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Non-perturbative matching of 3/4-flavor Wilson coefficients with a position-space procedure

Masaaki Tomii
Columbia University

The RBC & UKQCD collaborations

BNL and BNL/RBRC

Yasumichi Aoki (KEK)
Taku Izubuchi
Yong-Chull Jang
Chulwoo Jung
Meifeng Lin
Aaron Meyer
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni

UC Boulder

Oliver Witzel

CERN

Mattia Bruno

Columbia University

Ryan Abbot
Norman Christ
Duo Guo
Christopher Kelly
Bob Mawhinney
Masaaki Tomii
Jiqun Tu
Bigeng Wang
Tianle Wang
Yidi Zhao

University of Connecticut

Tom Blum
Dan Hoying (BNL)
Luchang Jin (RBRC)
Cheng Tu

Edinburgh University

Peter Boyle
Luigi Del Debbio
Felix Erben
Vera Gülpers
Tadeusz Janowski
Julia Kettle
Michael Marshall
Fionn Ó hÓgáin
Antonin Portelli
Tobias Tsang
Andrew Yong
Azusa Yamaguchi

KEK

Julien Frison

University of Liverpool

Nicolas Garron

MIT

David Murphy

Peking University

Xu Feng

University of Regensburg

Christoph Lehner (BNL)

University of Southampton

Nils Asmussen
Jonathan Flynn
Ryan Hill
Andreas Jüttner
James Richings
Chris Sachrajda

Stony Brook University

Jun-Sik Yoo
Sergey Syritsyn (RBRC)

N_f in Weak Hamiltonian

$$H_W = \Sigma_i \, w_i^{4f}(\mu) \, O_i^{4f}(\mu)$$

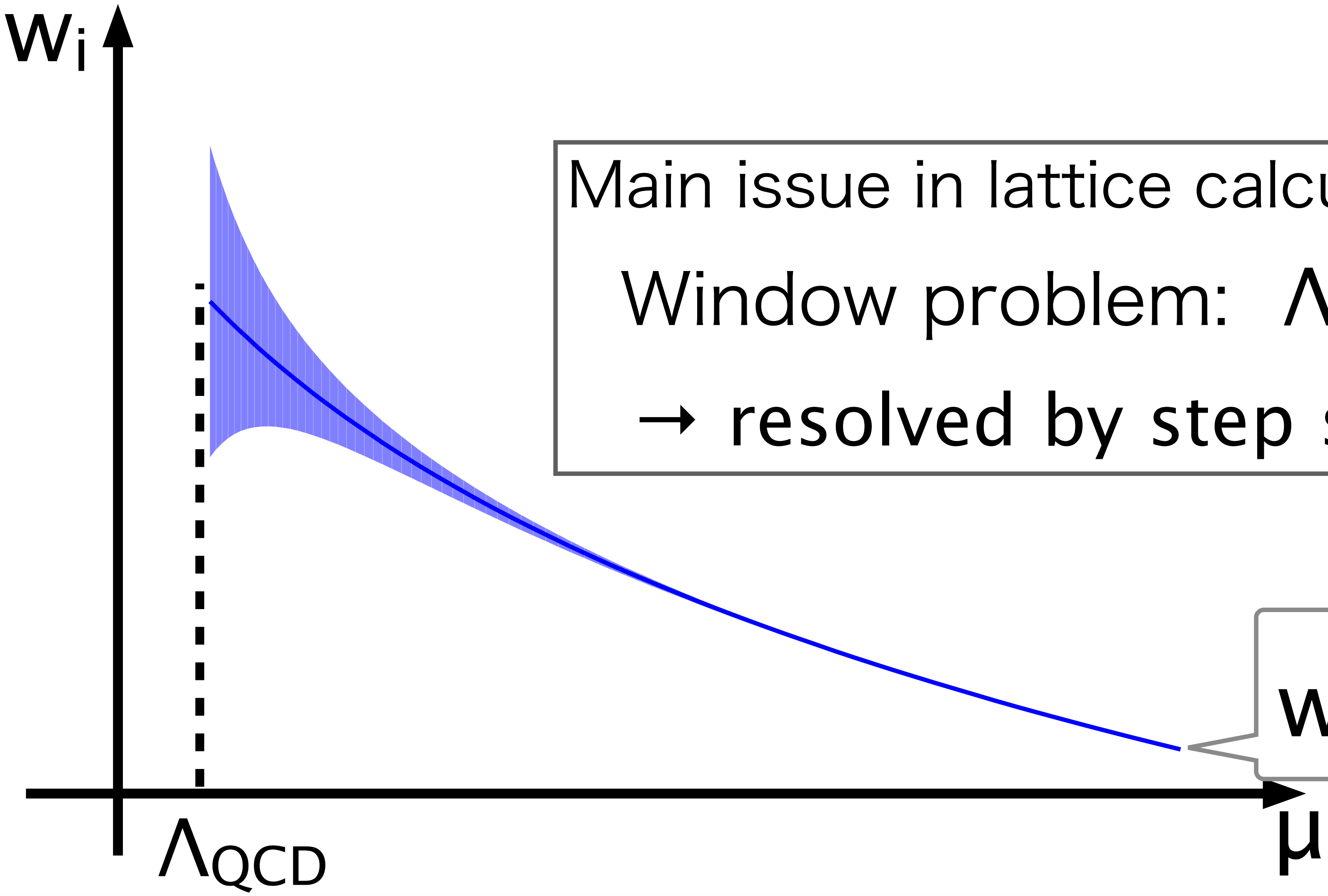
$$= \Sigma_i \, w_i^{3f}(\mu) \, O_i^{3f}(\mu)$$

$$= \ldots$$

We can use either 3f or 4f for WMEs

WMEs w/ 4-flavor operators

$$<f|H_W|i> \; = \; \Sigma_i \; \underbrace{w_i^{4f}(\mu)}_{\text{pQCD}} \; \underbrace{<f|O_i^{4f}(\mu)|i>}_{\text{LQCD}}$$



Main issue in lattice calculation is

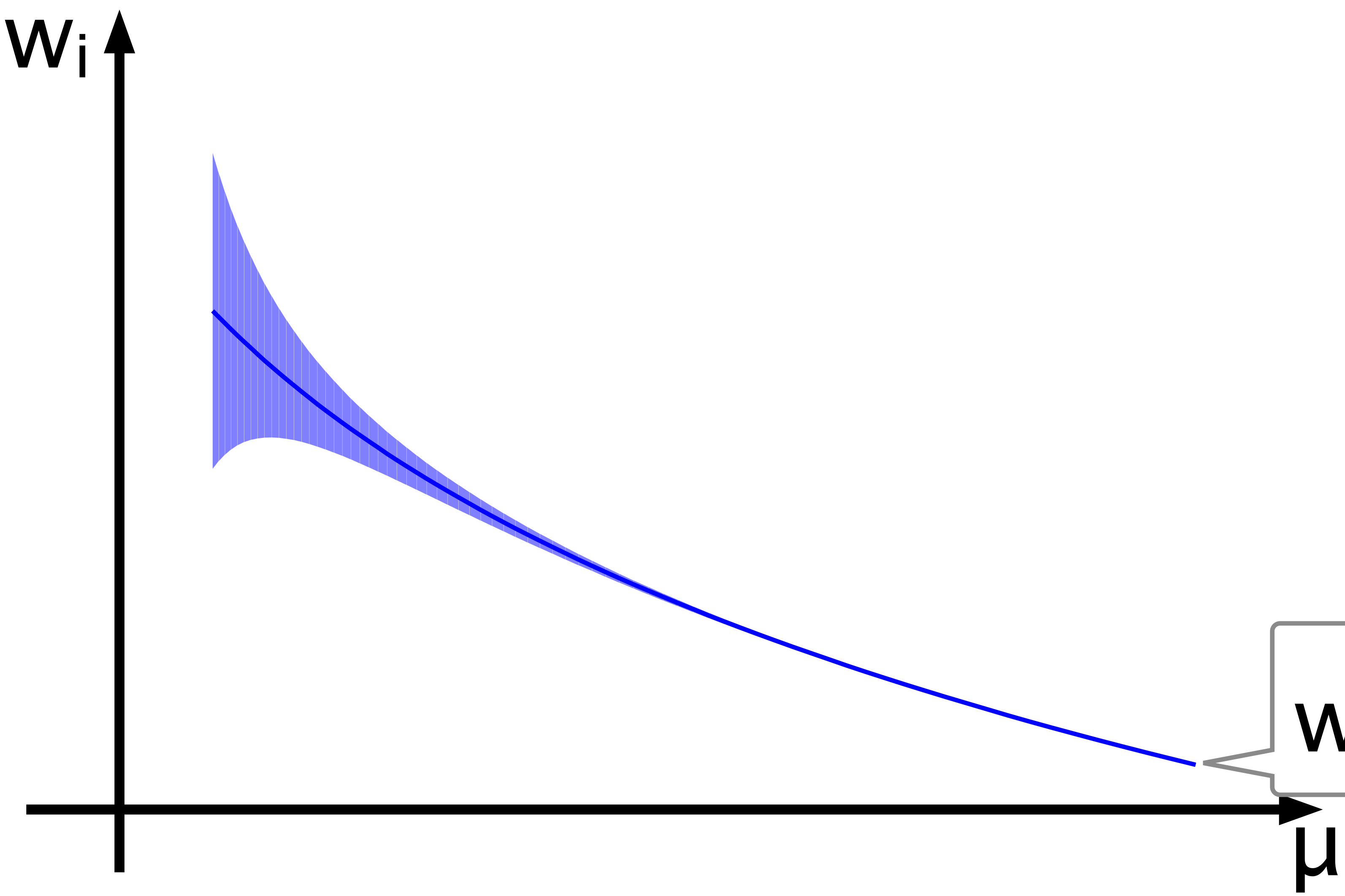
Window problem: $\Lambda_{QCD} \ll \mu \ll a^{-1}$

→ resolved by step scaling, etc...

$w_i^{4f}(\mu)$: precise at HEs

WMEs w/ 3-flavor operators

$$\langle f|H_W|i\rangle = \Sigma_i \underbrace{w_i^{3f}(\mu)}_{\text{pQCD}} \underbrace{\langle f|O_i^{3f}(\mu)|i\rangle}_{\text{LQCD}}$$



$w_i^{4f}(\mu)$: precise at HEs

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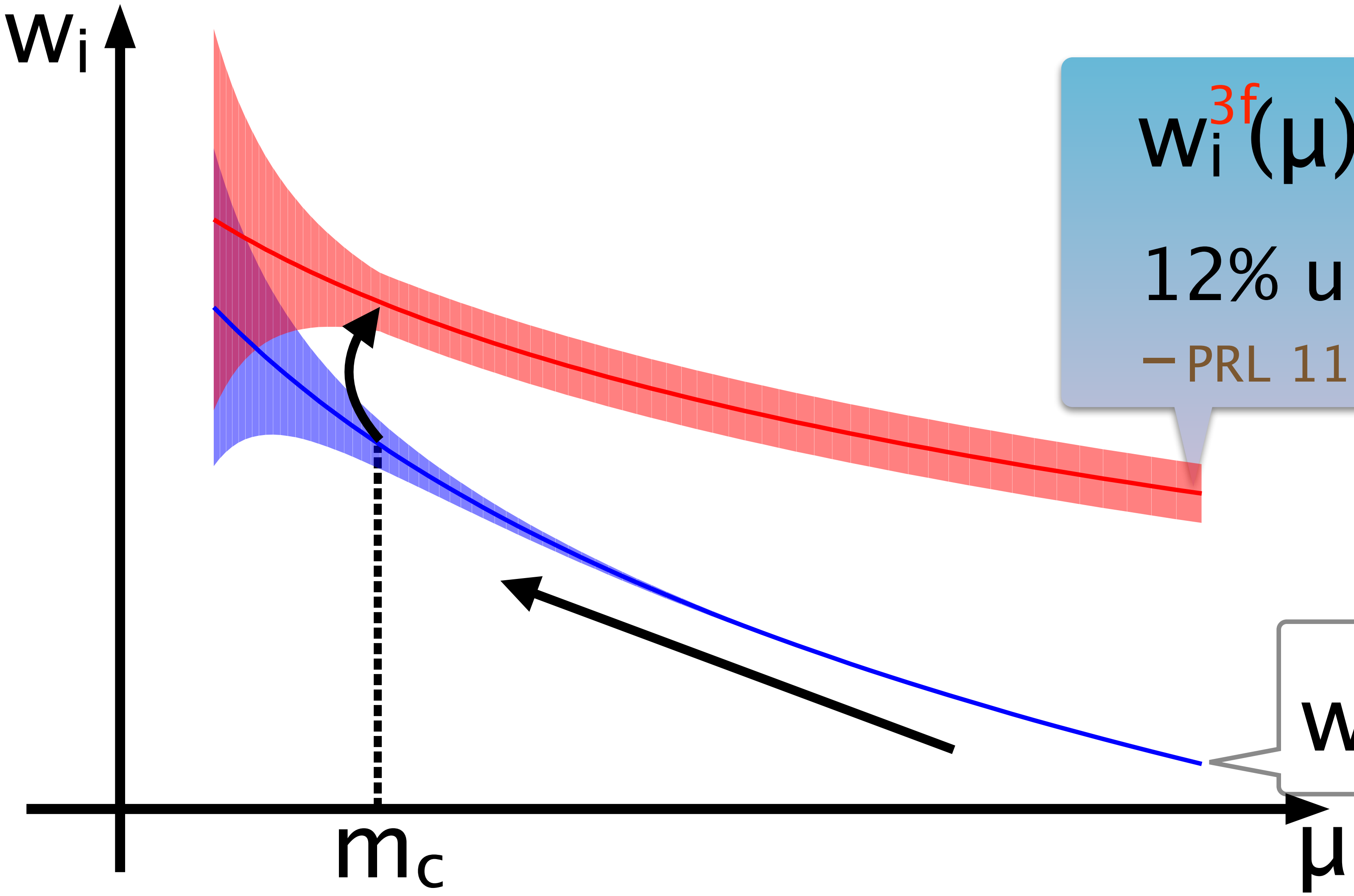
$$\langle f|H_W|i\rangle = \Sigma_i \underbrace{w_i^{3f}(\mu)}_{\text{pQCD}} \underbrace{\langle f|O_i^{3f}(\mu)|i\rangle}_{\text{LQCD}}$$



$w_i^{3f}(\mu)$: large uncertainty

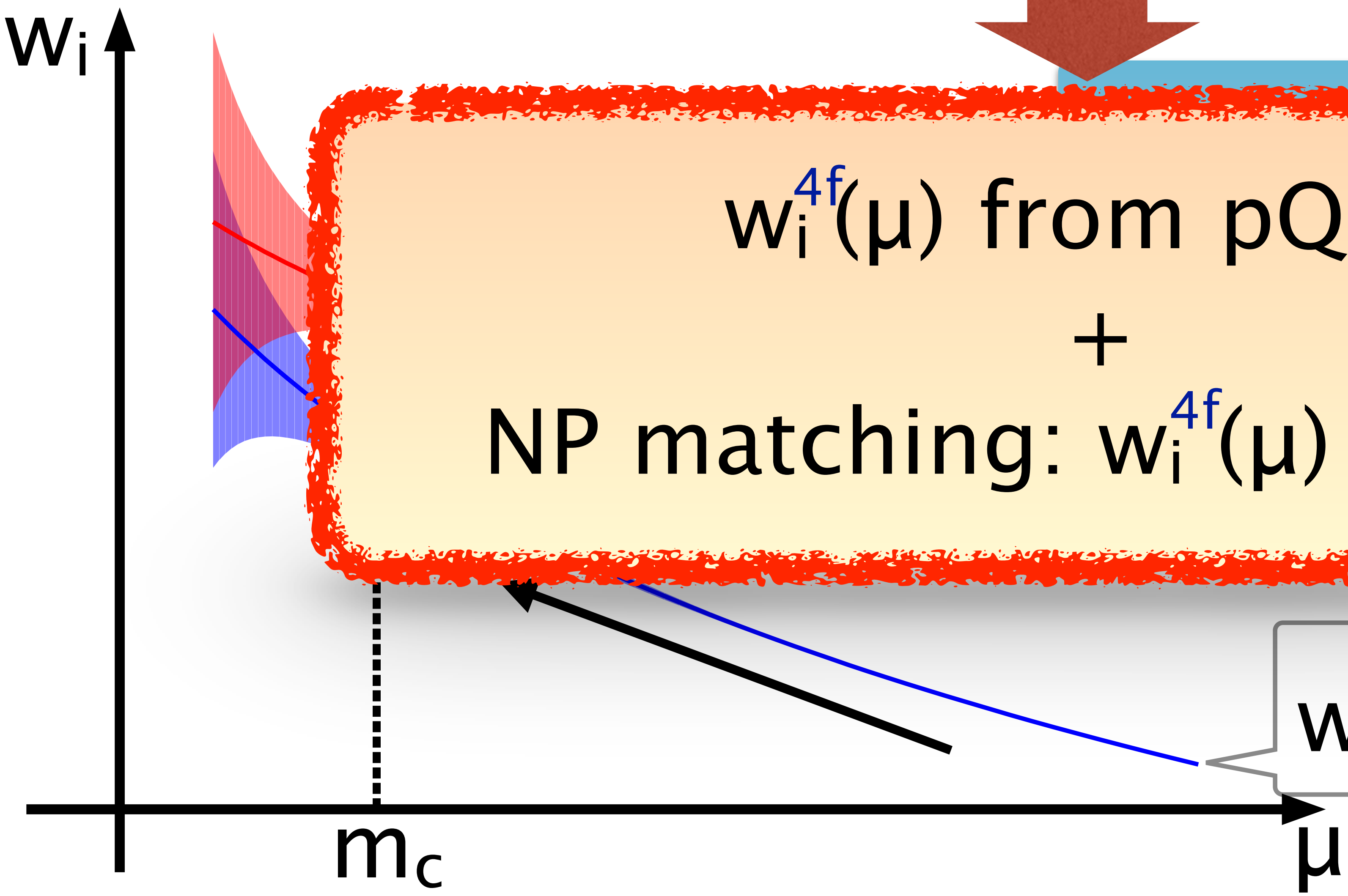
12% uncertainty for K → ππ

– PRL 115,212001(2015)

$w_i^{4f}(\mu)$: precise at HEs

WMEs w/ 3-flavor operators

$$<f|H_W|i> \; = \; \Sigma_i \; \underbrace{w_i^{3f}(\mu)}_{\cancel{\text{pQCD}}} \; \underbrace{<f|O_i^{3f}(\mu)|i>}_{\text{LQCD}}$$



rtainty

→ ππ

$w_i^{4f}(\mu)$ from pQCD

+

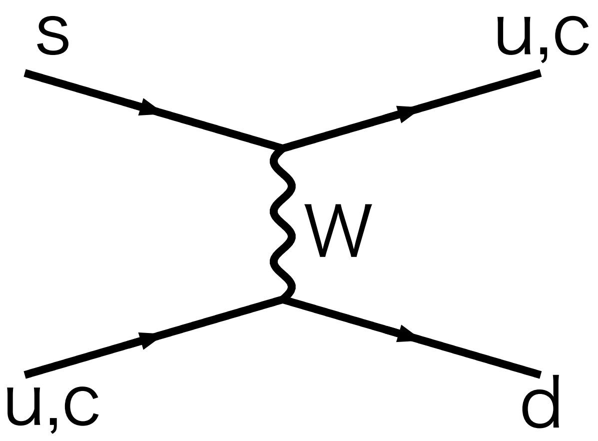
NP matching: $w_i^{4f}(\mu) \rightarrow w_i^{3f}(\mu)$

$w_i^{4f}(\mu)$: precise at HEs

$w_i^{3f}(\mu) \neq w_i^{4f}(\mu)$?

- Of coarse sea charm effects ⇒ $w_i^{3f}(\mu) \neq w_i^{4f}(\mu)$
  - Maybe small difference → neglect in this project
- If O_i^{4f} can contain charm…

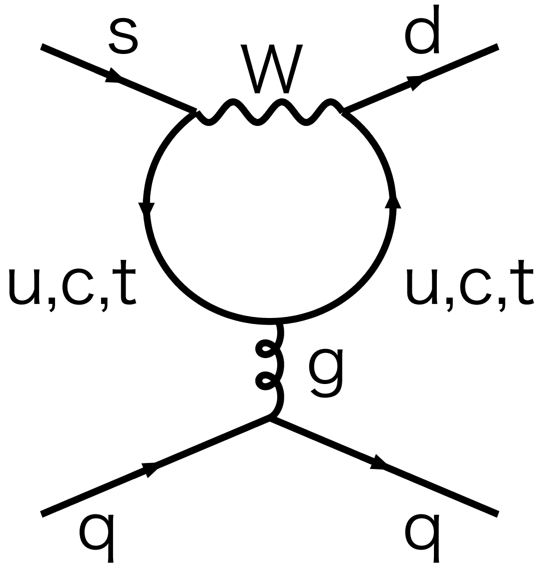
Ex)



current-current

$O_i^u = (\bar{s}d)_{V-A}\,(\bar{u}u)_{V-A}$

$O_i^c = (\bar{s}d)_{V-A}\,(\bar{c}c)_{V-A}$



QCD penguin

$O_i = (\bar{s}d)_{V-A}\,\Sigma_q\,(\bar{q}q)_{V\pm A}$

- O_i^{4f}'s w/ charm turn to a combination of O_i^{3f}'s in $\mu \ll m_c$
- w_i^{3f} necessary if MEs calculated with O_i^{3f}

$K \to \pi\pi$ by RBC/UKQCD (2015)

- 2+1 DWF / Iwasaki + DSDR gauge action
- a^{-1} = 1.38 GeV

⇒ too coarse to introduce charm

⇒ 3-flavor operators for MEs
& perturbative 4/3-flavor matching

⇒ 12% systematic uncertainty

- NP matching (obtained from finer lattices) is desired

Outline

- Introduction
- NP matching strategy
  - Two-point functions in position space
  - Gauge invariant
- Technique for reducing discretization errors
  - Average over spheres
  - Enables an appropriate a → 0 limit
- Result of exploratory calculation
  - 16^3 x 32
  - Statistical error significant

NP 4f–3f matching in position Sp.

$$H_W = \Sigma_i\, w_i^{4f}(\mu)\, O_i^{4f}(\mu) = \Sigma_i\, w_i^{3f}(\mu)\, O_i^{3f}(\mu)$$

- This means:

$$\Sigma_i <\bar{O}(x)\, O_i^{4f}(\mu;y)^\dagger> w_i^{4f}(\mu) = \Sigma_i <\bar{O}(x)\, O_i^{3f}(\mu;y)^\dagger> w_i^{3f}(\mu)$$

for any operator $\bar{O}(x)$

at LDs: $1/|x-y| \ll m_c$

- Relation b/w w_i^{4f} & w_i^{3f} can be obtained by choosing appropriate number of $\bar{O}(x)$'s

⇒ We choose

$$\bar{O}(x) = O_i^{3f}(\mu;x)$$

NP 4f−3f matching in position Sp.

$$H_W = \Sigma_i\, w_i^{4f}(\mu)\, O_i^{4f}(\mu) = \Sigma_i\, w_i^{3f}(\mu)\, O_i^{3f}(\mu)$$

$$\langle O_j^{3f}(\mu;x)\, H_W(y)^\dagger \rangle$$

$$\Sigma_j\, G_{ij}^{3f-4f}(\mu;x-y)\, w_j^{4f}(\mu) = \Sigma_j\, G_{ij}^{3f-3f}(\mu;x-y)\, w_j^{3f}(\mu)$$

$$G_{ij}^{nf-n'f}(\mu;x-y) = \langle O_i^{nf}(\mu;x)\, O_j^{n'f}(\mu;y)^\dagger \rangle$$

$$\boxed{w_i^{3f}(\mu) = \Sigma_{jk}\, (G^{3f-3f}(\mu;x-y))_{ij}^{-1}\, G_{jk}^{3f-4f}(\mu;x-y)\, w_k^{4f}(\mu)}$$

☆ Gauge invariant & free from contact terms

⇒ can prevent mixing with irrelevant operators

Matching matrix & 3f operators

- $M_{ik} = \Sigma_j \, (G^{3f-3f}(\mu;x-y))^{-1}_{ij} \, G^{3f-4f}_{jk}(\mu;x-y)$
- Inverse matrix $(G^{3f-3f}(\mu;x-y))^{-1}_{ij}$ exists ONLY IF O_i^{3f}'s are independent with each other
  - Ex: ΔS = 1 weak operators not the case!

Type	Q_i
current-current	$Q_1 = (\bar{s}_\alpha d_\alpha)_L (\bar{u}_\beta u_\beta)_L$ $Q_2 = (\bar{s}_\alpha d_\beta)_L (\bar{u}_\beta u_\alpha)_L$
QCD penguin	$Q_3 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} (\bar{q}_\beta q_\beta)_L$ $Q_4 = (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} (\bar{q}_\beta q_\alpha)_L$ $Q_5 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} (\bar{q}_\beta q_\beta)_R$ $Q_6 = (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} (\bar{q}_\beta q_\alpha)_R$
EW penguin	$Q_7 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\beta)_R$ $Q_8 = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\alpha)_R$ $Q_9 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\beta)_L$ $Q_{10} = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\alpha)_L$

Fierz symmetry

→ 3 relations among Q_i's

→ 7 independent operators

Matching matrix

- If we choose
  - $O_i^{3f} = (Q_1, Q_2, \dots, Q_{n3})$
  - $O_i^{4f} = (Q_1, Q_2, \dots, Q_{n3}, P_1, P_2, \dots, P_{nc})$
  - P_i's contain charm / Q_i's don't

We now redefine Q_i's as independent operators

- Then

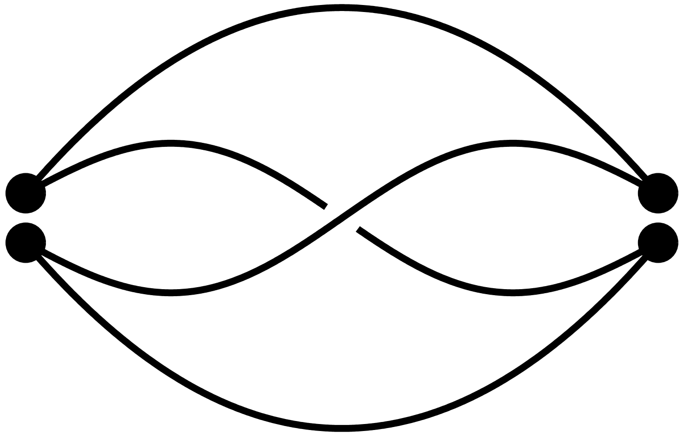
$$M_{ij} = \left(1_{n3\times n3} \;\vdots\; \overbrace{\begin{matrix} \bullet & \bullet & \cdots & \bullet \\ \bullet & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \bullet & \cdots & \cdots & \bullet \end{matrix}}^{nc\ (= 4 \text{ for } K \to \pi\pi)} \right)$$

Represents how P_i's turn to Q_i's below charm threshold

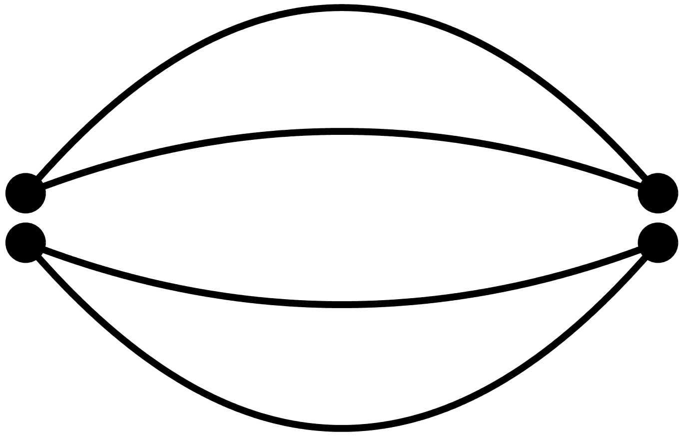
Contractions

- 4/3-flavor matching should be independent of m_{ud} & m_s

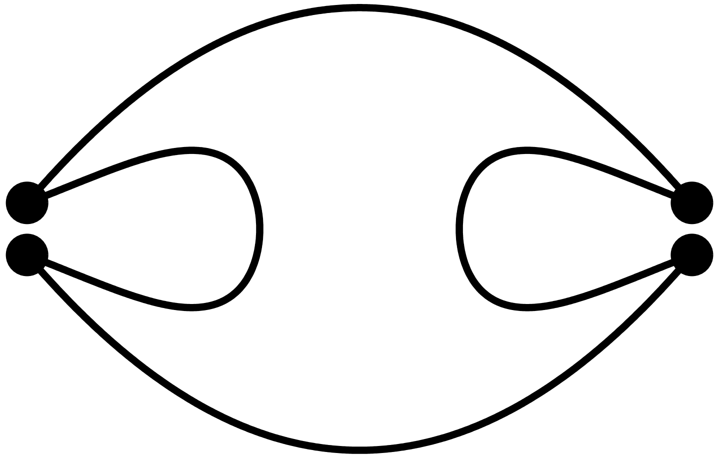
  ⇒ Calculate w/ SU(3) valence quarks + 1 heavier quark



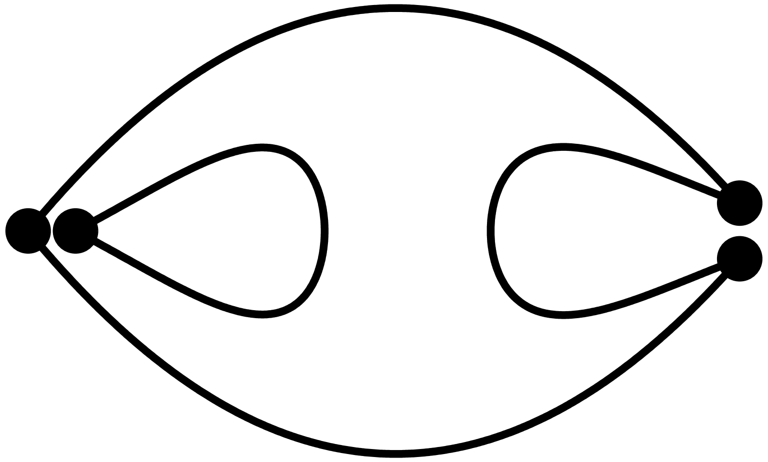
6 contractions



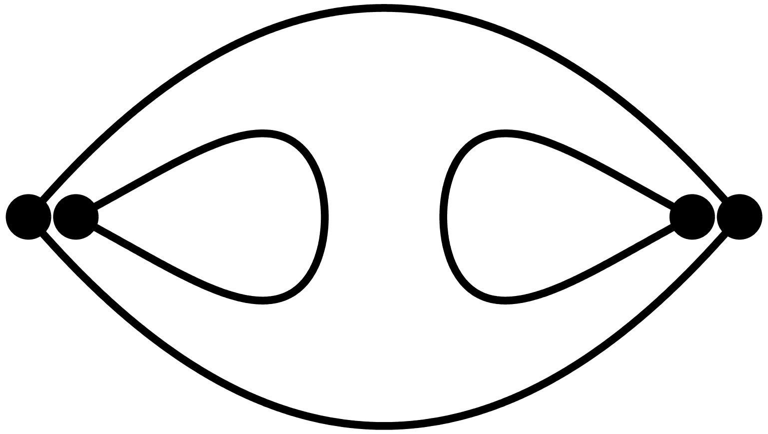
6 contractions



18 contractions



32 contractions



18 contractions

Subtraction of power divergence

- Loop diagram can contain power divergence
  - from power divergence of operators

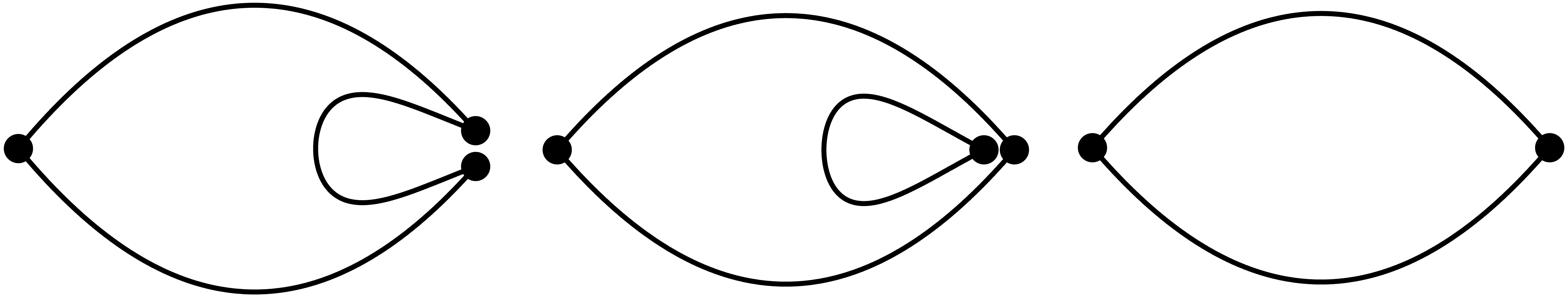
$$O_i \sim \frac{m_q}{a^2}$$

  - Eliminate by redefining

$$O_i' = O_i - C_- \bar{s}(1-\gamma_5)d - C_+ \bar{s}(1+\gamma_5)d$$

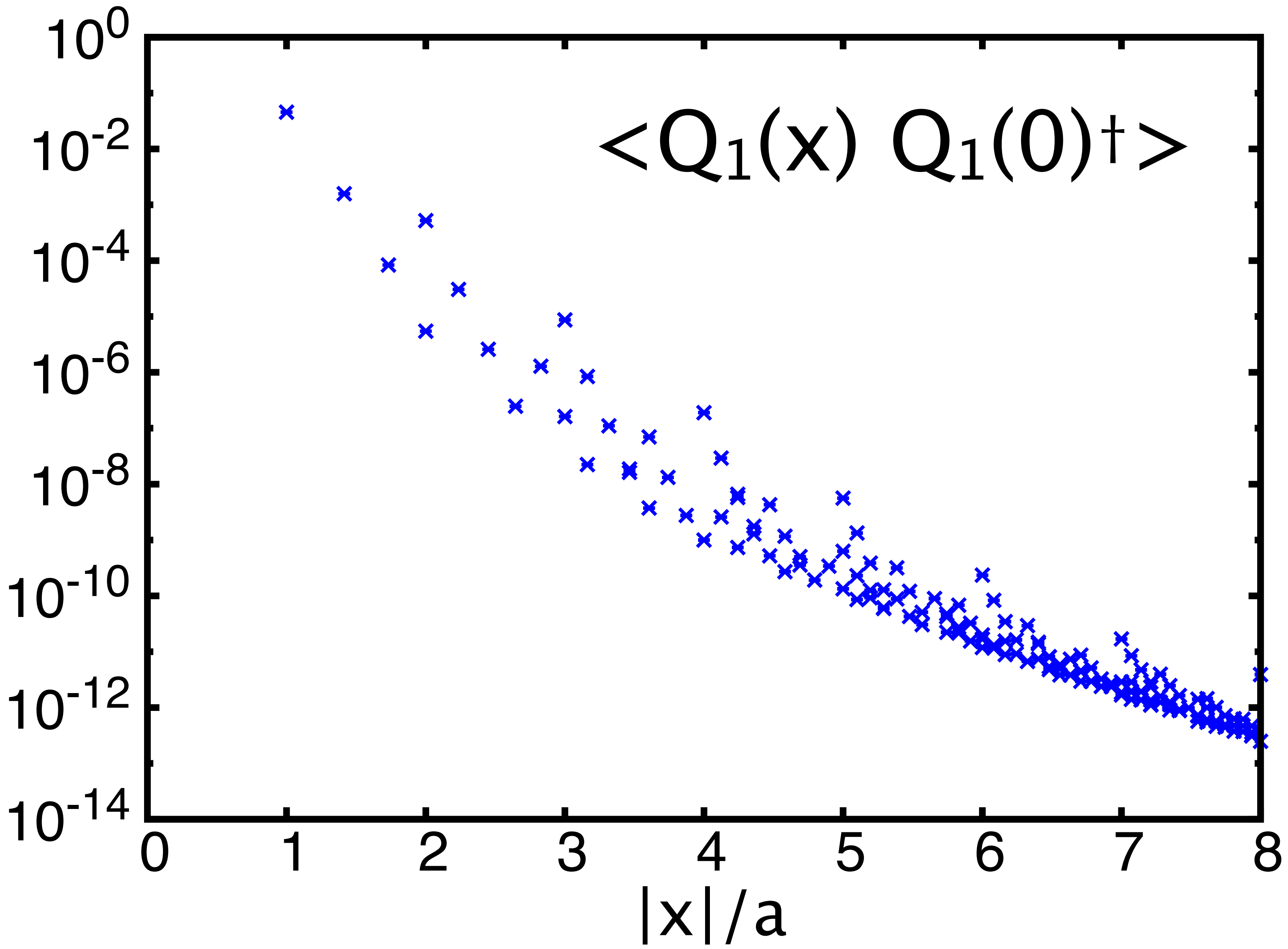
with a condition

$$\left\langle \bar{s}(1 \pm \gamma_5)d(x) \cdot O_i'(y)^\dagger \right\rangle \Big|_{x-y=x_0} = 0$$



12 contractions 12 contractions 3 contractions

Representative result



- $16^3 \times 32$
- $a^{-1} = 1.78$ GeV
- $m_{ud}^{val} = m_s^{val} = m_s^{sea}$
- Plotted in lattice units
- Unrenormalized

- Different lattice points distinguished ($(1,1,1,1) \neq (0,0,0,2)$)

How to a → 0?

- Our final goal: continuum limit of M_{ij}
  - to apply it to our K → ππ result on 1.38 GeV lattice

- a-dependence of M_{ij} may depend on x
  - Necessity to take a → 0 at mutual x (in physical units) for different a
  - Practically no or few such mutual x

- We propose an idea to take continuum limit in a more appropriate way
  - Averaging correlators over sphere to get O(4)-symmetric ones at any physical distance |x|
  - Example for Z_m [MT & N. Christ, PRD99,014515]

Average over spheres

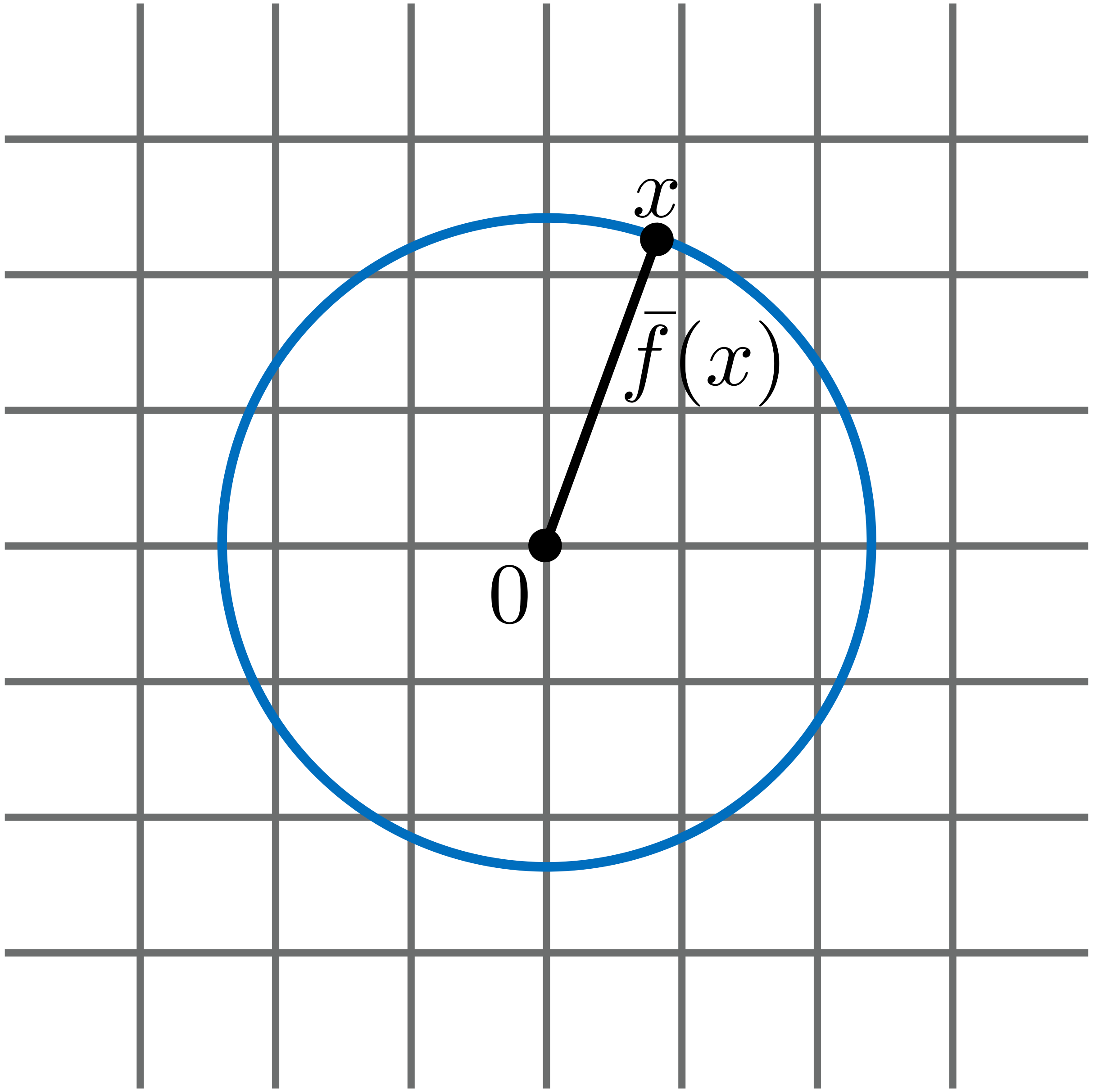
- Evaluate the value of a quantity at each 4d point from values at lattice points, with a guideline

$$\bar{f}(x) = \eta(f^{\mathrm{lat}}; x)$$

※ details in following slides

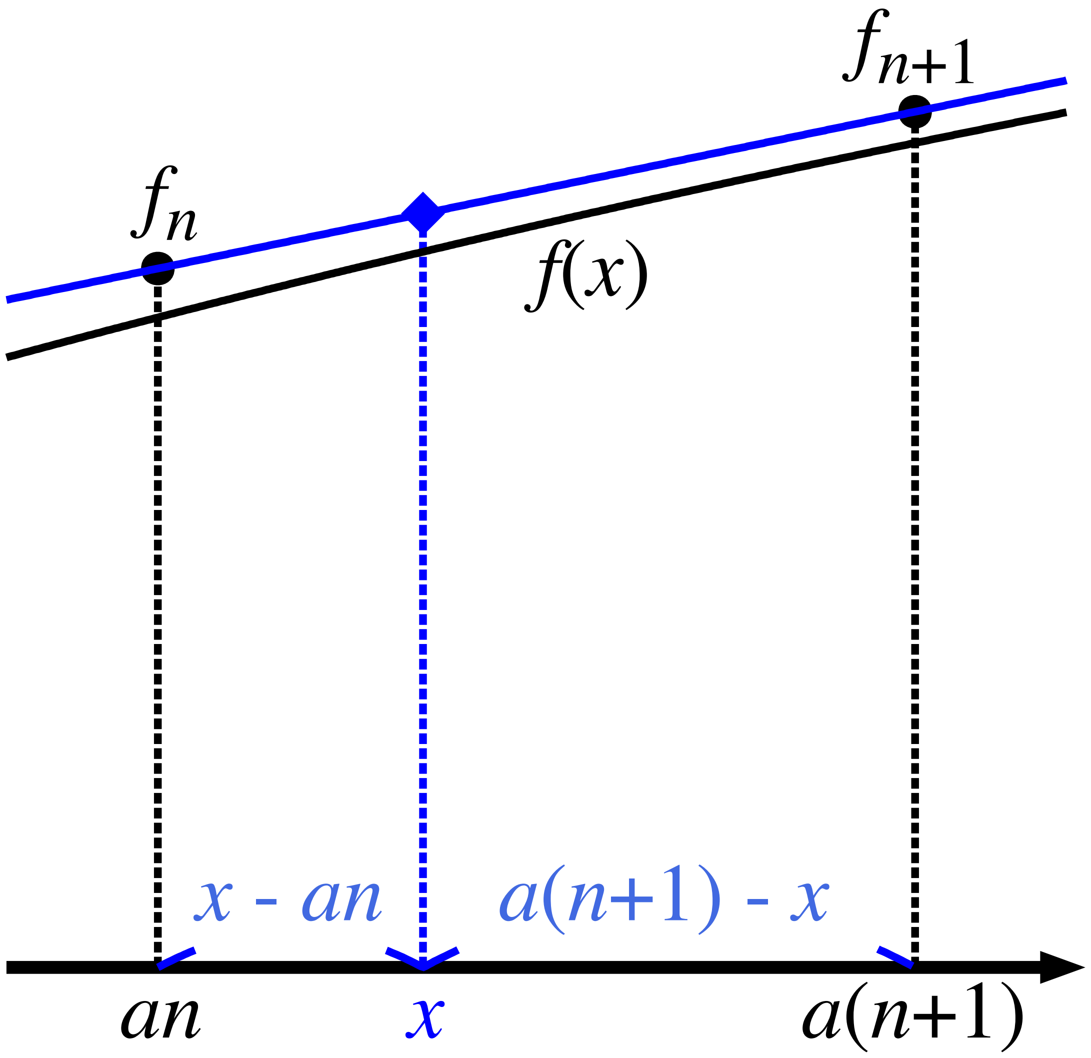
- Take the average over the sphere for each distance |x|

$$\hat{f}(|x|) = \frac{1}{2\pi^2} \oint_{S^3(|x|)} \mathrm{d}\Omega\, \bar{f}(x)$$



Evaluation of $\bar{f}(x)$ (1-dim)

- Linear interpolation

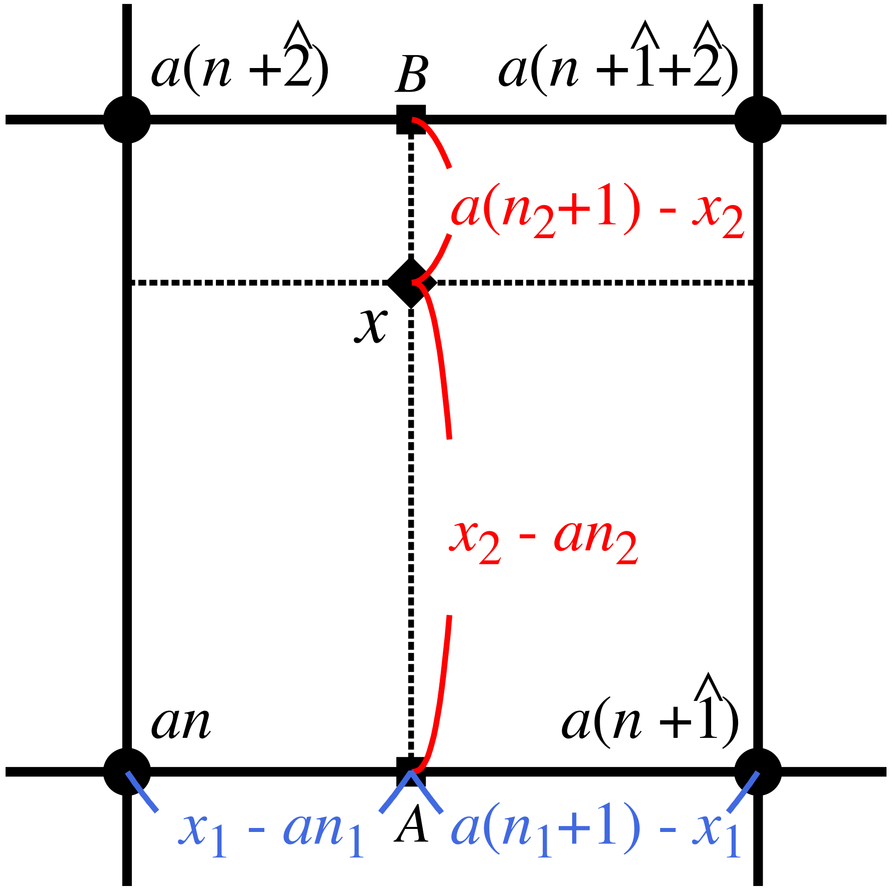


$$\bar{f}(x) = \frac{(a(n+1) - x) f_n + (x - an) f_{n+1}}{a} = f(x) \underline{+ O(a^2)}$$

Accurate up to $O(a^2)$

Evaluation of $\bar{f}(x)$ (2-dim)

- Bilinear interpolation



$$\bar{f}(x) = \frac{(a(n_2+1)-x_2)\bar{f}(A) + (x_2 - an_2)\bar{f}(B)}{a}$$

$$= a^{-2}\begin{pmatrix} a(n_1+1)-x_1 & x_1 - an_1 \end{pmatrix}\begin{pmatrix} f_n & f_{n+\hat{2}} \\ f_{n+\hat{1}} & f_{n+\hat{1}+\hat{2}} \end{pmatrix}\begin{pmatrix} a(n_2+1)-x_2 \\ x_2 - an_2 \end{pmatrix}$$

$$= f(x) + \underline{O(a^2)}$$

Evaluation of $\bar{\mathrm{f}}(\mathrm{x})$ (4-dim)

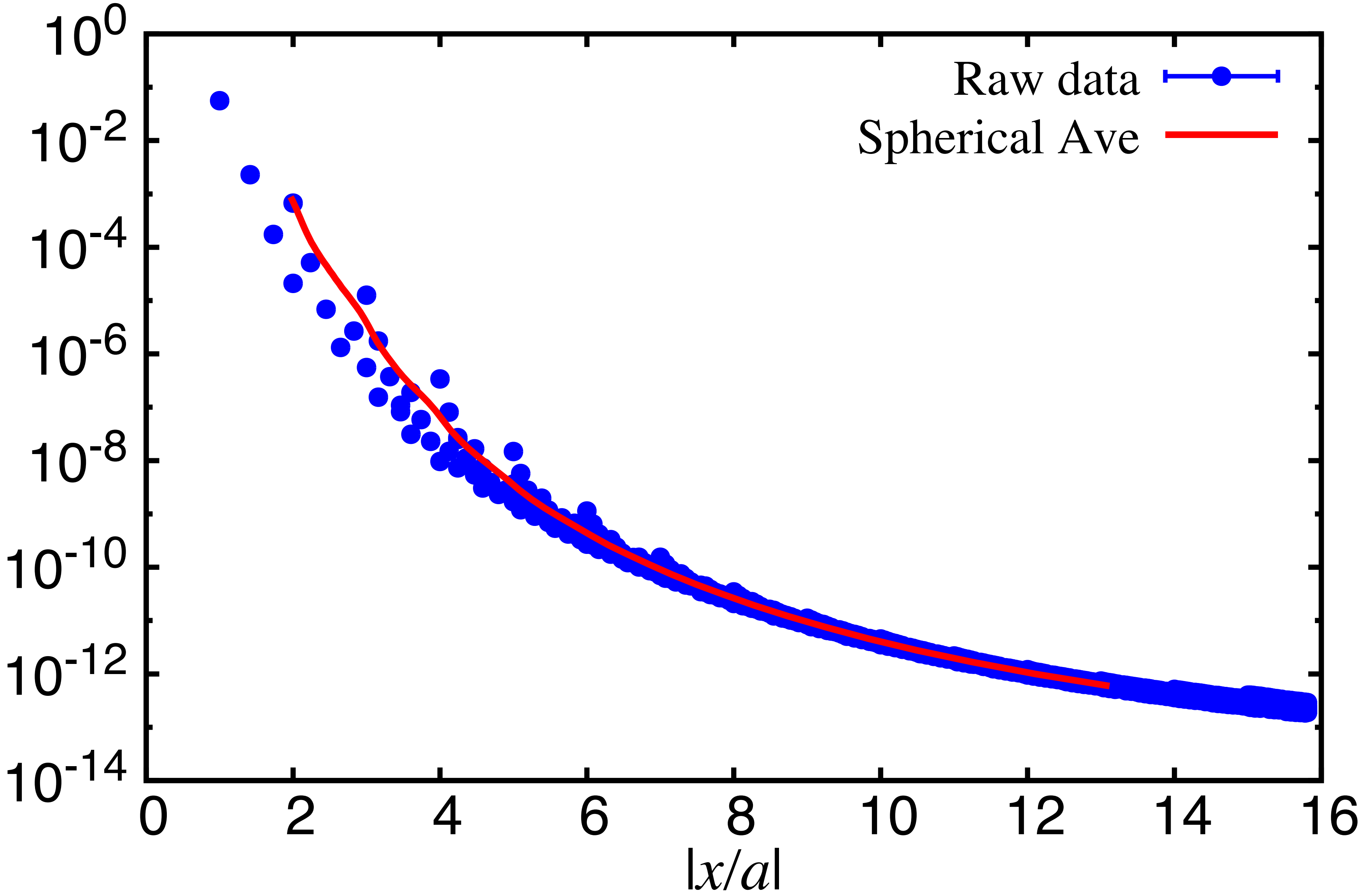
- Quadrilinear interpolation

$$\bar{f}(x) = a^{-4} \sum_{i,j,k,l=0}^{1} \Delta_{1,i}\Delta_{2,j}\Delta_{3,k}\Delta_{4,l} f_{n+i\hat{1}+j\hat{2}+k\hat{3}+l\hat{4}}$$

$$\Delta_{\mu,i} = |a(n_\mu + 1 - i) - x_\mu|$$

  - Accurate up to O(a^2)

Spherical Ave. for 2pt func

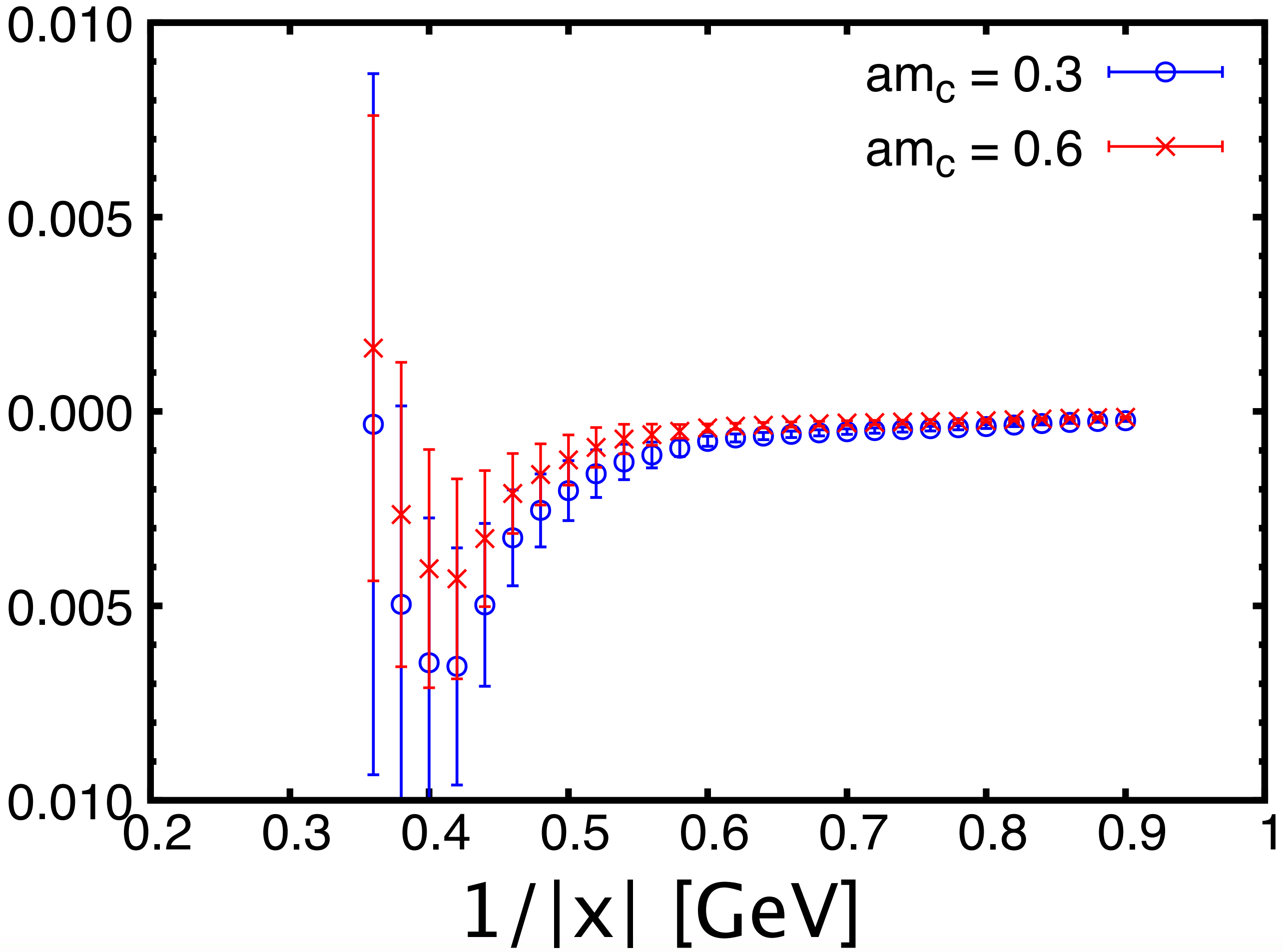


Results for M_{ij}

$$M_{ik} = \Sigma_j \, (G^{3f-3f}(x))^{-1}_{ij} \, G^{3f-4f}_{jk}(x)$$

$$= \left(1_{n3\times n3} \;\vdots\; \begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \bullet & \cdots & \cdots & \bullet \end{matrix} \right)$$

– Valid and should be independent of x at LDs $|x| \gg 1/m_c$



- $16^3 \times 32$
- $a^{-1} = 1.78$ GeV
- 88 confs in 3,500 MD time
- $m_{ud}^{val} = m_s^{val} = m_s^{sea}$
- Unrenormalized

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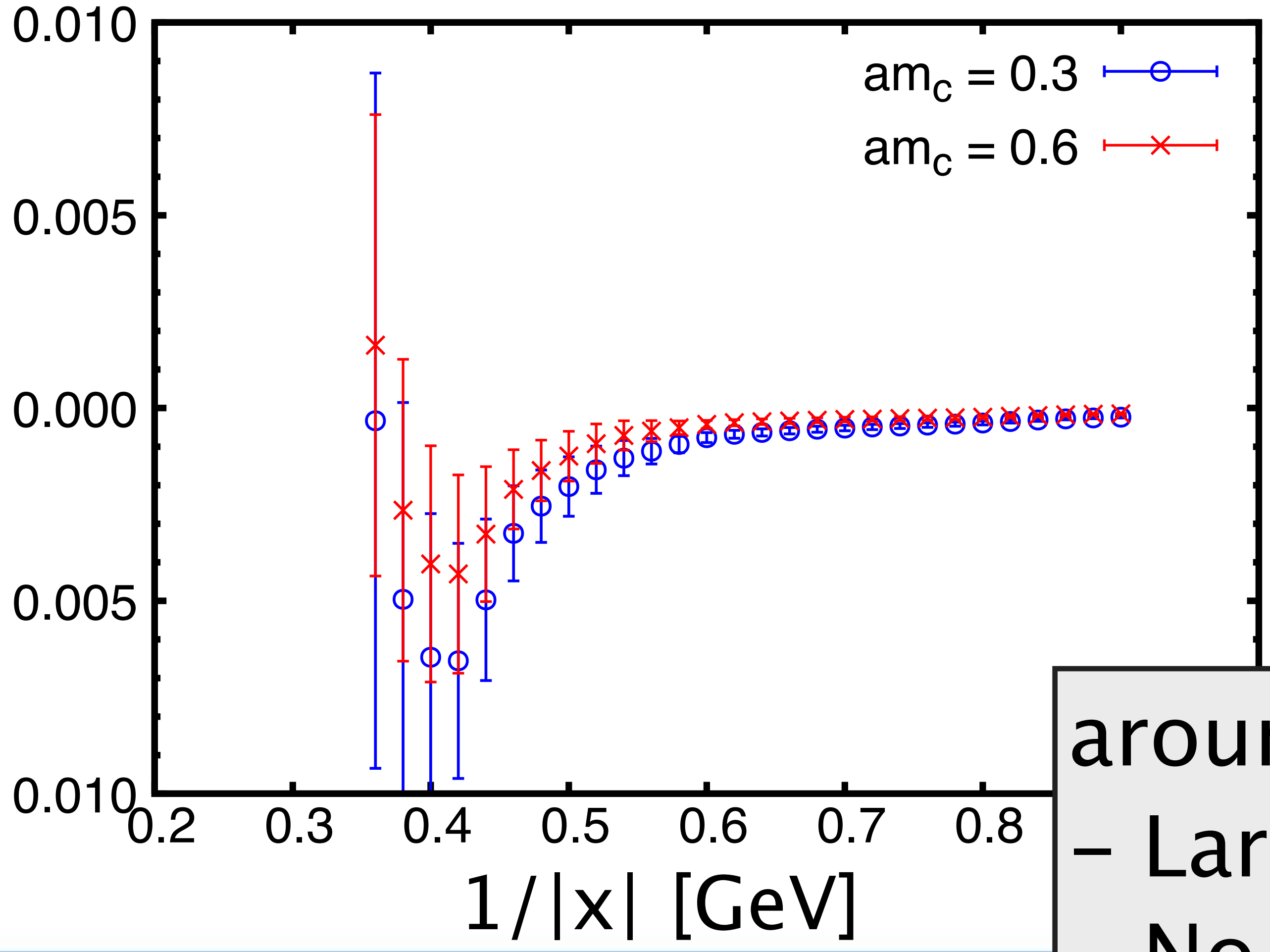


- 16^3 x 32
- a^{-1} = 1.78 GeV
- 88 confs in 3,500 MD time
- $m_{ud}^{val} = m_s^{val} = m_s^{sea}$
- Unrenormalized

around 1/|x| = 0.4 GeV…

– Large statistical error

– No clear plateau

Summary

- NP 4f/3f matching desired for K → ππ calculation

- Position-space procedure
  - Gauge invariant
  - Free from mixing w/ irrelevant operators

- Exploratory calculation on 16^3 lattice
  - Large statistical error

- To do
  - Seek ways to reduce statistical error (Lanczos A2A, …)
  - Main calculation on finer lattices (2.35 GeV, 3.15 GeV,…)

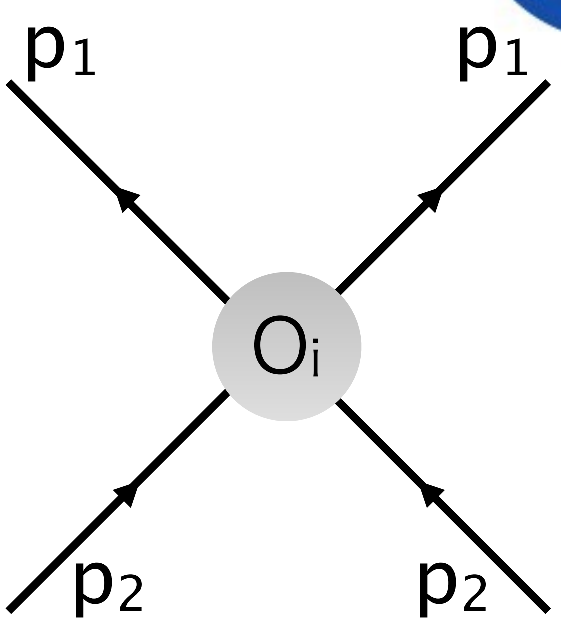
Previous effort in mom Sp.

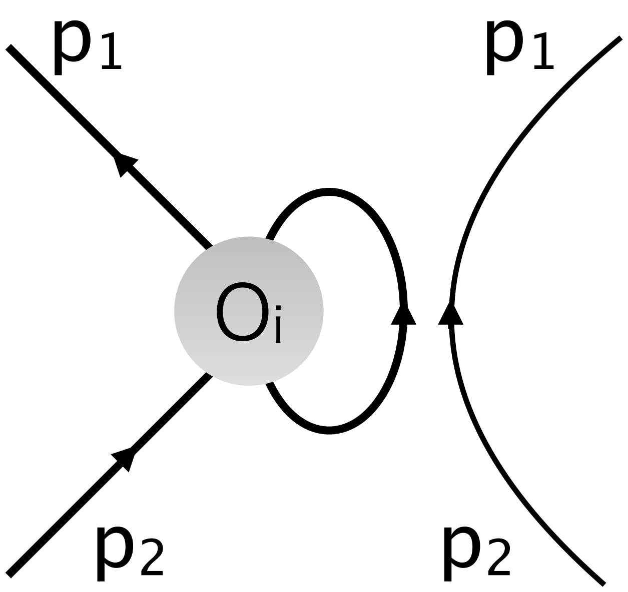
- Condition

$$P^{abcd}_{\alpha\beta\gamma\delta}\ \Lambda^{abcd}_{\alpha\beta\gamma\delta}(O^{3f}_i(\mu);p_1,p_2)\ w^{3f}_i(\mu)$$

$$\parallel$$

$$\underbrace{P^{abcd}_{\alpha\beta\gamma\delta}}_{\text{flavor, color and spin projector}}\ \underbrace{\Lambda^{abcd}_{\alpha\beta\gamma\delta}(O^{4f}_i(\mu);p_1,p_2)}_{\text{G-fixed amputated Green's function}}\ w^{4f}_i(\mu)$$





- Condition valid in $|p_{1,2}| \ll m_c$
- Statistical error
  - $|p_{1,2}| = 1.2$ GeV → 10%
  - $|p_{1,2}| = 0.6$ GeV → 50%
-

Why mom procedure so bad?

- Gauge fixing
  - Large Gribov noise
    - Gauge condition does not have a unique solution on the gauge orbit
    - Gauge-dependent quantities have some ambiguity
  - Mixing with gauge-noninvariant operators
- Off-shell condition
  - Mixing with operators that vanish by EoM

☆ All significant at small $p_{1,2}$

☆ Position-space procedure is free from all of these

ΔS = 1 4-quark operators

3-flavor

Type	Q_i
current-current	$Q_1 = (\bar{s}_\alpha d_\alpha)_L(\bar{u}_\beta u_\beta)_L$ $Q_2 = (\bar{s}_\alpha d_\beta)_L(\bar{u}_\beta u_\alpha)_L$
QCD penguin	$Q_3 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} (\bar{q}_\beta q_\beta)_L$ $Q_4 = (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} (\bar{q}_\beta q_\alpha)_L$ $Q_5 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} (\bar{q}_\beta q_\beta)_R$ $Q_6 = (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} (\bar{q}_\beta q_\alpha)_R$
EW penguin	$Q_7 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} e_q(\bar{q}_\beta q_\beta)_R$ $Q_8 = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_q^{3f} e_q(\bar{q}_\beta q_\alpha)_R$ $Q_9 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} e_q(\bar{q}_\beta q_\beta)_L$ $Q_{10} = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_q^{3f} e_q(\bar{q}_\beta q_\alpha)_L$

7 independent operators

4-flavor

Type	P_i
current-current	$P_1 = (\bar{s}_\alpha d_\alpha)_L(\bar{u}_\beta u_\beta)_L$ $P_1^c = (\bar{s}_\alpha d_\alpha)_L(\bar{c}_\beta c_\beta)_L$ $P_2 = (\bar{s}_\alpha d_\beta)_L(\bar{u}_\beta u_\alpha)_L$ $P_2^c = (\bar{s}_\alpha d_\beta)_L(\bar{c}_\beta c_\alpha)_L$
QCD penguin	$P_3 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{4f} (\bar{q}_\beta q_\beta)_L$ $P_4 = (\bar{s}_\alpha d_\beta)_L \sum_q^{4f} (\bar{q}_\beta q_\alpha)_L$ $P_5 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{4f} (\bar{q}_\beta q_\beta)_R$ $P_6 = (\bar{s}_\alpha d_\beta)_L \sum_q^{4f} (\bar{q}_\beta q_\alpha)_R$
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9 independent operators

Color trivialization by Fierz trf.

- Def: $(\bar{s}d)_L(\bar{q}q)_{R/L} = \bar{s}\gamma_\mu(1-\gamma_5)d \cdot \bar{q}\gamma_\mu(1 \pm \gamma_5)q$

- Left-Left operators

$$(\bar{s}_\alpha d_\beta)_L(\bar{q}_\beta q_\alpha)_L = (\bar{s}_\alpha q_\alpha)_L(\bar{q}_\beta d_\beta)_L$$

- Left-Right operators

$$(\bar{s}_\alpha d_\beta)_L(\bar{q}_\beta q_\alpha)_R = -2\bar{s}_\alpha(1+\gamma_5)q_\alpha \cdot \bar{q}_\beta(1-\gamma_5)d_\beta$$

Potential $O(a^1)$ error (1-dim)

- Defs:
  - f_n: lattice value at site n
  - $f(x)$: "continuum limit" : $f_n = f(an) + O(a^2)$

- Estimation $\bar{f}(x)$ should satisfy
  - $\bar{f}(x) = f(x) + O(a^2)$

- Potential $O(a^1)$ error in $\bar{f}(x)$
  - $f_n = f(an) + O(a^2)$

$$= f(x) + \underbrace{f'(x)\cdot(an-x)}_{O(a^1)} + O(a^2)$$

  - $\bar{f}(x)$ is calculated using f_n's ⇒ $O(a^1)$ can appear
  - Balanced combination needed

Quark mass renormalization

- $Z_m = Z_S^{-1}$
- Position-space renormalization of scalar current

$$Z_S^{\overline{\mathrm{MS}}/\mathrm{lat}}(\mu,1/a)^2 \ \Pi_S^{\mathrm{lat}}(1/a;x) = \Pi_S^{\overline{\mathrm{MS}}}(\mu;x)$$

$$Z_S^{\overline{\mathrm{MS}}/\mathrm{lat}}(\mu,1/a) = \sqrt{\frac{\Pi_S^{\overline{\mathrm{MS}}}(\mu;x)}{\Pi_S^{\mathrm{lat}}(1/a;x)}}$$

  - ✦ Π_S: two-point function of scalar currents
  - ✦ $Z_S = Z_P$ if chirally symmetric lattice action (DWF in this work)

- We analyze

$$\widetilde{m}_q^{\mathrm{MS}}(\mu;x;a) = m_q^{\mathrm{bare,phys}}(a)\sqrt{\frac{\frac{1}{2}(\Pi_S^{\mathrm{lat}}(1/a;x)+\Pi_P^{\mathrm{lat}}(1/a;x))}{\Pi_S^{\overline{\mathrm{MS}}}(\mu;x)}}$$

Lattice calculation

- Ensembles
  - 2+1 Domain-wall fermions
  - 3 lattice spacings: 1.7−3.1 GeV
  - Pion masses: 300−420 MeV

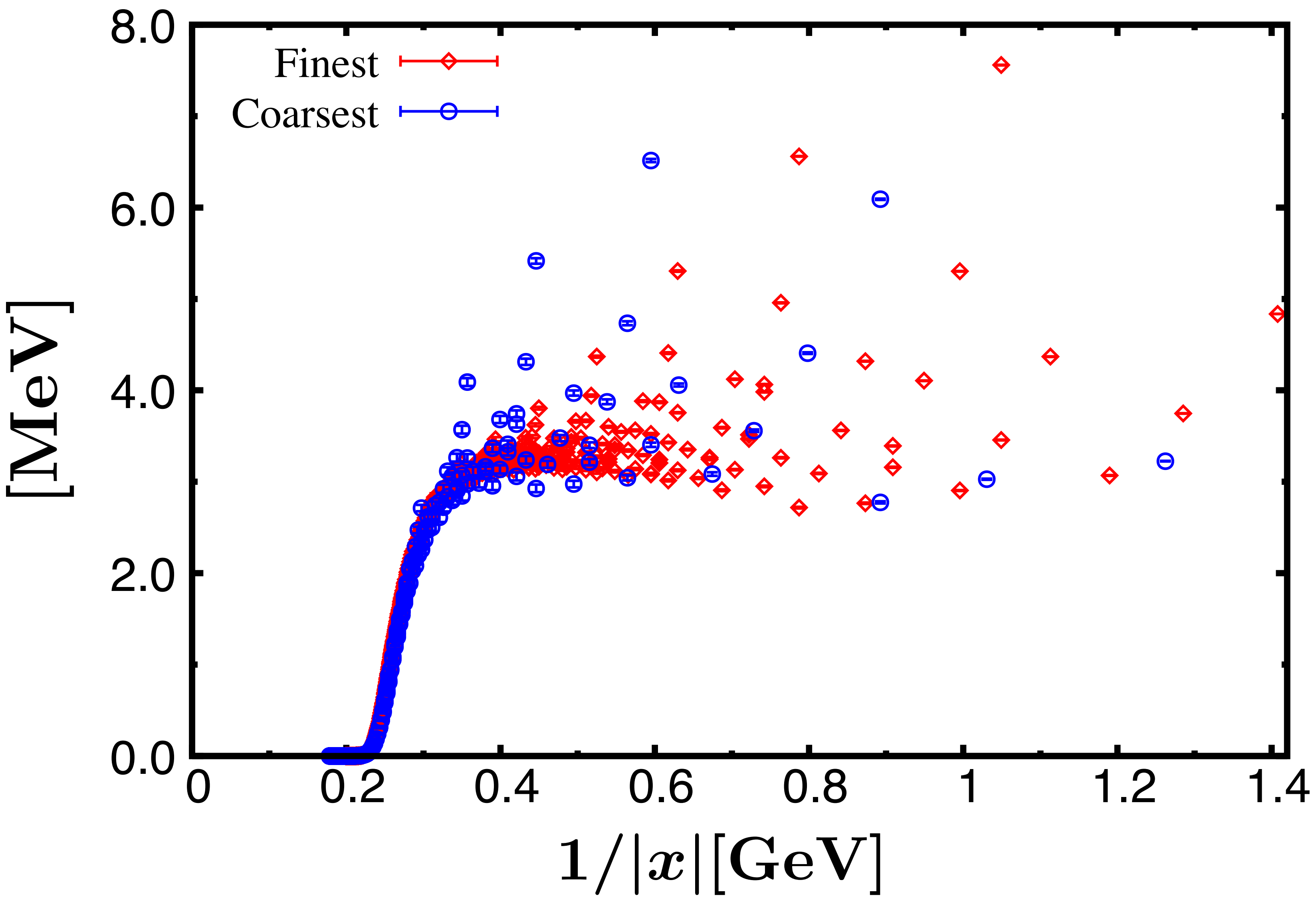
- For each ensemble, we analyze

$$\widetilde{m}_q^{\overline{\mathrm{MS}}}(\mu; x; a) = \underline{m_q^{\mathrm{bare,phys}}(a)}\,\underline{\sqrt{\frac{\frac{1}{2}\left(\Pi_S^{\mathrm{lat}}(1/a; x) + \Pi_P^{\mathrm{lat}}(1/a; x)\right)}{\Pi_S^{\overline{\mathrm{MS}}}(\mu; x)}}}$$

[RBC/UKQCD (2016)]

Supposed to be a constant Z_m
if in renormalization window

$\widetilde{m}_q^{\overline{MS}}(3\,\mathrm{GeV};x)$



- Different lattice points distinguished ((1,1,1,1) vs (0,0,0,2))
- Large discretization errors

Result for spherical average

- Sphere average of

$$\widetilde{m}_q^{\overline{\mathrm{MS}}}(\mu;x;a) = m_q^{\mathrm{bare,phys}}(a)\sqrt{\frac{\frac{1}{2}(\Pi_S^{\mathrm{lat}}(1/a;x)+\Pi_P^{\mathrm{lat}}(1/a;x))}{\Pi_S^{\overline{\mathrm{MS}}}(\mu;x)}}$$

- Able to calculate at any distance
- Plateau seen better for finer lattices