Lattice study on the twisted $\mathbb{CP}^{N-1}$ model on $\mathbb{R} \times S^1$

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\textbf{CP}^{N-1} \text{ sigma model}

\textit{2D CP}^{N-1} \text{ model is not only a toy model of QCD, but also effectively describes gauge theory!}

- Effective theory on vortex in U(N) + Higgs model is CP$^{N-1}$ Eto, et.al.(05)
- Effective theory on long strings in YM is CP$^{N-1}$ Aharony, Komargodski(13)
- It is also notable that CP$^{1}$ describes spin chain systems Haldane(83)

Lattice study on CP$^{N-1}$ model is of physical significance

\textbf{Lagrangian of CP}^{N-1} \text{ models}

\[
S = \frac{1}{2g^2} \int |D\phi|^2 \quad |\phi|^2 = 1, \quad D\phi = (d + ia)\phi, \quad a = i\bar{\phi} \cdot d\phi
\]

\text{discretized on the lattice}

\[
S = -N\beta \sum_{n,\mu} \left( \bar{z}_{n+\mu} \cdot z_n \lambda_{n,\mu} + \bar{z}_n \cdot z_{n+\mu} \bar{\lambda}_{n,\mu} - 2 \right)
\]
**CP^{N-1} sigma model on R x S^1**

- Global symmetry : PSU(N) flavor symmetry + Time reversal
- $Z_N$ symmetry is not exact for periodic b. c. (cf. QCD)

- $Z_N$-twisted b. c.
  \[ \phi(x_1, x_2 + L) = \Omega \phi(x_1, x_2) \quad \Omega = \text{diag.} \left[ 1, e^{2\pi i/N}, e^{4\pi i/N}, \ldots, e^{2(N-1)\pi i/N} \right] \]

**Exact $Z_N$-symmetric model**
- $Z_N$ flavor shift + $Z_N$ center
  (irrelevant with decompactifying)

- Fractional instantons (Q=1/N, S=S_i/N)
  - BPS eq. \[ D\phi \pm i \star D\phi = 0 \]
  - BPS sol. \[ \phi = \frac{(1, e^{z-z_0}, \ldots)}{\sqrt{1 + |e^{z-z_0}|^2}} \]
  - cf. $N=2$

\*It is shown to have resurgent structure (pert. vs non-pert. relation)

- Lee, Yi(97) Kraan, van Baal(97) Eto, et.al. (04~) Bruckmann, et.al. (05~)

- Distribution of P-loop for N=5

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**Notes:**
- For a fixed topological charge, the minimal action is given by the (anti-)BPS derivative, and
- The model is defined by $D_124$, component complex fields with $D_125$.
- The fractional instantons ($Q=1/N, S=S_i$) correspond to the vacuum with the gauge symmetry breaking $\omega_i$.
- The Wilson-loop holonomy in the compactified direction is given by $\frac{\sqrt{N}}{N}$ after $I=0$(a)
- The action and charge with the factor $\frac{1}{N}$ after $I=0$(b)
- $Z_N$ is symmetric model and reduced sine-Gordon model.
- The first and second homotopy groups for instantons in the sine-Gordon model and reduced sine-Gordon model.
- The configuration in $I=0$ (c) $Z_N$ twisted boundary conditions in a compactified space.
- The configuration in $I=0$ (d) $Z_N$ twisted boundary conditions.
- The configuration in $I=0$ (e) $Z_N$ twisted boundary conditions.
- Exact $Z_N$-symmetric model $Z_N$ flavor shift + $Z_N$ center (irrelevant with decompactifying)
- Fractional instantons ($Q=1/N, S=S_i/N$)
- BPS eq. $D\phi \pm i \star D\phi = 0$
- BPS sol. $\phi = \frac{(1, e^{z-z_0}, \ldots)}{\sqrt{1 + |e^{z-z_0}|^2}}$
- cf. $N=2$
- It is shown to have resurgent structure (pert. vs non-pert. relation)
Resurgent structure in QM and QFT

\[ S_+ \Phi_0(z) - S_- \Phi_0(z) \approx 5 e^{-Az} S \Phi_1(z) \]

Perturbative imaginary ambiguity  Non-perturbative effect

Resurgent structure is expected to be in quantum theory, thus perturbative series could include nonpert. information!

Zinn-Justin(01), Marino(07) Marino, Schiappa, Weiss(09), Argyres,Unsal(12), Dunne, Unsal(12)

In a certain class of QFT as twisted CP^{N-1} models QFT can be defined based on the structure.
Main questions

Question 1: $\mathbb{Z}_N$ (phase) transition for pbc
- 2nd-order phase transition expected in large-$N$
- it should be crossover for finite $N$ since $\mathbb{Z}_N$

$\langle P \rangle \sim 0$ for small $\beta$ $\Rightarrow$ $\langle P \rangle \neq 0$ for large $\beta$

We will check it directly in numerical study

Question 2: Continuity and fractional instantons for $\mathbb{Z}_N$-tbc
- Fractional instantons yield transition between classical $N$-vacua
- makes $\mathbb{Z}_N$ stable, leading to volume indep. of vacuum structure

$\langle P \rangle \sim 0$ for small $\beta$ $\Rightarrow$ still $\langle P \rangle \sim 0$ for large $\beta$

We will show quite suggestive results on fractional instantons and adiabatic continuity
Setup of lattice simulation

cf.) Berg,Luscher(81), Campostrini,et.al.(92), Alles,et.al.(00), Flynn,et.al.(15), Abe,et.al.(18)

• Lattice formulation
  \[ S = -N\beta \sum_{n,\mu} \left( \bar{z}_{n+\mu} \cdot z_n \lambda_{n,\mu} + \bar{z}_n \cdot z_{n+\mu} \bar{\lambda}_{n,\mu} - 2 \right) \]

Vector field \( \Phi \) is introduced:
\[ \phi_{2j} = \Re[z_{n,j}], \quad \phi_{2j+1} = \Im[z_{n,j}], \quad j = 0, \ldots, N - 1 \]
\[ \phi^R_\mu = \Re[\lambda_\mu], \quad \phi^I_\mu = \Im[\lambda_\mu], \]

\[ s_\phi = -N\beta \phi \cdot F_\phi = -N\beta |F_\phi| \cos \theta \quad \text{updated just by updating } \theta \]

Over heat-bath algorithm is adopted to update this \( \theta \)

• Parameters and quantities
  \( N_x = 40-400, \quad N_\tau = 8,12, \quad \beta = 0.1-4.0, \quad N = 3-20, \quad N_{\text{sweep}} = 200000,400000 \)
  
  • Expectation values of Polyakov loop and its susceptibility
  
  • Thermal entropy \( s = \beta(N\tau)^2(<T_{xx}> - <T_{\tau\tau}>) \)

(1)\( Z_N \) transition(pbc)  (2)\( Z_N \) continuity(tbc)  (3)Thermal entropy
Polyakov-loop of $\mathbb{C}P^{N-1}$ models on $\mathbb{R} \times S^1$ with pbc.

$N=3,5,10,20$  \hspace{1cm} (Nx,Nt) = (200, 8)  \hspace{1cm} N\text{sweep} = 200,000$
Low-β : around the origin
→ approximate $Z_N$ symmetry

High-β : moves to one of $Z_N$ vacua
→ $Z_N$ breaking transition

Note that $Z_N$ symmetry is not exact for PBC
VEV of Polyakov loop $|<P>|$

- $|<P>| \sim 0$ at low $\beta$, then $|<P>|$ undergoes crossover-like transition
- Peak of Polyakov-loop susceptibility $\chi$ gets sharper with $N$

Crossover transition for finite $N$ is checked, which would be 2nd-order phase transition for large $N$ limit
Volume dependence of $\chi$-peak

$\chi\langle |P| \rangle$

$\chi_{\text{max}} = c + aV^p$

$p=1 : 1\text{st}, \quad 0<p<1: 2\text{nd or crossover}$

• Volume dependence of the peak is not linear $\rightarrow$ not 1st-order
• $\chi$ for $N=20$ is larger than that for $N=10$ $\rightarrow$ 2nd-order in large $N$? it supports crossover transition for finite $N$ (2nd-order in large-$N$)

Fukugita, et al. (90)
Polyakov loop of $\mathbb{CP}^{N-1}$ models on $\mathbb{R} \times S^1$ with $\mathbb{Z}_N$ tbc.

$N=3,5,10,20$ $(N_x,N_t) = (200, 8), (400, 12)$ $N_{\text{sweep}}=200000, 400000$
Low-β: around the origin → $Z_N$ symmetry at the action level

Intermediate-β: Transition between $N$ vacua → quantum $Z_N$ symmetry
Distribution plot of P-loop

\[ \text{Im}[P] \quad N=3, \beta=0.1 \]

\[ \begin{align*}
|\langle P \rangle| & \sim 0 \\
\end{align*} \]

\[ \begin{align*}
\text{Re}[P] \\
\end{align*} \]

\[ \begin{align*}
\text{Low-}\beta : \text{around the origin} \rightarrow \\
\text{Z}_N \text{ symmetry at the action level} \\
\end{align*} \]

\[ \begin{align*}
\text{N}=3, \beta=1.8 \\
|\langle P \rangle| \neq 0 \\
\end{align*} \]

\[ \begin{align*}
\text{High-}\beta : \text{One of Z}_N \text{ vacua selected} \rightarrow \text{SSB of Z}_N \text{ symmetry}....? \\
\end{align*} \]
Low-$\beta$: around the origin $\rightarrow$ $Z_N$ symmetry at the action level

High-$\beta$: One of $Z_N$ vacua selected $\rightarrow$ SSB of $Z_N$ symmetry...?
Distribution plot of P-loop

|<P>| $\sim$ 0

Low-\(\beta\) : around the origin $\rightarrow$ Z\(_N\) symmetry at the action level

|<P>| $\neq$ 0

High-\(\beta\) : One of Z\(_N\) vacua selected $\rightarrow$ SSB of Z\(_N\) symmetry….?
VEV of Polyakov loop $|\langle P \rangle|$

- Low $\beta \rightarrow |\langle P \rangle| = 0$: distribution around origin
- Mid $\beta \rightarrow |\langle P \rangle|$ highly fluctuates: distribution forms polygons
- High $\beta \rightarrow$ suddenly gets $|\langle P \rangle| \neq 0$: but more stat. can form polygon

This peculiar P-loop could imply something special ($Z_N$ stability?). We still need larger volume or more statistics to judge continuity.
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- **Low** $\beta \rightarrow |\langle P \rangle| = 0$: distribution around origin
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VEV of Polyakov loop $|\langle P \rangle|$  

$N=10$  

$N=20$  

- Low $\beta \rightarrow |\langle P \rangle|=0$: distribution around origin  
- Mid $\beta \rightarrow |\langle P \rangle|$ highly fluctuates: distribution forms polygons  
- High $\beta \rightarrow$ suddenly gets $|\langle P \rangle|\neq 0$: but more stat. can form polygon  

This peculiar P-loop could imply something special ($Z_N$ stability?). We still need larger volume or more statistics to judge continuity.
Polygon-shaped distributions of Polyakov loop ($|\langle P \rangle| \sim 0$) appear more often with more statistics.

It may indicate $Z_N$ stability (continuity)....

Furthermore,
**Distribution plot of P-loop** (very high $\beta$, large volume)

Independent configurations for very high $\beta$ ($\beta=4.0$) with large volume include a quantum $Z_N$ symmetric case as below!

**Im[$P$] $N=3$, $\beta=4.0$, (400\times12)**

|\langle P\rangle| \sim small

**Re[$P$]**

**Hysteresis of arg[$P$]**

Any of $Z_N$ vacua is not selected

Very high-$\beta$ : quantum $Z_N$ symmetric case found with certain probability

it seems we need larger volume or more statistics for $Z_N$ continuity....
Fractional instantons

Pick up two of configurations and look into the $x$-dependence of $\text{arg}[P]$

$\frac{1}{3}$ fractional antiinstanton $+ \frac{1}{3}$ fractional instanton $= \text{bion}$

$3 \times \frac{1}{3}$ fractional instantons $= \text{instanton}$

implies fractional instantons cause transition between classical vacua at high $\beta$, which lead to quantum $Z_N$ symmetry and could yield adiabatic continuity
**Fractional instantons**

Pick up two of configurations and look into the x-dependence of arg[P]

\[
\text{arg}[P]
\]

1/3 fractional antiinstanton + 1/3 fractional instanton = bion

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Fractional instantons

Pick up two of configurations and look into the x-dependence of \( \arg[P] \)

\[
\frac{1}{3} \text{ fractional antiinstanton} + \frac{1}{3} \text{ fractional instanton} = \text{ bion}
\]

implies fractional instantons cause transition between classical vacua at high \( \beta \), which lead to quantum \( Z_N \) symmetry and could yield adiabatic continuity

\* we are on the way of calculating topological charge density directly.
Fractional instantons
Thermal entropy for pbc and $Z_N$ tbc.

$N=3,5,10,20$  \hspace{1cm} (Nx,Nt) = (200, 8)  \hspace{1cm} N_{\text{sweep}}=200000, 400000$

◆ Question 3 : Thermal entropy

- Free energy of $\mathbb{CP}^\infty$ and free energy of free scalar for small $L$ also indicate

$$s = \beta N^2_\tau (\langle T_{xx} \rangle - \langle T_{\tau\tau} \rangle) = \frac{2\pi(N - 1)}{3N}$$

We will check thermal entropy for large $\beta$ (small $L$)

Monin, Shifman, Yung (15)
Thermal entropy is in agreement with the analytical prediction.

This is also consistent with the prediction from YM+Higgs model.

Our numerical results successfully confirm the predicted thermal entropy.
Thermal entropy for tbc

\[ \beta (N \tau)^2 (<T_{xx}> - <T_{\tau \tau}>) \]

- Thermal entropy behaves 1/N smaller than that of PBC.
- This observation should be checked analytically.

Prediction from numerical study which should be reproduced analytically.
Summary

- Lattice simulation of $\text{CP}^{N-1}$ model on $\mathbb{R} \times S^1$
- $\mathbb{Z}_N$ crossover transition is confirmed for pbc
- Thermal entropy agrees with the prediction for pbc
- Characteristic $\beta$ dependence of P-loop for tbc, which inspires more study on adiabatic continuity
- A pivotal role of fractional instantons is implied for tbc