Investigating Rare Kaon Decays with the All-to-All Method.

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Lattice 2019 Wuhan
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Outline

- $K \to \pi \ell^+ \ell^-$
- All-to-All method
- Exploratory $24^3$ results
- Preliminary $48^3$ results
- Future outlook & conclusions
$K \rightarrow \pi \ell^+ \ell^-$
Motivations

• $K \to \pi \ell^+ \ell^-$ are FCNC processes
• Forbidden at tree level
  $\implies$ Sensitive to new physics

• Long distance effects dominate: use lattice QCD
• NA62 has collected $O(30k)$ $K \to \pi \mu^+ \mu^-$ samples
  • Potential for $e^+e^-$ samples in 2021-2023 runs discussed at the Rare Kaon Decays Forum 2019 [C. Parkinson: RKF 2019]
In terms of EM transition form factor $V_j(z)$ \((j = +, S)\)


\[ \mathcal{A}_\mu(q^2) = -iG_F \frac{V_j(z)}{(4\pi)^2} (q^2(k + p)_\mu - (M_K^2 - M_\pi^2)) \]

\[ V_j(z) = a_j + b_j z + V_{j\pi\pi}(z), \quad z = \frac{q^2}{M_K^2}, \quad q \equiv k - p \]

Phenomenological predictions:

\[ V. \text{Cirigliano et al. arXiv:1107.6001} \]

\[
\begin{array}{c|c|c|c}
\text{e} & |a_S| & a_+ & b_+ \\
\hline
\text{\mu} & |a_S| = 1.54^{+0.40}_{-0.32} & a_+ = -0.575 \pm 0.039 & b_+ = -0.813 \pm 0.145 \\
\end{array}
\]

NA62 working towards new $a_+, b_+$ measurements for the muon: [A. Sturgess: 2018 Thesis]
Long-Distance Minkowski amplitude:

\[ A^j_\mu(q^2) = \int d^4 x \, \langle \pi^j(p) | T[J_\mu(0)H_W(x)] | K^j(k) \rangle \]

\[ \Delta S = 1 \text{ effective weak Hamiltonian:} \]

\[ H_W(x) = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left( C_1 (Q^u_1 - Q^c_1) + C_2 (Q^u_2 - Q^c_2) \right) \]

Wilson coefficients \( C_1 \) and \( C_2 \) are much larger than \( C_3,...,8 \)

\[ Q^q_1 = (\bar{s}_a \gamma^L_\mu d_a)(\bar{q}_b \gamma^L_\mu q_b) \quad \text{and} \quad Q^q_2 = (\bar{s}_a \gamma^L_\mu q_a)(\bar{q}_b \gamma^L_\mu d_b) \]

where \( \gamma^L_\mu = \gamma_\mu (1 - \gamma_5) \).

Current can be either the local or conserved lattice vector current.
• The Wick contractions for $K \rightarrow \pi H_W$ 3-pt functions gives:

- **W** (Wing)
- **C** (Connected)
- **S** (Saucer)
- **E** (Eye)

• Including the current insertions:
All-to-All Method
Eigenvectors of $D(x, y)$, $\lambda_i$ and $\phi_i$, are used to solve the low modes exactly. For a set of random noise source vectors that have the property:

\[
\lim_{N_h \to \infty} \sum_{h=1}^{N_h} \eta_h \eta_h^\dagger = 1
\]

and deflating the Dirac operator

\[
D^{-1}_{\text{defl}} = D^{-1} - \sum_{j=1}^{N_l} \frac{\phi_j \phi_j^\dagger}{\lambda_j}
\]

we can write

\[
D^{-1}_{\text{A2A}} = \sum_{i=1}^{N_l} \frac{\phi_i \phi_i^\dagger}{\lambda_i} + \sum_{h=1}^{N_h} D^{-1}_{\text{defl}} \eta_h \eta_h^\dagger
\]

All-to-all formalism has been implemented in the C++ library, “Grid,” and the framework based on Grid, “Hadrons.”

See Antonin Portelli’s talk (Wed 09:00).
The “all-to-all vectors” are defined as

\[ v_i = \begin{cases} 
\frac{1}{\lambda_i} \phi_i \\
D_{defl}^{-1} \eta_i 
\end{cases} \quad w_i = \begin{cases} 
\phi_i : 1 \leq i < N_l \\
\eta_i : N_l \leq i < N_l + N_h 
\end{cases} \]

such that

- \( D_{A2A}^{-1}(x, y) = \sum_i v_i(x) w_i^{\dagger}(y) \)

We can now write the meson correlator functions as

- \( C(t) = \sum_{i,j} \Pi_{ji}^{(q',q)}(t_x; \Gamma_2) \Pi_{ij}^{(q,q')} (t_y; \Gamma_1) \)

where \( \Pi_{ij}^{(q,q')} (t_x; \Gamma) = \sum_{\vec{x}} w_i^{\dagger q}(x) \Gamma v_j^{q'}(x) \) are “Meson Fields.”

Arbitrary \( n \)-point functions can be made with appropriate MF multiplication and contraction. [J.Foley arXiv:hep-lat/0505023]
$24^3$ Tests
$24^3$ **Setup & $K \rightarrow \pi$ 3-pt Function**

$24^3 \times 64$ Domain Wall Fermion

- $M_\pi \approx 340 \text{MeV}$
- $a^{-1} = 1.78 \text{GeV}$
- 2 + 1 flavor
- Light quarks: 600 low modes
- Spin/color/time diluted high modes

Diagrams were found by computing contractions:

- $\mathcal{P}(x_H; t_y, \Gamma) = \sum_{i,j} v_i^{(q)}(x_H) \tilde{\Pi}_{ij}(t_y; \Gamma) v_j^\dagger(q')(x_H)$

where $\tilde{\Pi}(t_y; \Gamma)$ is some product of MFs.
All-to-All Current Insertion

Stochastic solves:
- All propagators treated in A2A fashion
- Noise on current insertion unavoidable

Sequential solves:
- Removes noise at current insertion
- No $t_J$ translation $\Rightarrow$ loss of statistics
Physical Point Runs
Physical Point Setup

$48^3 \times 96$ gauge configuration [T. Blum et al. arXiv:1411.7017]

- $M_\pi \approx 140$ MeV, $M_K \approx 500$ MeV
- $a^{-1} = 1.73 GeV$
- 2 + 1 flavor
- Scaled DWF action for strange quarks
- ZMöbius DWF action for light quarks
- 2000 low modes for light quarks $\implies$ deflated solves
- Spin/color/time diluted high modes
- $p_K = (0, 0, 0)$, $p_\pi = \frac{2\pi}{L} (1, 0, 0)$
Physical Point Preliminary Results

- Same source/sink separation as exploratory $24^3$ runs
- Local current insertions
- Coulomb gauge fixed wall sources

- With Wilson coefficients and quark charges
- Analysis: remove unphysical exponential terms and integrate over $t_H$
Future Outlook and Conclusions
\( N_t = 96 \) gives more leeway for source/sink separation

- Expected range already investigated

[P.A. Boyle et al. arXiv:1504.01692]

For Eye diagrams:

- Use A2A vector for the loops
- For loop divergence we can:
  - Compute charm quark loop & employ GIM mechanism
  - Calculate in 3-flavor theory & use NPR

For disconnected diagrams:

Same approach as James Richings’ QED talk (Wed 11:50)
Summary

• A2A vectors have been implemented in Grid, with modularization in Hadrons
• They are a powerful tool for approximating quark propagators
  • But they are not always appropriate to use for a full calculation
  • A2A propagators will be used to supplement the $K \rightarrow \pi \ell^+ \ell^-$ calculation
• Physical point simulations have already begun
  • Eye diagrams and more kinematics to follow
Thank you.
C. Parkinson: $K \rightarrow \pi \mu^+ \mu^-$ at NA62 - The 2nd Forum on Rare Kaon Decays 2019

G. D’Ambrosio et al.: The Decays $K \rightarrow \pi \ell^+ \ell^-$ beyond Leading Order in the Chiral Expansion - arXiv:hep-ph/9808289


A. Sturgess: Tracking Optimisation and the Measurement of $K \rightarrow \pi \mu^+ \mu^-$ at NA62 - Thesis (University of Birmingham)

Gino Isidori et al.,: Rare Kaon Decays on the Lattice - arXiv:hep-lat/0506026

J. Foley et al.,: Practical all-to-all propagators for lattice QCD - arXiv:hep-lat/0505023

T. Blum et al.: Domain wall QCD with physical quark masses - arXiv:1411.7017

P.A. Boyle et al.: The kaon semileptonic form factor in $N_f = 2 + 1$ domain wall lattice QCD with physical light quark masses - arXiv:1504.01692