Calculating the two-photon contribution to the real part of $\pi^0 \to e^+e^-$ decay amplitude

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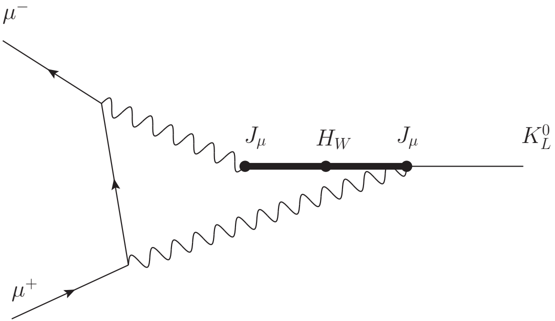
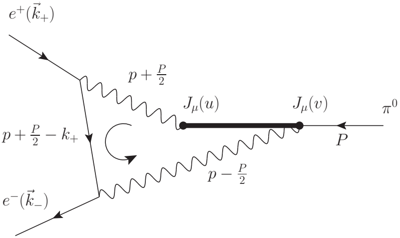
Two-photon contribution to $K_L \to \mu^+\mu^-$



- Intermediate states with lower energy than kaon mass: $\gamma\gamma, |0\rangle, \eta, \pi, \pi\pi(\gamma)$, etc.
- Calculation in Euclidean space would result in exponentially divergent behavior.

$\pi^0 \to e^+e^-$



$$\langle e^+(k_+)e^-(k_-)|\pi^0(P)\rangle = \int d^4u\ d^4v\ H_{\mu\nu}(u,v)L^{\mu\nu}(u,v)$$

$$H_{\mu\nu}(u,v) = \langle 0|J_\mu(u)J_\nu(v)|\pi\rangle$$

$$L^{\mu\nu}(u,v) = \int d^4p\ e^{-ip\cdot(u-v)} \left[\frac{g_{\mu\mu'}}{(p+\frac{P}{2})^2 - i\epsilon}\right] \left[\frac{g_{\nu\nu'}}{(p-\frac{P}{2})^2 - i\epsilon}\right]$$

$$\overline{u}(k_-)\gamma_{\mu'} \left[\frac{\gamma\cdot(p+\frac{P}{2}-k_-)+m_e}{(p+\frac{P}{2}-k_-)^2+m_e^2-i\epsilon}\right] \gamma_{\nu'} v(k_+)$$

$\pi^0 \to e^+e^-$

Let w be the relative coordinate of two EM currents i.e. $w = u - v$. Decay amplitude:

$$\begin{aligned}
\mathcal{A} \;=\; & \int d^4w \, \langle 0 | T\{J_\mu(\frac{w}{2}) J_\nu(-\frac{w}{2})\} | \pi^0 \rangle \qquad (1) \\
& \int d^4p \; e^{-ip\cdot w} \left[\frac{g_{\mu\mu'}}{(p+\frac{P}{2})^2 + m_\gamma^2 - i\epsilon} \right] \left[\frac{g_{\nu\nu'}}{(p-\frac{P}{2})^2 + m_\gamma^2 - i\epsilon} \right] \\
& \overline{u}(k_-)\gamma_{\mu'} \left[\frac{\gamma \cdot (p + \frac{P}{2} - k_-) + m_e}{(p+\frac{P}{2} - k_-)^2 + m_e^2 - i\epsilon} \right] \gamma_{\nu'} v(k_+)
\end{aligned}$$

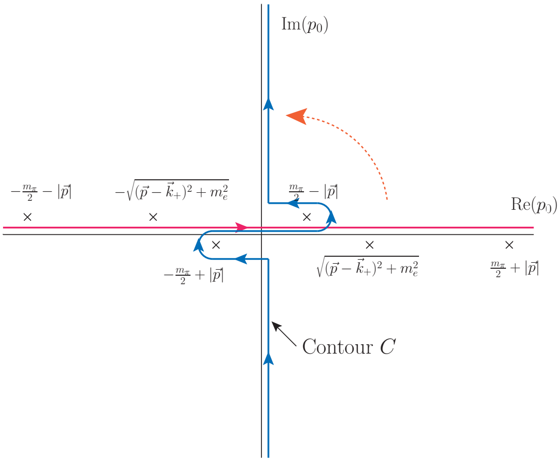
Here the matrix elment is a Minkowski-space quantity.
To calculate it using Lattice QCD, it is necessary to do a Wick rotation $w^0 \to iw^0$.

Wick Rotation

$$L_{\mu\nu}(w) \;=\; \int d^4p \; e^{-ip\cdot w} \left[\frac{g_{\mu\mu'}}{(p+\frac{P}{2})^2 - i\epsilon}\right]\left[\frac{g_{\nu\nu'}}{(p-\frac{P}{2})^2 - i\epsilon}\right]$$
$$\overline{u}(k_-)\gamma_{\mu'}\left[\frac{\gamma\cdot(p+\frac{P}{2}-k_-)+m_e}{(p+\frac{P}{2}-k_-)^2+m_e^2-i\epsilon}\right]\gamma_{\nu'}v(k_+)$$

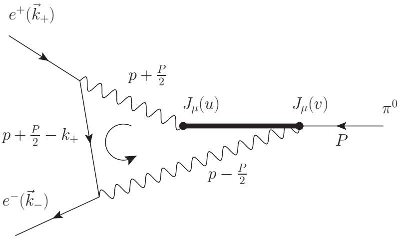
- After w^0 is Wick-rotated, the exponential in $L_{\mu\nu}(w)$ introduces exponential growth.
- The p^0 contour also needs to be rotated.
- Because of the presence of intermediate states with lower energy than the mass of pion, a naive Wick rotation $p^0 \to ip_E^0$ will make the integral blow up.

Wick Rotation



- When $|\vec{p}| < \frac{m_\pi}{2}$, the p^0 contour needs to be deformed to circumvent the two poles that crosses the imaginary axis.

Wick Rotation

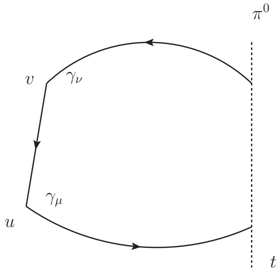


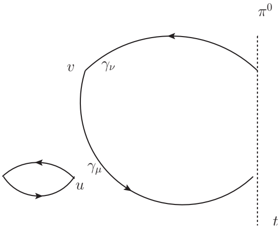
- $L_{\mu\nu} \propto e^{M_\pi \frac{|w^0|}{2}}$, where $w^0 = u^0 - v^0$
- The lightest intermediate state between J_μ and J_ν is $\pi\pi$ state $\rightarrow H_{\mu\nu} = \langle 0|J_\mu(\frac{w}{2})J_\nu(-\frac{w}{2})|\pi\rangle \propto e^{-(E_n - \frac{M_\pi}{2})|w^0|}$, $E_n > 2M_\pi$
- Amplitude converges exponentially at least as fast as $e^{-M_\pi|w^0|}$. We can safely perform the Euclidean space calculation.

Hadronic part

$$\langle 0|TJ_\mu(0)J_\nu(x)|\pi\rangle = \lim_{t\to-\infty} \frac{2m_\pi}{N_\pi} Z_V^2 e^{m_\pi|t|}\langle 0|TJ_\mu(0)J_\nu(x)\pi(t)|0\rangle$$

where N_π is the pion ground state amplitude $N_\pi = \langle\pi|\pi(0)|0\rangle$, Z_V is the coefficient of EM current that connects the local non-conserved current with global conserved current.





- Contribution from the second graph is small (suppressed by SU(3) flavor symmetry).

Summary

- Leptonic part integral is calculated numerically with CUBA library.
- Hadronic part can be extracted from three point function.

$$\langle 0|TJ_\mu(0)J_\nu(x)|\pi\rangle = \lim_{t\to-\infty}\frac{2m_\pi}{N_\pi}Z_V^2 e^{m_\pi|t|}\langle 0|TJ_\mu(0)J_\nu(x)\pi(t)|0\rangle$$

KTeV Experimental result

- Branching ratio $B(\pi^0 \to e^+e^-) = (6.87 \pm 0.36) \times 10^{-8}$
- Absorptive (imaginary) part of branching ratio is analytically known (Optical theorem)
  $B_{imag}(\pi^0 \to e^+e^-) = 4.69 \times 10^{-8}$
- Dispersive (real) part of branching ratio (Total branching ratio - absorptive part)
  $B_{real}(\pi^0 \to e^+e^-) = (2.18 \pm 0.36) \times 10^{-8}$

What we have calculated:

- Absorptive part of branching ratio.
  It can be compared with analytic result to help us estimate error in lattice calculation.
- Dispersive part of branching ratio. (Our real goal)

Ensembles

Gauge action: Iwasaki + DSDR

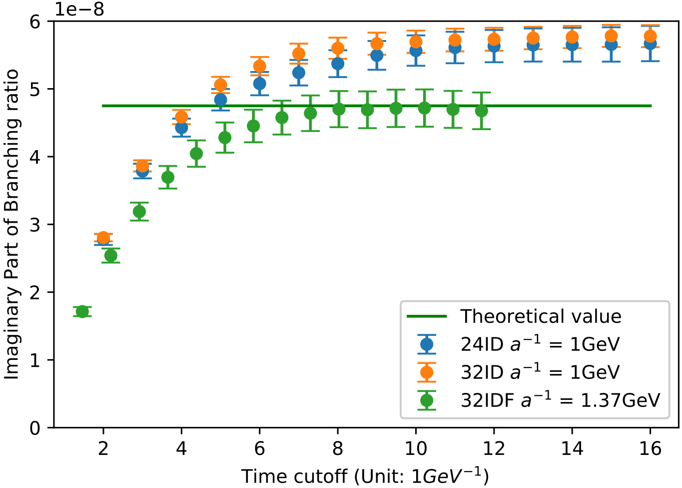
1. 24^3 **Coarse ensemble**
Lattice volume: $24^3 \times 64$
$a^{-1} = 1 GeV$
$am_\pi = 0.13975(10)$
Number of trajectories: 35

2. 32^3 **Coarse ensemble**
Lattice volume: $32^3 \times 64$
$a^{-1} = 1 GeV$
$am_\pi = 0.139474(96)$
Number of trajectories: 40

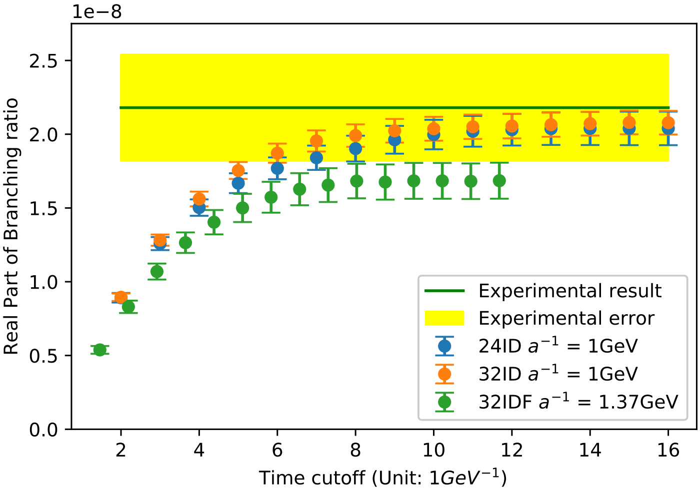
3. 32^3 **Fine ensemble**
Lattice volume: $32^3 \times 64$
$a^{-1} = 1.37 GeV$
$am_\pi = 0.10468(32)$
Number of trajectories: 17

- Ensemble 1 & 3 have almost the same physical spatial volume.
- Compare 1 & 2 $\rightarrow$ Finite spatial volume error
- Compare 2 & 3 $\rightarrow$ Finite lattice spacing error

Imaginary Part Contribution to Decay Rate



Real Part Contribution to Decay Rate



Lattice Results

- Absorptive Part
  - Coarse ensemble: calculated amplitude is $\sim 10\%$ larger than theoretical value.
  - Fine ensemble: calculated amplitude and theoretical value agree within error.
- Dispersive Part
  - Coarse ensemble: calculated amplitude and central experimental value (almost) agree within error.
  - Fine ensemble: calculated amplitude is $\sim 11\%$ smaller than central experimental value.
- One possible problem
  - Value of EM current coefficient Z_V might be problematic.

Conclusion

- We developed method for dealing with $\gamma\gamma$ intermediate state and for combining QED part with QCD matrix element.
- We carried out a first-principles calculation of dispersive part of $\pi^0 \to e^+e^-$ decay amplitude.
- We have obtained preliminary results. The different results from coarse and fine ensemble require more careful examination.