The chirally rotated Schrödinger functional (\(\chi SF\)) with massless Wilson fermions is a lattice regularization which endows the Schrödinger functional (SF) with the property of automatic \(O(a)\)-improvement. The \(\chi SF\) framework is effective in reducing lattice artefacts in correlation and step scaling functions, but especially it offers new strategies to study and simplify the patterns of renormalization. The price to pay for this gain in the automatic \(O(a)\)-improvement is the nonperturbative tuning of coefficients of new boundary contributions. This tuning is the first phase of a long-term project, aiming at the computation of \(B_{\psi}^0\), low-energy contributions, but Wilson fermion \(N_f = 2 + 1\) lattice QCD in a non-unitary (mixed-action) framework.

The \(\chi SF\) setup

The fermion flavour doublet \(\psi = \begin{pmatrix} \bar{c} \\ \bar{s} \end{pmatrix} \bar{c} \) satisfies bc's.

\[ Q_{\psi} = \frac{1}{2} \left( \gamma \cdot p + m \right) \psi, \text{ in time and periodics}\ in\ space. \]

The action is

\[ S_f = a^4 \sum_{\bar{x}} \bar{c}(\bar{x}) D_{\bar{x}} m_{\bar{x}} c(\bar{x}), \]

with \(D_{\bar{x}} \) the standard Wilson fermion matrix and the boundary term

\[ D_{\bar{x}} \psi(\bar{x}) = \left( \gamma_{\bar{x}} + a_{\bar{x}} \right) \psi(\bar{x}), \]

Performing the chiral flavour rotation

\[ R = \exp \left( a \gamma^5 \right) \]

we obtain the following universality relation for SF and \(\chi SF\) correlation functions

\[ \langle O(1) \rangle_{\chi SF} = \lim_{\nu \to 0} \left[ Z_{\chi SF}(\nu) \langle O(1) \rangle_{SF} + O(a) \right], \]

where composite operators \(O \bar{Q}\) are defined in the bulk and \(\bar{O} \bar{Q}\) are defined on a time boundary. Note that \(\chi SF\) incorporates automatic \(O(a)\)-improvement. The price to pay is the nonperturbative tuning of the boundary counterterm coefficient \(a\).

Correlation functions

We consider the set of fermion bilinear operators; e.g.,

\[ \langle \bar{c}(x) c(y) \rangle_{\chi SF}, \quad \langle \bar{c}(x) c(y) \rangle_{SF}, \]

with flavours \(f_1, f_2 \in \{ u, d, s, c, t \} \), and determine the SF bulk-to-boundary correlation functions

\[ g_{\chi SF}(x,y) = -\frac{1}{\nu} \langle k(x) \rangle_{\chi SF}, \quad X = \nu A_0, S P \]

\[ g_{SF}(x,y) = -\frac{1}{\nu} \langle k(x) \rangle_{SF} \]

Up to discretization effects and boundary fields renormalization they are related to the standard SF correlation functions \(f_1 \) and \(s_1 \) by universality

\[ g_{\chi SF}(x,y) = Z_{\chi SF} g_{SF}(x,y) = Z_{\chi SF} g_{SF}(x,y), \quad \bar{J}_{\chi SF} = Z_{\chi SF} \bar{J}_{SF}, \quad \bar{J}_{SF} \rightarrow \bar{J}_{\chi SF}, \quad \bar{J}_{SF} \rightarrow \bar{J}_{\chi SF} \]

The SF correlation functions \(f_2 \) and \(s_2 \) are non-zero, while \(f_3 \) and \(s_3 \), being parity odd, are \(0\). The local vector current can be replaced by the exactly conserved one \(\bar{J}_{\chi SF}(x) \) with normalization \(Z_{\chi SF} = 1\). Therefore \(\bar{J}_{\chi SF} \) may be obtained from the ratio

\[ Z_{\chi SF}^{-1} = \frac{\langle k(x) \rangle_{\chi SF}}{\langle k(x) \rangle_{SF}}, \quad \text{or} \quad Z_{\chi SF} = \frac{\langle k(x) \rangle_{SF}}{\langle k(x) \rangle_{\chi SF}}. \]

Computational setup and results

We obtain results for \(N_f = 3\) QCD in a non-unitary setup. Valence quark propagators are inverted with \(\chi SF\) boundaries on the configuration ensembles of \([1655, 025243]\), generated on lattices with standard SF boundary conditions. These configurations have been used for the RG-running of the quark mass in a range of scales \(Z_{\psi} \mu < 125\, \text{GeV}\), in the standard framework of finite-size scaling \(L \rightarrow 2L\).

**Tuning**

We must ensure that massless QCD with SF boundary conditions is correctly regularized. This is achieved by tuning the bare mass parameter \(m_0\) to its critical value, \(m_\text{crit}\), where the axial current is conserved, and by tuning the boundary counterterm coefficient \(s_2\) so that physical parity is restored. At present we choose the conditions

\[ m_0 = \frac{1}{2} \left( \frac{2 \Delta m}{\Delta m + 1} \right), \quad \text{with} \quad \Delta m = m_\text{crit} - m_0, \]

By requiring that eq. (2) be satisfied, \(s_2\) is tuned for each ensemble as shown in Fig. 1.

**O(a)-improvement**

Using eqs. (2), we obtain two estimates for \(Z_{\chi SF}\) which differ by discretization errors. These estimates are computed across the full range of ensembles available. A fit to the data matches onto the asymptotic perturbative result in the limit \(N_f \rightarrow \infty\).

**Outlook for 4 fermions**

Renormalization

Automatic \(O(a)\) improvement is especially valuable in simplifying the renormalization of four fermion operators. They enter the most general expression of the effective Hamiltonian which describes flavour physics processes at low energy in the Standard Model (SM) and its extensions (BSM). Here we focus on \(\Delta F = 4\) transitions. The 4-quark operators with four distinct flavours

\[ O_{ij}^T \equiv \langle \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \rangle \quad (i,j,k) \]

can be classified as purely even and purely odd:

\[ O_{ij}^T \equiv \langle \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \rangle = (\bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4) \quad \text{purely even}, \]

Due to the explicit breaking of chiral symmetry of the Wilson regularization, the operators in general mix as follows:

\[ O_{ij}^T = \langle \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \rangle = (\bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4) + (\bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4) \]

The parity-odd sector has a simpler, continuum-like mixing pattern \((\Delta F = 0)\). Renormalization conditions are imposed on parity odd operators by setting suitable renormalized correlation functions equal to their tree level values at the scale \(\mu = L^{-1}\). In a standard SF setup parity conservation imposes the use of 4-point correlation functions; e.g.

\[ T_{\chi SF}(x) = \langle \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \rangle \]

These are statistically noisy and suffer from \(O(a)\) discretization errors. The most convenient renormalization scheme is \(\chi SF\) imposed on 3 point correlation functions; e.g.

\[ G_{\chi SF}(x) = \langle \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \rangle \]

They are statistically noisy and automatically \(O(a)\) improved.

Performing suitable chiral rotations, the \(\chi SF\) flavours are rotated into the physical flavours. Thus we can map the renormalized \(G_{\chi SF}(x)\) into the continuum correlation function

\[ G_{\chi SF}(x) \rightarrow T_{\chi SF}(x) = \langle \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \rangle \]

**Physical determinations**

The renormalization program will be employed in the lattice computation of the physical \(B_{\psi}^0\) parameter which controls the \(K^0 \rightarrow K^\ast \) meson oscillations, towards a better understanding of the physics of CP violation. To accommodate physical meson states the configurations of \(N_f = 2 + 1\) CLS ensembles are now characterised by large physical volumes with open boundary conditions and by non-zero quark masses \([1411, 2382, 161474, 05808005]\). The sea quarks are Wilson/Clover. Valence fermions are fully twisted \([1812, 12474]\), with three flavours twisted at tuned angle \(\alpha = \pi/2\) and the fourth one at \(\alpha = \pi/2\). Uniaxially frozen at finite lattice spacing, but it is recovered in the continuum limit. Performing distinct Overwalder-Seiler chiral rotations for each flavour, correlation functions with parity-odd operators (renormalized in the \(\chi SF\) scheme as described above) are mapped onto the 3-point correlation functions of parity even operators with pseudoscalar sources, from which the \(U\)-parameters are readily extracted:

\[ B_{\chi SF}(\mu) \equiv \langle K^0 \rangle \langle \bar{Q}_1 \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \rangle \].