Calculating the two-photon contribution to the real part of $\pi^0 \rightarrow e^+ e^-$ decay amplitude

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Two-photon contribution to $K_L \rightarrow \mu^+ \mu^-$

Intermediate states with lower energy than kaon mass:
$\gamma\gamma, |0\rangle, \eta, \pi, \pi\pi(\gamma)$, etc.

Calculation in Euclidean space would result in exponentially divergent behavior.
\[ \pi^0 \rightarrow e^+ e^- \]

\[ \langle e^+(k_+)e^-(k_-)|\pi^0(P)\rangle = \int d^4 u \ d^4 v \ H_{\mu\nu}(u, v)L^{\mu\nu}(u, v) \]

\[ H_{\mu\nu}(u, v) = \langle 0| J_\mu(u) J_\nu(v)|\pi\rangle \]

\[ L^{\mu\nu}(u, v) = \int d^4 p \ e^{-ip\cdot(u-v)} \left[ \frac{g_{\mu\mu'}}{(p + \frac{P}{2})^2 - i\epsilon} \right] \left[ \frac{g_{\nu\nu'}}{(p - \frac{P}{2})^2 - i\epsilon} \right] \]

\[ \bar{u}(k_-)\gamma_\mu' \left[ \frac{\gamma \cdot (p + \frac{P}{2} - k_-) + m_e}{(p + \frac{P}{2} - k_-)^2 + m_e^2 - i\epsilon} \right] \gamma_{\nu'} v(k_+) \]
Let $w$ be the relative coordinate of two EM currents i.e. $w = u - v$. Decay amplitude:

$$\mathcal{A} = \int d^4w \langle 0 | T \{ J_{\mu} \left( \frac{w}{2} \right) J_{\nu} \left( - \frac{w}{2} \right) \} | \pi^0 \rangle$$

$$= \int d^4p \ e^{-ip \cdot w} \left[ \frac{g_{\mu\mu'}}{(p + \frac{P}{2})^2 + m^2_\gamma - i\epsilon} \right] \left[ \frac{g_{\nu\nu'}}{(p - \frac{P}{2})^2 + m^2_\gamma - i\epsilon} \right] \bar{u}(k_-) \gamma_{\mu'} \left[ \frac{\gamma \cdot (p + \frac{P}{2} - k_-) + m_e}{(p + \frac{P}{2} - k_-)^2 + m^2_e - i\epsilon} \right] \gamma_{\nu'} v(k_+)$$

Here the matrix element is a Minkowski-space quantity. To calculate it using Lattice QCD, it is necessary to do a Wick rotation $w^0 \rightarrow i\omega^0$. 

$\pi^0 \rightarrow e^+ e^-$
Wick Rotation

\[
L_{\mu\nu}(w) = \int d^4 p \ e^{-ip\cdot w} \left[ \frac{g_{\mu\mu'}}{(p + \frac{P}{2})^2 - i\epsilon} \right] \left[ \frac{g_{\nu\nu'}}{(p - \frac{P}{2})^2 - i\epsilon} \right] \overline{u}(k_-) \gamma_\mu' \left[ \frac{\gamma \cdot (p + \frac{P}{2} - k_-) + m_e}{(p + \frac{P}{2} - k_-)^2 + m_e^2 - i\epsilon} \right] \gamma_{\nu'} v(k_+) \]

▶ After \( w^0 \) is Wick-rotated, the exponential in \( L_{\mu\nu}(w) \) introduces exponential growth.
▶ The \( p^0 \) contour also needs to be rotated.
▶ Because of the presence of intermediate states with lower energy than the mass of pion, a naive Wick rotation \( p^0 \rightarrow ip^0_E \) will make the integral blow up.
When $|\vec{p}| < \frac{m}{2}$, the $p^0$ contour needs to be deformed to circumvent the two poles that crosses the imaginary axis.
Wick Rotation

\[ e^+(\vec{k}_+)^{\mu} \left( v + (\vec{k}^+) \right)^{\nu} \cdot (\vec{k} - (\vec{k}^-))^{\mu} \left( u - P \right)^{\nu} + \]

\[ e^-(\vec{k}_-) \cdot (\vec{k} + P)^{\mu} \left( u + P \right)^{\nu} \\
\]

\[ \Rightarrow \quad L_{\mu\nu} \propto e^{M_\pi \left| w_0 \right|} \cdot \frac{w_0}{2}, \quad \text{where} \quad w_0 = u^0 - v^0 \]

\[ \Rightarrow \quad \text{The lightest intermediate state between } J_\mu \text{ and } J_\nu \text{ is } \pi\pi \text{ state} \\
\]

\[ \Rightarrow \quad H_{\mu\nu} = \langle 0 | J_\mu \left( \frac{w}{2} \right) J_\nu \left( -\frac{w}{2} \right) | \pi \rangle \propto e^{-\left( E_n - \frac{M_\pi}{2} \right) \left| w_0 \right|}, \quad E_n > 2M_\pi \]

\[ \Rightarrow \quad \text{Amplitude converges exponentially at least as fast as} \quad e^{-M_\pi \left| w_0 \right|}. \quad \text{We can safely perform the Euclidean space calculation}. \]
Hadronic part

\[ \langle 0 | T J_\mu(0) J_\nu(x) | \pi \rangle = \lim_{t \to -\infty} \frac{2m_\pi}{N_\pi} Z_V^2 e^{m_\pi t} | \langle 0 | T J_\mu(0) J_\nu(x) \pi(t) | 0 \rangle | \]

where \( N_\pi \) is the pion ground state amplitude \( N_\pi = \langle \pi | \pi(0) | 0 \rangle \), \( Z_V \) is the coefficient of EM current that connects the local non-conserved current with global conserved current.

▶ Contribution from the second graph is small (suppressed by SU(3) flavor symmetry).
Summary

- Leptonic part integral is calculated numerically with CUBA library.

- Hadronic part can be extracted from three point function.

\[
\langle 0 | T J_\mu(0) J_\nu(x) | \pi \rangle = \lim_{t \to -\infty} \frac{2 m_\pi}{N_\pi} Z_V^2 e^{m_\pi |x|} t \langle 0 | T J_\mu(0) J_\nu(x) \pi(t) | 0 \rangle
\]
Branching ratio $B(\pi^0 \to e^+ e^-) = (6.87 \pm 0.36) \times 10^{-8}$

Absorptive (imaginary) part of branching ratio is analytically known (Optical theorem)
$B_{imag}(\pi^0 \to e^+ e^-) = 4.69 \times 10^{-8}$

Dispersive (real) part of branching ratio (Total branching ratio - absorptive part)
$B_{real}(\pi^0 \to e^+ e^-) = (2.18 \pm 0.36) \times 10^{-8}$
What we have calculated:

- Absorptive part of branching ratio.
  It can be compared with analytic result to help us estimate error in lattice calculation.
- Dispersive part of branching ratio. (Our real goal)
Ensembles

Gauge action: Iwasaki + DSDR

1. **24³ Coarse ensemble**
   - Lattice volume: $24^3 \times 64$
   - $a^{-1} = 1\text{GeV}$
   - $am_\pi = 0.13975(10)$
   - Number of trajectories: 35

2. **32³ Coarse ensemble**
   - Lattice volume: $32^3 \times 64$
   - $a^{-1} = 1\text{GeV}$
   - $am_\pi = 0.139474(96)$
   - Number of trajectories: 40

3. **32³ Fine ensemble**
   - Lattice volume: $32^3 \times 64$
   - $a^{-1} = 1.37\text{GeV}$
   - $am_\pi = 0.10468(32)$
   - Number of trajectories: 17

- Ensemble 1 & 3 have almost the same physical spatial volume.
- Compare 1 & 2 $\rightarrow$ Finite spatial volume error
- Compare 2 & 3 $\rightarrow$ Finite lattice spacing error
Imaginary Part Contribution to Decay Rate

![Graph showing the imaginary part of branching ratio as a function of time cutoff. The graph includes data points for theoretical values and various decay rates.](image-url)
Real Part Contribution to Decay Rate

![Graph showing real part contribution to decay rate with data points and error bars. The x-axis represents time cutoff (Unit: 1GeV\(^{-1}\)) and the y-axis represents real part of branching ratio. The graph includes experimental results and error bars for different scenarios, such as 24ID, 32ID, and 32IDF with different time cutoff values.](image-url)
Lattice Results

- Absorptive Part
  - Coarse ensemble: calculated amplitude is $\sim 10\%$ larger than theoretical value.
  - Fine ensemble: calculated amplitude and theoretical value agree within error.

- Dispersive Part
  - Coarse ensemble: calculated amplitude and central experimental value (almost) agree within error.
  - Fine ensemble: calculated amplitude is $\sim 11\%$ smaller than central experimental value.

- One possible problem
  - Value of EM current coefficient $Z_V$ might be problematic.
Conclusion

- We developed method for dealing with $\gamma\gamma$ intermediate state and for combining QED part with QCD matrix element.
- We carried out a first-principles calculation of dispersive part of $\pi^0 \rightarrow e^+e^-$ decay amplitude.
- We have obtained preliminary results. The different results from coarse and fine ensemble require more careful examination.