\Big|_{\text{tree}} \\ B_i^{(3,3)}\Big|_{\text{renorm.}} = B_i^{(3,3)}\Big|_{\text{tree}} \end{cases}$$

we set $m_b = m_c = m_s (= m_c)$

typical scale is given by m_c

$$A_1 : \left\langle (\bar{b}\gamma_5 s)\, O_1\, (\bar{c}\gamma_5 c) \right\rangle$$

$$= \left\langle (\bar{b}\gamma_5 s) \left(\bar{s}_i \gamma_\mu P_- c_j\right) \left(\bar{c}_j \gamma_\mu P_- b_i\right) (\bar{c}\gamma_5 c) \right\rangle$$

$$= \left\langle (\bar{b}\gamma_5 s) \left(\bar{s}_i \gamma_\mu P_- b_i\right) \left(\bar{c}_j \gamma_\mu P_- c_j\right) (\bar{c}\gamma_5 c) \right\rangle$$

Fierz identity

$$B_2 : \left\langle (\bar{b}\gamma_5 c)\, O_2\, (\bar{c}\gamma_5 s) \right\rangle$$

$$= \left\langle (\bar{b}\gamma_5 c) \left(\bar{c}_j \gamma_\mu P_- b_j\right) \left(\bar{s}_i \gamma_\mu P_- c_i\right) (\bar{c}\gamma_5 s) \right\rangle$$

$(m_c = m_s)$

These have the same structure in color and dirac spaces.

identities

$$A_1 \quad \left\langle (\bar{b}\gamma_5 s)\, O_1\, (\bar{c}\gamma_5 c) \right\rangle = \left\langle (\bar{b}\gamma_5 c)\, O_2\, (\bar{c}\gamma_5 s) \right\rangle \quad B_2$$

$$A_2 \quad \left\langle (\bar{b}\gamma_5 s)\, O_2\, (\bar{c}\gamma_5 c) \right\rangle = \left\langle (\bar{b}\gamma_5 c)\, O_1\, (\bar{c}\gamma_5 s) \right\rangle \quad B_1$$

By using Fierz identities

$$A_1 = B_2,\ A_2 = B_1$$

$$\tilde{A}_1 = \tilde{B}_2,\ \tilde{A}_2 = \tilde{B}_1$$

$$Z_{11} = Z_{22},\ Z_{12} = Z_{21}$$

$$Z_{11}A_1^{(3,3)} + Z_{12}A_2^{(3,3)} = \tilde{A}_1^{(3,3)}$$

$$Z_{11}B_1^{(3,3)} + Z_{12}B_2^{(3,3)} = \tilde{B}_1^{(3,3)}$$

$$Z_{21}A_1^{(3,3)} + Z_{22}A_2^{(3,3)} = \tilde{A}_2^{(3,3)}$$

$$Z_{21}B_1^{(3,3)} + Z_{22}B_2^{(3,3)} = \tilde{B}_2^{(3,3)}$$

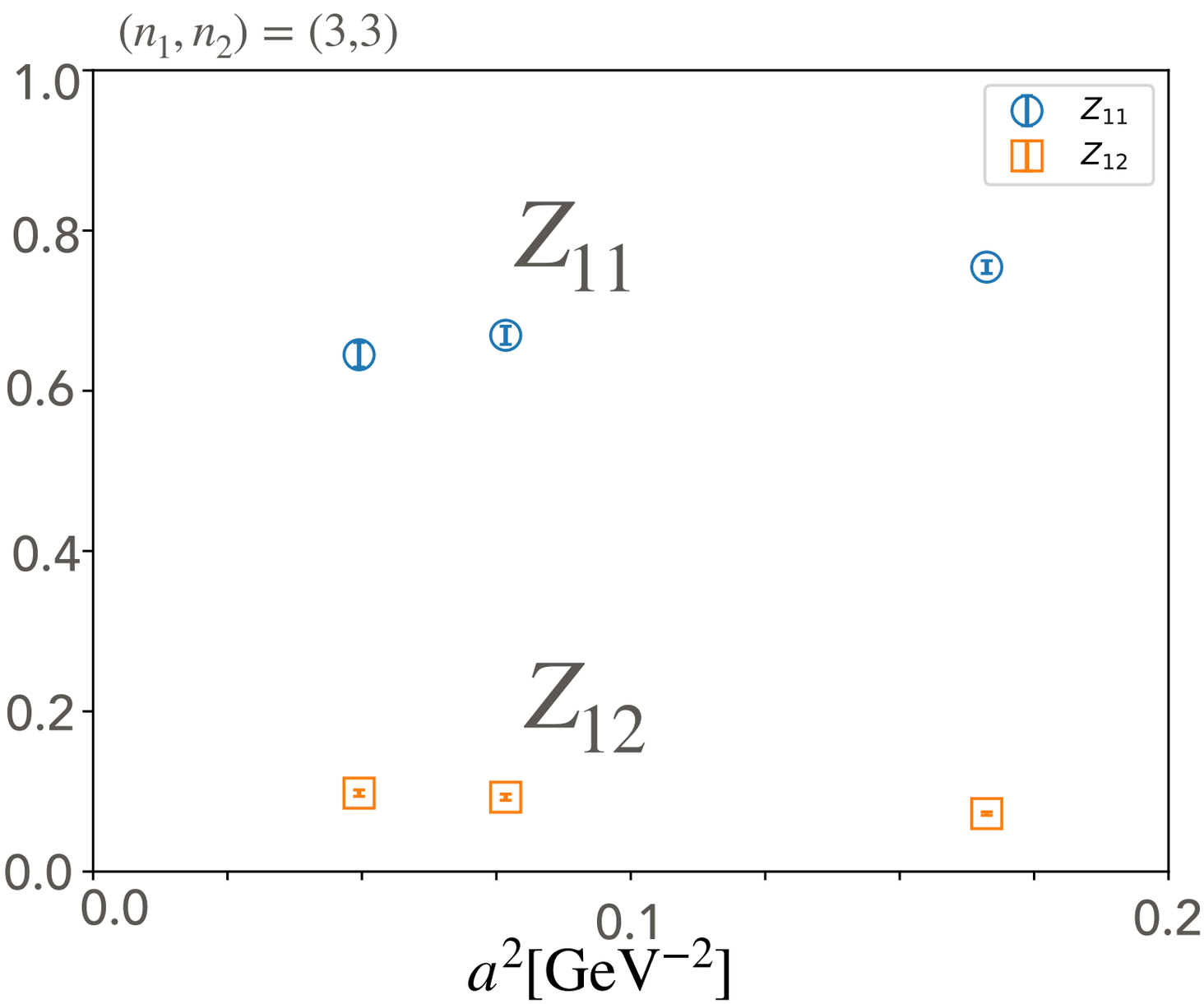
tree

We solve linear equations

$$\begin{cases} Z_{11}A_1 + Z_{12}A_2 = \tilde{A}_1 \\ Z_{11}A_2 + Z_{12}A_1 = \tilde{A}_2 \end{cases}$$

Results

Result



β	a^{-1}[GeV]	Z_{11}	Z_{12}
4.17	2.453(4)	0.754(8)	0.072(2)
4.35	3.610(9)	0.669(11)	0.093(4)
4.47	4.496(9)	0.645(15)	0.098(4)

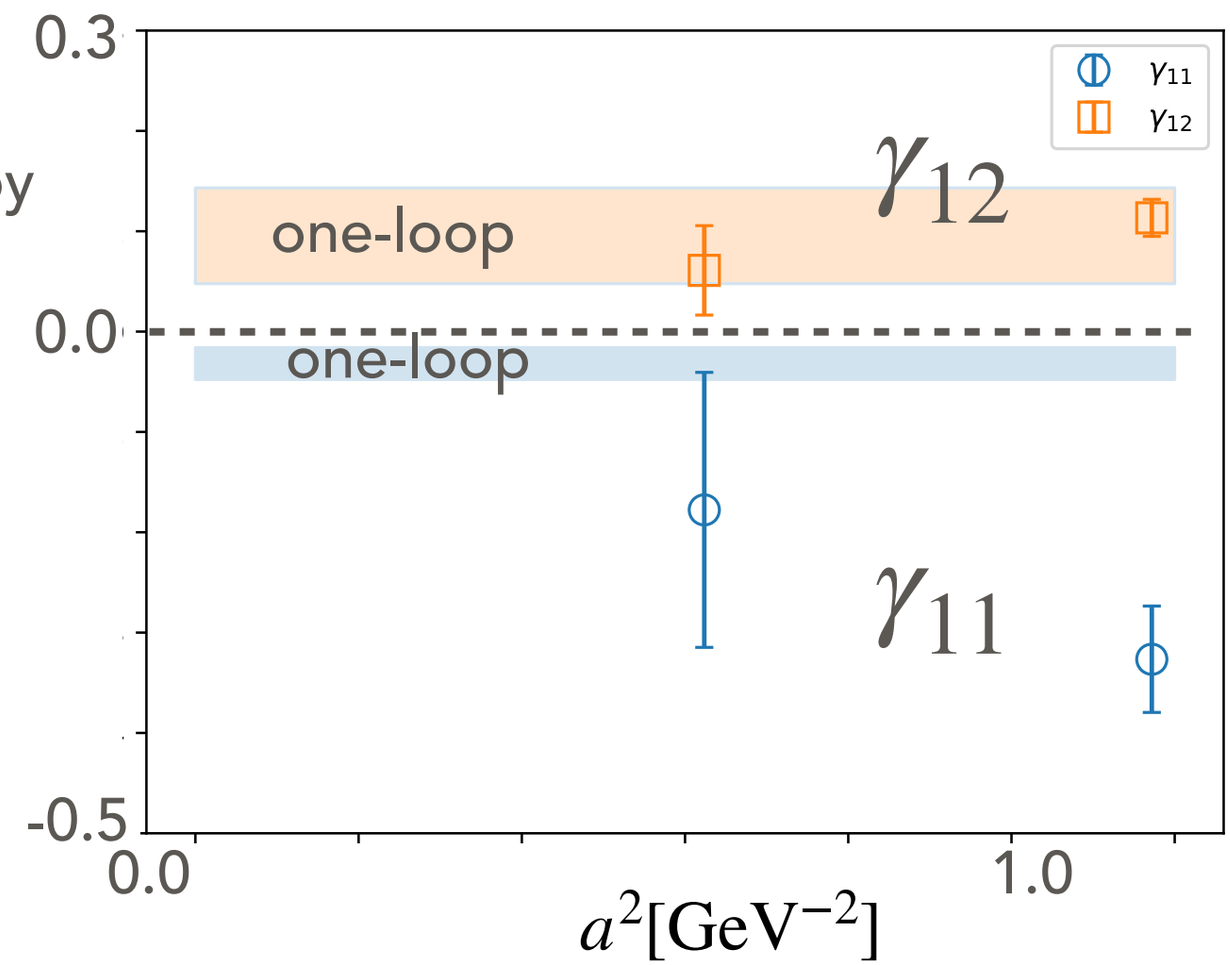
Anomalous Dimension

We calculate the anomalous dimension by taking a difference between two nearby lattice spacings

$$\gamma_{ij} = -a\frac{\partial}{\partial a}\log Z_{ij}$$
$$= -Z_{ik}^{-1}a\frac{\partial}{\partial a}Z_{kj}$$

The signs are consistent with the one-loop result

$$\gamma = \frac{\alpha_s}{4\pi}\begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix}$$



estimate with $\alpha_s = 0.1 \sim 0.3$

Summary

We propose a renormalization scheme.

- Based on charmonium temporal moments.
- No gauge fixing is necessary. (unlike RI/MOM)
- Matching to $\overline{\mathrm{MS}}$ has to be calculated.
- With a finite quark mass, the scale is set by $\mu = m_c$.
- Mixing with different chirality operators should be taken into account. Or we could introduce a renormalization in the chiral limit (but $q^2 \neq 0$).