Two-photon decay of the neutral pion from a coordinate-space method

Norman H. Christ[1], Xu Feng[2], Luchang Jin[3], Cheng Tu[3], Yidi Zhao[1]

[1]Columbia University [2]Peking University [3]University of Connecticut

Introduction

A lot of interesting decays with non-QCD final states

- photonic decays: $\pi^0 \to \gamma\gamma$, $\eta_c \to \gamma\gamma$, $\chi_{c0} \to \gamma\gamma$, $J/\Psi \to \gamma\gamma\gamma$
- leptonic decays: $\pi^0 \to e^+e^-$, $K_L \to \mu^+\mu^-$ [N. Christ and Y. Zhao's talks on Tuesday]
- radiative leptonic decays: $\pi^0 \to \gamma e^+e^-$, $B \to \gamma\mu^+\mu^-$ [S. Meinel's talk on Tuesday]

Conventional method

- Study momenta dependence of the form factor, e.g. $F_{\pi\gamma\gamma}(m_\pi^2, p_1^2, p_2^2)$

Can the calculation be simpler?

We propose to calculate the on-shell amplitude in coordinate space

$$A = \int d^4x \underbrace{\omega(x)}_{\text{Non-QCD}} \underbrace{H(x)}_{\text{Hadronic}}$$

Methodology

Take $\pi^0 \to \gamma\gamma$ as an example

- Step 1 - Calculate hadronic matrix element in coordinate space

$$\mathcal{H}_{\mu\nu}(x) = \langle 0|T[J_\mu(x)J_\nu(0)]|\pi^0(q)\rangle$$

- Step 2 - Choose on-shell momentum

$$\mathcal{F}_{\mu\nu}(q,p,p') = \int d^4x\, e^{-ipx}\mathcal{H}_{\mu\nu}(x)$$

with

$$p = (im_\pi/2, \vec{p}), \quad p' = (im_\pi/2, -\vec{p}), \quad q = (im_\pi, \vec{0}), \quad |\vec{p}| = m_\pi/2.$$

We have

$$\mathcal{F}_{\mu\nu}(q,p,p') = \varepsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta F_{\pi\gamma\gamma}(m_\pi^2, 0, 0)$$

- Step 3 - Obtain a Lorentz scalar amplitude

$$\begin{aligned}\mathcal{I} &= \varepsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta \int d^4x\, e^{-ipx}\mathcal{H}_{\mu\nu}(x) \\ &= \varepsilon_{\mu\nu\alpha\beta}q_\beta \int d^4x\, e^{-ipx}\left(-i\frac{\partial}{\partial x_\alpha}\right)\mathcal{H}_{\mu\nu}(x) \\ &= m_\pi \int d^4x\, e^{-ipx}\varepsilon_{\mu\nu\alpha 0}\frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha} \\ &= m_\pi \int dt\, e^{m_\pi t/2}\int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\varepsilon_{\mu\nu\alpha 0}\frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha}\end{aligned}$$

- Step 4 - Average over the spatial direction for $\vec{p}$

$$\begin{aligned}\mathcal{I} &= 2\int dt\, e^{m_\pi t/2}\int d^3\vec{x}\frac{\sin(m_\pi|\vec{x}|/2)}{|\vec{x}|}\varepsilon_{\mu\nu\alpha 0}\frac{\partial \mathcal{H}_{\mu\nu}(x)}{\partial x_\alpha} \\ &= \int dt\, e^{m_\pi t/2}\int d^3\vec{x}\frac{-m_\pi|\vec{x}|\cos(m_\pi|\vec{x}|/2) + 2\sin(m_\pi|\vec{x}|/2)}{|\vec{x}|^3}\varepsilon_{\mu\nu\alpha 0}x_\alpha \mathcal{H}_{\mu\nu}(x)\end{aligned}$$

- Step 5 - Master formula

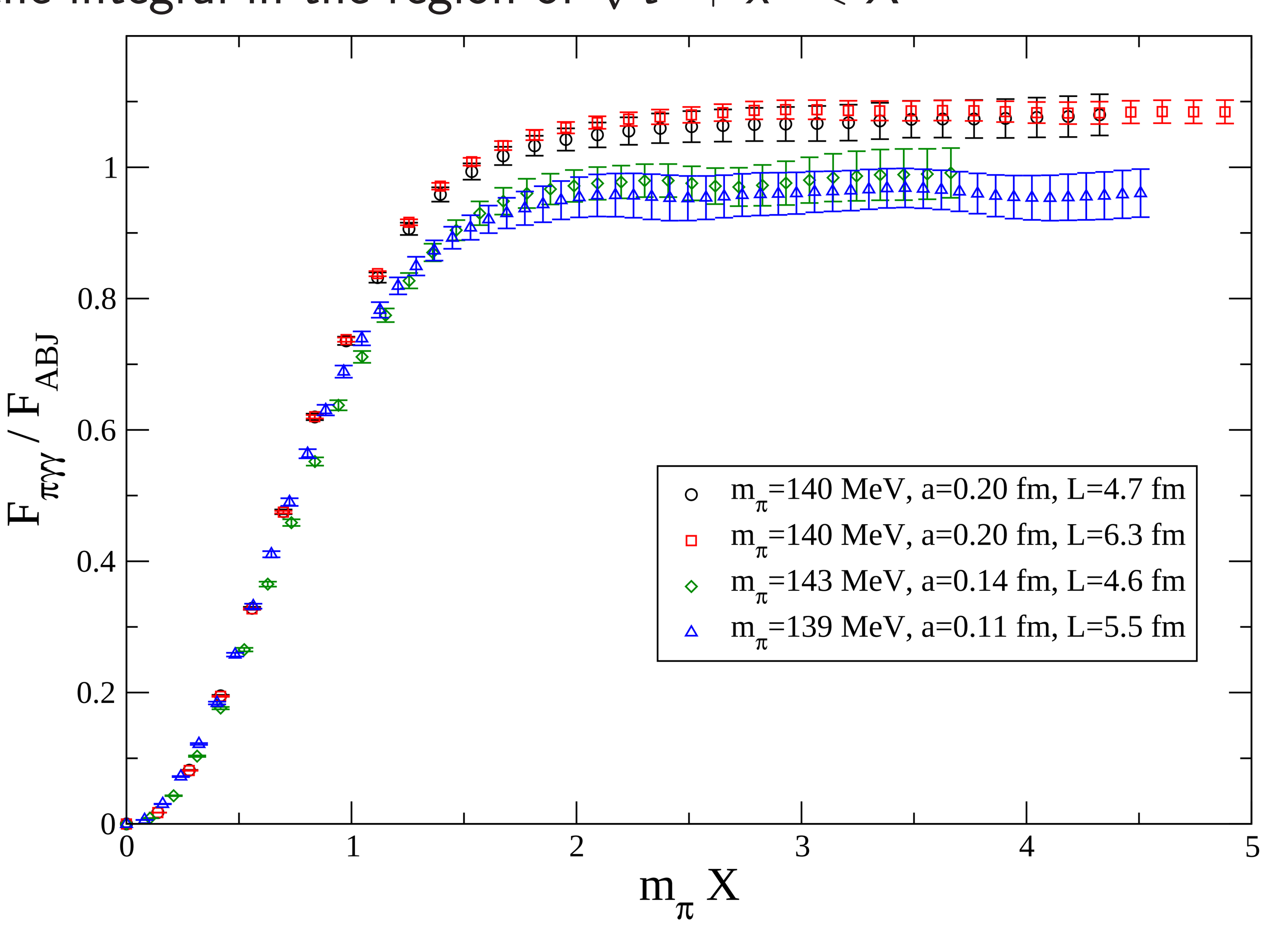
$$F_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0) = \frac{\mathcal{I}}{[\varepsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta][\varepsilon_{\mu\nu\rho\sigma}p_\rho q_\sigma]} = \frac{2}{m_\pi^4}\mathcal{I}$$

Key quantity required from lattice QCD is $H(x) = \varepsilon_{\mu\nu\alpha 0}x_\alpha \mathcal{H}_{\mu\nu}(x)$

Remark: Finite-volume effects are exponentially suppressed in $H(x)$

Results for $\pi^0 \to \gamma\gamma$

Perform the integral in the region of $\sqrt{t^2 + \vec{x}^2} < X$



(Ensemble $m_\pi = 139$ MeV uses Iwasaki action while the others use Iwasaki + DSDR action.)

Results for $\pi^0 \to \gamma e^+e^-$

Branching ratio given by Kroll-Wada formula

$$\frac{\Gamma_{\pi\to\gamma e^+e^-}}{\Gamma_{\pi\to\gamma\gamma}} = \frac{\alpha}{3\pi}\int_r^1 \frac{d\rho}{\rho}(1-\rho)^3\left(1-\frac{r}{\rho}\right)^{\frac{1}{2}}\left(2+\frac{r}{\rho}\right)\frac{F^2_{\pi^0\gamma\gamma}(m_\pi^2, s, 0)}{F^2_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0)}$$

where $r = 4m_e^2/m_\pi^2$, $\rho = s/m_\pi^2$.

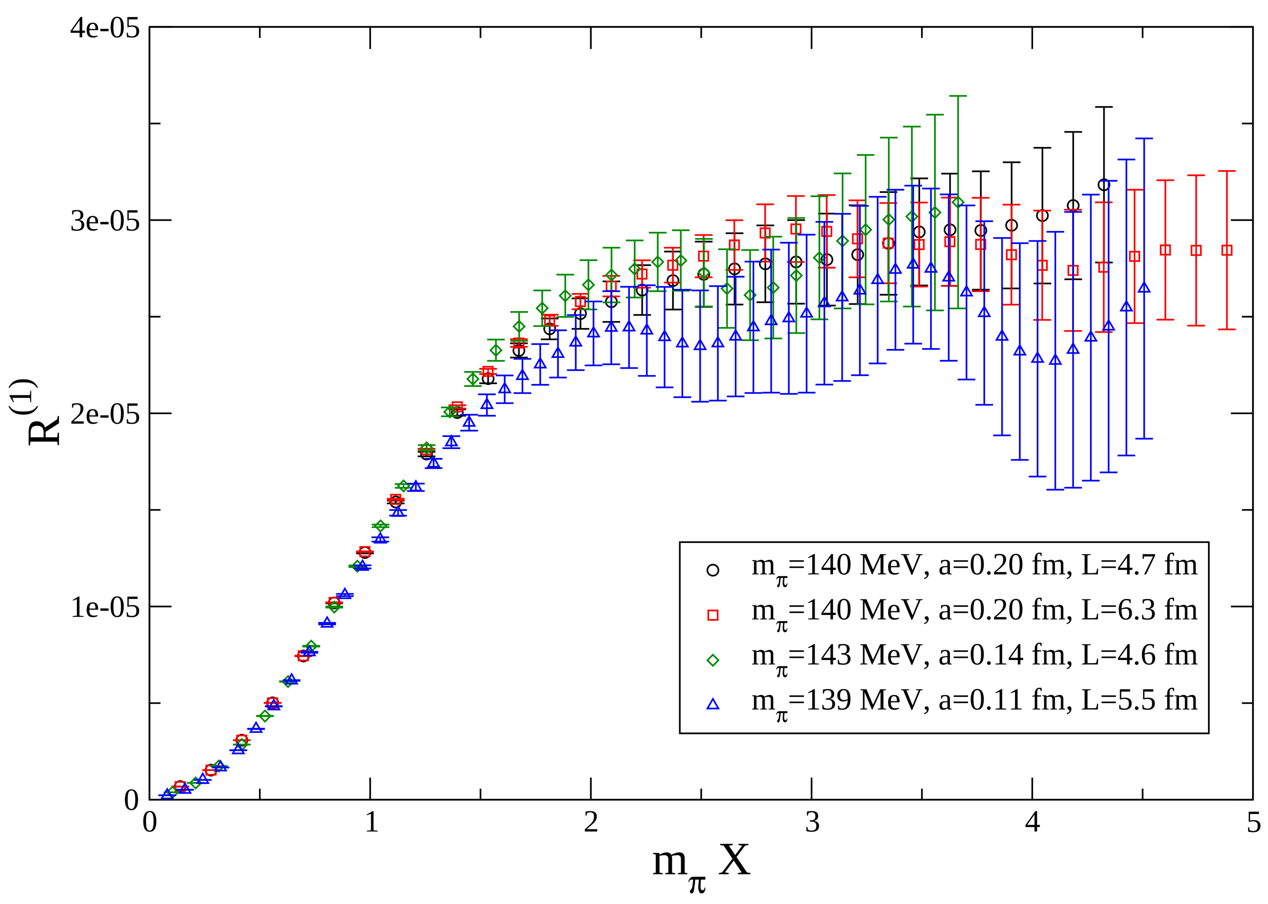
$$\frac{F^2_{\pi^0\gamma\gamma}(m_\pi^2, s, 0)}{F^2_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0)} = 1 + 2\left(\frac{F(\rho,0)}{F(0,0)} - 1\right) + \left(\frac{F(\rho,0)}{F(0,0)} - 1\right)^2$$

Correspondingly, the branching ratio can be written as

$$\frac{\Gamma_{\pi\to\gamma e^+e^-}}{\Gamma_{\pi\to\gamma\gamma}} = R^{(0)} + R^{(1)} + R^{(2)}$$

where $R^{(0)} = 0.01185$ is irrelevant for QCD correction.

$$R^{(1)} = \int d^4x\, \omega^{(1)}(t, |\vec{x}|)\varepsilon_{\mu\nu\alpha 0}x_\alpha \mathcal{H}_{\mu\nu}(x)$$

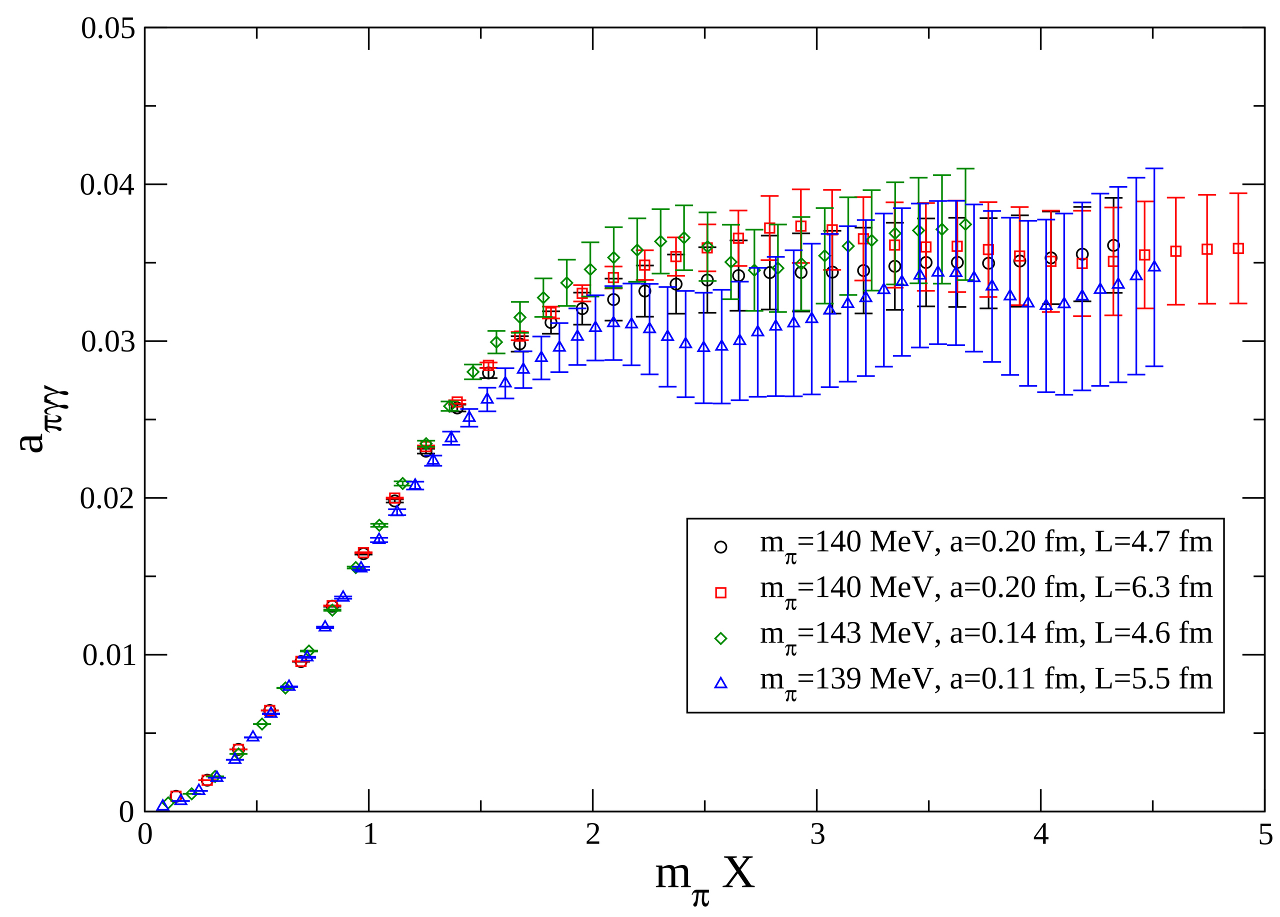


Results for form factor slope

Taylor expansion of form factor $F(\rho, 0)$ at small ρ is

$$\frac{F(\rho,0)}{F(0,0)} = 1 + a_{\pi\gamma\gamma}\rho + \cdots$$

$$a_{\pi\gamma\gamma} = \int d^4x\, \omega^{(a)}(t, |\vec{x}|)\varepsilon_{\mu\nu\alpha 0}x_\alpha \mathcal{H}_{\mu\nu}(x)$$



Summary