E&M corrections to $\pi^- \to \mu^- \bar{\nu}$ using infinite volume reconstruction method

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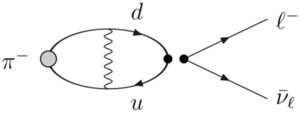
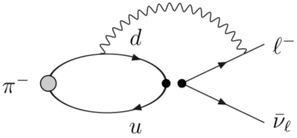
(RBC and UKQCD collaborations)

Lattice 2019 @ Wuhan, 06/19/2019





$m_\gamma = 0 \Rightarrow$ long-range propagator enclosed in the lattice box

$\Rightarrow$ power-law finite-size effects

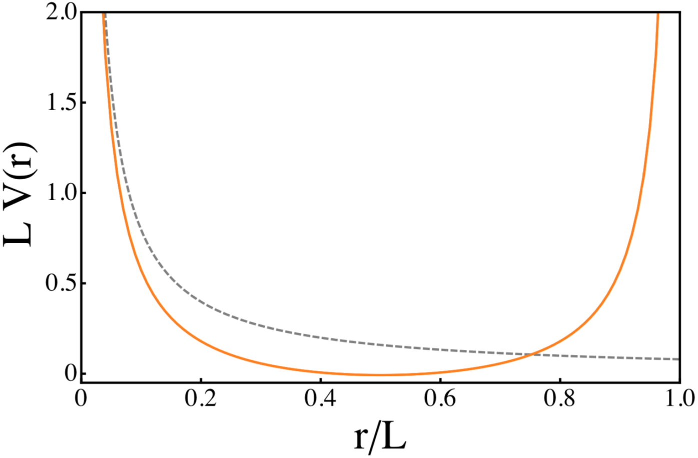
pioneering calculation done by RM123, PRL 2018

See also A. Portelli's talk in this session

Remove zero mode - QED_L

Infinite volume propagator $\Rightarrow$ finite-volume propagator

$$S_\infty(x) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2} = \frac{1}{4\pi^2 x^2} \quad \Rightarrow \quad S_L(x) = \frac{1}{VT}\sum_p{}' \frac{e^{ipx}}{p^2}, \quad p = \frac{2\pi}{L} n \neq 0$$

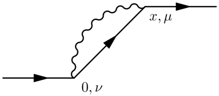


[Davoudi, Savage, PRD90 (2014) 054503]

Infinite volume reconstruction method

XF, Luchang Jin [arXiv:1812.09817]

QED self energy



- We start with infinite volume [QED$_\infty$ method, used in HVP & HLbL]

$$\mathcal{I} = \frac{1}{2} \int d^4x\, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x)$$

- The hadronic part $\mathcal{H}_{\mu,\nu}(x)$ is given by

$$\mathcal{H}_{\mu,\nu}(x) = \mathcal{H}_{\mu,\nu}(t, \vec{x}) = \langle N | T[J_\mu(x) J_\nu(0)] | N \rangle$$

  - $\langle N | J_\mu(t, \vec{x}) \quad \to \quad e^{Mt}$
  - $J_\mu(t, \vec{x}) J_\nu(0) \quad \to \quad e^{-M\sqrt{t^2 + \vec{x}}}$

For small $|t|$, we have exponentially suppressed FV effects:

$$\mathcal{H}_{\mu,\nu}(t, \vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2} - t\right)} \sim e^{-M|\vec{x}|} \sim e^{-ML}$$

For large $|t|$, we shall have:

$$\mathcal{H}_{\mu,\nu}(t, \vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2} - t\right)} \sim e^{-M\frac{\vec{x}^2}{2t}} \sim O(1)$$

Exponentially suppressed FV effects

Realizing at large $t > t_s$ we have ground state dominance:

- Reconstruct $\mathcal{H}_{\mu,\nu}(t,\vec{x})$ at large t using $\mathcal{H}_{\mu,\nu}(t_s,\vec{x})$ at modest t_s

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}') \approx \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(t_s,\vec{x}) \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} e^{-(E_{\vec{p}}-M)(t-t_s)} e^{-i\vec{p}\cdot\vec{x}'}$$

We then split the integral $\mathcal{I}$ into two parts

$$\begin{aligned}
\mathcal{I} &= \mathcal{I}^{(s)} + \mathcal{I}^{(l)} \\
\mathcal{I}^{(s)} &= \frac{1}{2}\int_{-t_s}^{t_s} \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \\
\mathcal{I}^{(l)} &= \int_{t_s}^{\infty} \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \\
&= \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(t_s,\vec{x}) L_{\mu,\nu}(t_s,\vec{x})
\end{aligned}$$

At $t \le t_s$,

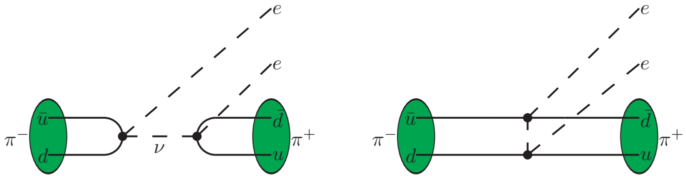
$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \quad \Rightarrow \quad \mathcal{H}^{L}_{\mu,\nu}(t,\vec{x})$$

The replacement only amounts for exponentially suppressed FV effects

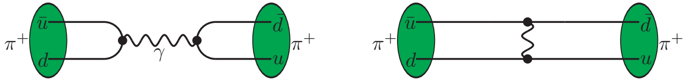
Example of applications

Similarity between $\pi^- \to \pi^+ ee$ and π^+-π^0 mass splitting

- $\pi^- \to \pi^+ ee$:



- $m_{\pi^+} - m_{\pi^0}$:

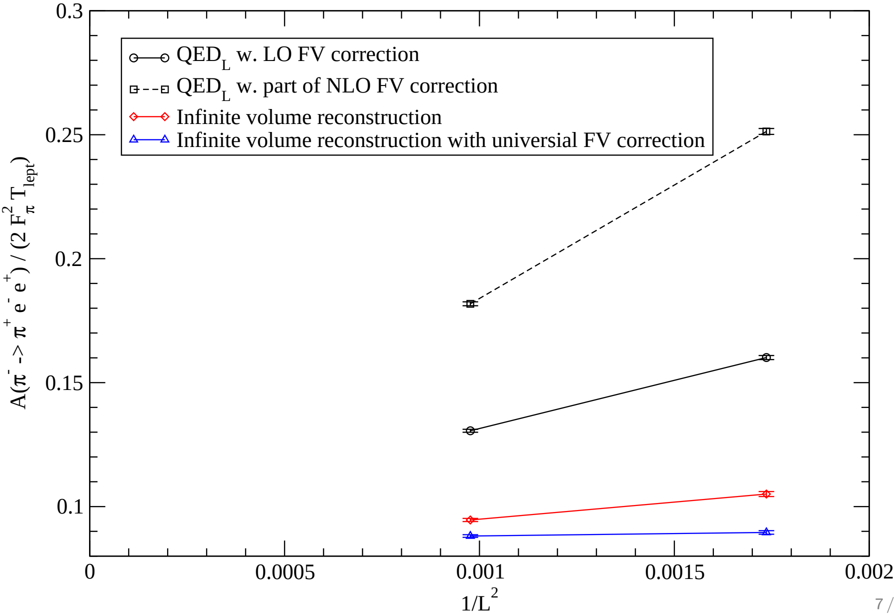


Different feature: $\pi^- \to \pi^+ ee$ invovles also axial vector current

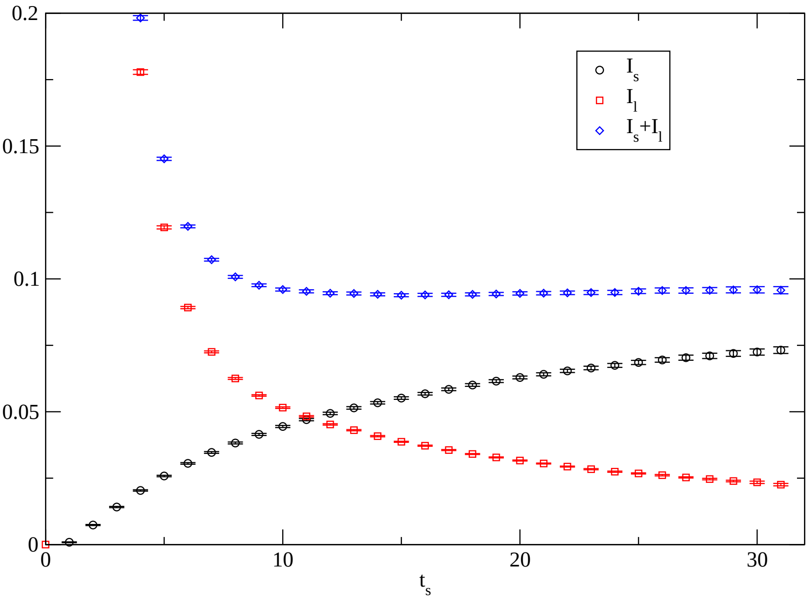
- For vector current, LO & NLO FV corrections are universial
  $\Rightarrow$ described by scalar QED
- For axial vector current, intermediate particle is a scalar state or $\pi\pi$
  $\Rightarrow$ non-universial FV effects

Results for $\pi^- \to \pi^+ ee$

XF, Luchang Jin, Xin-Yu Tuo, poster by X. Tuo



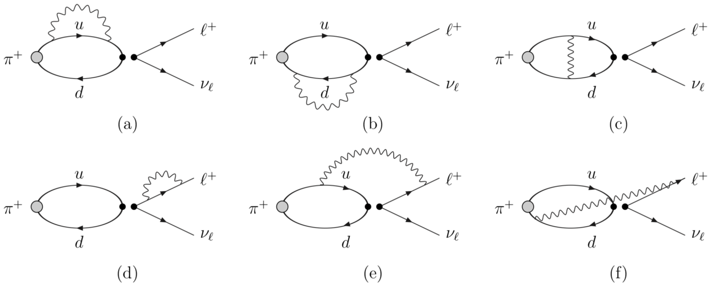
t_s dependence



Move to $\pi \to \mu\nu(\gamma)$

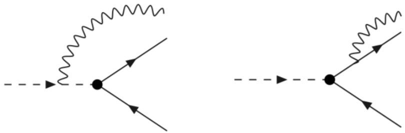
Norman Christ, XF, Luchang Jin, Chris Sachrajda

- 0 photon emission

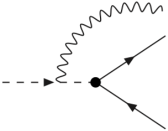


figures from RM123

- 1 photon emission



1 photon emission



- Decay amplitude: split into hadronic and leptonic parts

$$\mathcal{M} = \int d^4x \underbrace{\langle 0| T[J^W_\mu(0) J_\rho(x)] |\phi\rangle}_{H^{(1\gamma)}_{\mu\rho}(x)} e^{-ikx} \epsilon^*_\rho(\lambda, k) \cdot \underbrace{\bar{u}(p)\gamma_\mu(1-\gamma_5)v(-p-k)}_{L_\mu(p,k)}$$

  with photon momentum k and muon momentum p

- Hadronic part

$$\int_0^\infty dt \int d^3\vec{x}\, H^{(1\gamma)}_{\mu\rho}(x) e^{-ikx} + \int_{-\infty}^0 dt \int d^3\vec{x}\, H^{(1\gamma)}_{\mu\rho}(x) e^{-ikx}$$

$$= \quad -\sum_n \frac{\langle 0|J_\rho(0)|n(\vec{k})\rangle\langle n(\vec{k})|J^W_\mu(0)|\phi\rangle}{k - E_n} + \sum_n \underbrace{\frac{\langle 0|J^W_\mu(0)|n(-\vec{k})\rangle\langle n(-\vec{k})|J_\rho(0)|\phi\rangle}{k + E_n - m_\phi}}_{\text{singular when } k \to 0}$$

IR singularity from 1 photon emission

- Similar as self energy, we split the time integral over $(-\infty, 0]$ into two parts

$$\int_{t_s}^{0} dt \int d^3\vec{x}\, H_{\mu\rho}^{(1\gamma)}(x) e^{-ikx} + \int_{-\infty}^{t_s} dt \int d^3\vec{x}\, H_{\mu\rho}^{(1\gamma)}(x) e^{-ikx}$$

- For $t < t_s < 0$, $H_{\mu\rho}^{(1\gamma)}(t, \vec{x})$ can be written as

$$\begin{aligned} \int d^3\vec{x}\, H_{\mu\rho}^{(1\gamma)}(t, \vec{x}) e^{-ikx} &\approx \int d^3\vec{x}\, H_{\mu\rho}^{(1\gamma)}(t_s, \vec{x}) e^{-i\vec{k}\cdot\vec{x}} e^{(E_\phi - m_\phi)(t - t_s)} e^{kt} \\ &\approx \underbrace{\langle 0 | J_\mu^W(0) | n(-\vec{k}) \rangle \langle n(-\vec{k}) | J_\rho(0) | \phi \rangle}_{H_{\mu\rho}^{1\gamma}(\vec{k})} e^{(E_\phi - m_\phi)t} e^{kt} \end{aligned}$$

- IR singularity in decay width

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} \sum_\lambda |\mathcal{M}|^2 \sim \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} \frac{H_{\mu\rho}^{1\gamma}(\vec{k}) H_{\nu\rho}^{1\gamma *}(\vec{k})}{(k + E_\phi - m_\phi)^2} \cdot L_\mu(p,k) L_\nu^*(p,k)$$

- We use $H_{\mu\rho}^{1\gamma}(0 > t \geq t_s, \vec{x})$ and $H_{\mu\rho}^{1\gamma}(t_s, \vec{x})$ as inputs for SD and LD parts

LD contribution from 1 photon emission

LD contribution to the decay width is given by

$$\Gamma^{\mathrm{LD}}_{(1\gamma)} = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2k}\int d^3\vec{x}\,\frac{\langle H^{(1\gamma)}_{\mu\rho}(t_s,\vec{x})\rangle\langle H^{(1\gamma)}_{\nu\rho}(t_s,\vec{0})\rangle^*}{(k+E_\phi-m_\phi)^2} e^{-i\vec{k}\cdot\vec{x}}e^{2kt_s}\cdot L_\mu(p,k)L^*_\nu(p,k)$$

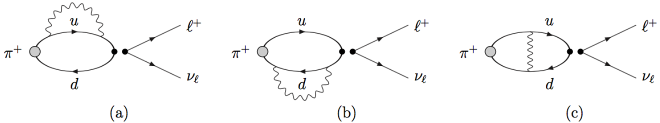
We split the LD contribution into two pieces

$$\Gamma^{\mathrm{LD}}_{(1\gamma)} = \Gamma^{\mathrm{LD,finite}}_{(1\gamma)} + \Gamma^{\mathrm{LD,div}}_{(1\gamma)}$$

$$\Gamma^{\mathrm{LD,finite}}_{(1\gamma)} = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2k}\int d^3\vec{x}\,\frac{\langle H^{(1\gamma)}_{\mu\rho}(t_s,\vec{x})\rangle\langle H^{(1\gamma)}_{\nu\rho}(t_s,\vec{0})\rangle^*}{(k+E_\phi-m_\phi)^2}\left(e^{-i\vec{k}\cdot\vec{x}}-1\right)e^{2kt_s}\cdot L_\mu(p,k)L^*_\nu(p,k)$$

$$\Gamma^{\mathrm{LD,div}}_{(1\gamma)} = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2k}\frac{\langle 0|J^W_\mu|\phi\rangle\langle 0|J^W_\nu|\phi\rangle^*\langle\phi|J_\rho|\phi\rangle\langle\phi|J_\rho|\phi\rangle^*}{(k+E_\phi-m_\phi)^2} e^{2kt_s}\cdot L_\mu(p,k)L^*_\nu(p,k)$$

0 photon emission



- Correlation function

$$\frac{1}{2}\int d^4x \int d^4y \, \langle 0|J_\nu^W(0)J_\rho(x)J_\sigma(y)\phi(-t)|0\rangle S_{\rho\sigma}^\gamma(x,y)$$

- The IR divergence appears in

$$\frac{1}{2}\int dt_x \int dt_y \int d^3\vec{x} \, \langle 0|J_\mu^W(0)|\phi\rangle \underbrace{\langle \phi|J_\rho(t_x,\vec{x})J_\sigma(t_y,0)|\phi\rangle}_{\sim e^{(E_\phi - m_\phi)(t_x-t_y)}} S_{\rho\sigma}^\gamma(t_x,\vec{x};t_y,\vec{0})$$

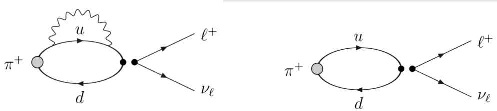
  - Integral over t_x and t_y yields $\frac{1}{(E_\phi+k-m_\phi)^2}$
  - Photon propagator produces another factor of $\frac{1}{2E_\gamma} = \frac{1}{2k}$

- Define a hadronic matrix element as

$$H_{\rho\sigma}^{(0\gamma)}(t,\vec{x}) = \langle \phi|J_\rho(t,\vec{x})J_\sigma(0,\vec{0})|\phi\rangle$$

For $|t| > t_s$, $H_{\rho\sigma}^{(0\gamma)}(t,\vec{x})$ can be written in terms of $H_{\rho\sigma}^{(0\gamma)}(t_s,\vec{x})$

Long-distance contribution to decay width



$\langle 0|J^W_\mu|\phi\rangle H^{0\gamma}_{\rho\sigma}(t,\vec{x}) S^{\gamma}_{\rho\sigma}(t,\vec{x})$ $\quad$ $\langle 0|J^W_\nu|\phi\rangle$

Long-distance contribution to decay width is

$$\Gamma^{\rm LD}_{(0\gamma)} = -\int \frac{d^3\vec{k}}{(2\pi)^3}\int d^3\vec{x}\, \frac{\langle 0|J^W_\mu|\phi\rangle\langle 0|J^W_\nu|\phi\rangle^* H^{0\gamma}_{\rho\rho}(t_s,\vec{x})}{(E_\phi+k-m_\phi)^2} e^{-i\vec{k}\cdot\vec{x}} e^{-(m_\phi-E_\phi)t_s}\frac{1}{2k}L_\mu(p,0)L^*_\nu(p,0)$$

We split the contribution into two pieces

$$\Gamma^{\rm LD,finite}_{(0\gamma)} = -\int \frac{d^3\vec{k}}{(2\pi)^3}\int d^3\vec{x}\, \frac{\langle 0|J^W_\mu|\phi\rangle\langle 0|J^W_\nu|\phi\rangle^* H^{0\gamma}_{\rho\rho}(t_s,\vec{x})}{(E_\phi+k-m_\phi)^2} \left(e^{-i\vec{k}\cdot\vec{x}}-1\right) e^{-(m_\phi-E_\phi)t_s}\frac{1}{2k}L_\mu(p,0)L^*_\nu(p,0)$$

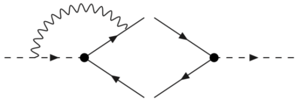
$$\Gamma^{\rm LD,div}_{(0\gamma)} = -\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\langle 0|J^W_\mu|\phi\rangle\langle 0|J^W_\nu|\phi\rangle^* \langle\phi|J_\rho|\phi\rangle\langle\phi|J_\rho|\phi\rangle^*}{(E_\phi+k-m_\phi)^2} e^{-(m_\phi-E_\phi)t_s}\frac{1}{2k}L_\mu(p,0)L^*_\nu(p,0)$$

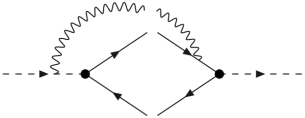
Combine 0γ and 1γ contributions

Cancellation between the divergent parts yields

$$\begin{aligned}
& \Gamma^{\mathrm{LD,div}}_{(1\gamma)} + \Gamma^{\mathrm{LD,div}}_{(0\gamma)} \\
= \;& \langle 0|J^W_\mu|\phi\rangle \langle 0|J^W_\nu|\phi\rangle^* \langle \phi|J_\rho|\phi\rangle \langle \phi|J_\rho|\phi\rangle^* \\
\times \;& \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2k} \frac{e^{2kt_s} L_\mu(p,k) L^*_\nu(p,k) - e^{-(m_\phi - E_\phi)t_s} L_\mu(p,0) L^*_\nu(p,0)}{(k + E_\phi - m_\phi)^2}
\end{aligned}$$

Similar cancellation can happens for example in





Conclusion

- It is appealing to have a method to include photon with exponentially suppressed FV effects
- In this talk, part of diagrams for $\pi^- \to \mu^- \bar{\nu}$ are used to show how the infinite volume reconstruction method works
- IR divergence cancellation is realized with a straightforward way
- Future work: perform the realistic lattice QCD calculation of E&M corrections in $\pi^- \to \mu^- \nu$