Tensor Network Approach to Real-Time Path Integral

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(Kanazawa U.)

Lattice 2019

2019.06.16-22 @Wuhan

Study of real-time dynamics in HEP

- Complex Langevin
  - Real-time correlator, 3+1d ϕ^4 theory PRL95,202003(2005) Berges et al.
  - 3+1d SU(2) gauge theory with Schwinger-Keldysh (SK) setup, non-equilibrium, PRD75,045007(2007) Berges et al.
  - convergence issue (difficult for $t >> \beta$)

- Algorithm inspired by Lefschetz thimble PRD95,114501(2017) Alexandru et al.
  - 1+1d ϕ^4 theory with SK setup
  - Small box (2x8+2)x8 (Larger time extent seems harder)

- Tensor networks
  - Hamiltonian approach See D. Lin's talk (next)
  - Lagrangian approach (=Path integral approach) Here!

Model

1+1dim. Scalar field theory with Minkowskian metric

$$S = \int d^2x \left[\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \right]$$

$$V(\phi) = \frac{m_0^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$(\partial_\mu \phi)^2 \rightarrow (\partial_0 \phi)^2 - (\partial_1 \phi)^2$

$$x = (x_0, x_1)$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

"purely" Minkowskian but not SK

$$\phi \in \mathbb{R}$$

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$$V(\phi) = \frac{m_0^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

$$(\partial_\mu \phi)^2 \longrightarrow (\partial_0 \phi)^2 - (\partial_1 \phi)^2$$

On lattice

$a = 1$ lattice units

$$\partial_\mu \phi(x) \longrightarrow \phi_{x+\hat{\mu}} - \phi_x$$

$$\int d^2x \longrightarrow \sum_{x \in \mathbb{Z}^2}$$

Path integral

$$Z = \int [d\phi] \exp[iS]$$

To use tensor network method, path integral should be written in terms of tensor network representation

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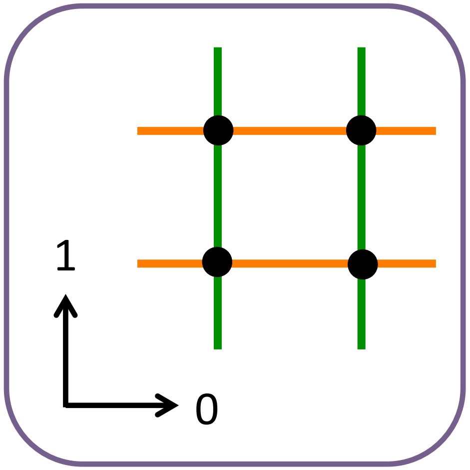
To use tensor network method, path integral should be written in terms of tensor network representation

$$= \int [d\phi] \prod_x H_0(\phi_x, \phi_{x+\hat{0}}) H_1(\phi_x, \phi_{x+\hat{1}})$$

$$H_0(\phi, \phi') = \exp\left[+\frac{i}{2}(\phi - \phi')^2 - \frac{i}{4}V(\phi) - \frac{i}{4}V(\phi')\right]$$

$$H_1(\phi, \phi') = \exp\left[-\frac{i}{2}(\phi - \phi')^2 - \frac{i}{4}V(\phi) - \frac{i}{4}V(\phi')\right]$$



Expansion of H

Lay 2002,
Shimizu 2012

For two-variable function

orthonormal basis

$$H_0(\phi, \phi') = \sum_{k=0}^{\infty} \Psi_k^{(0)}(\phi)\ \sigma_k^{(0)}\ \Phi_k^{(0)*}(\phi')$$

singular values $\qquad \sigma_0 > \sigma_1 > \sigma_2 > ... \geq 0$

If H_0 is a compact operator $\int |H_0(x, y)|^2 dxdy < \infty$

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If H_0 is a compact operator $\int |H_0(x,y)|^2 dxdy < \infty$

$$H_0(\phi,\phi') = \exp\left[\frac{i}{2}(\phi-\phi')^2 - \frac{i}{4}V(\phi) - \frac{i}{4}V(\phi')\right]$$ pure phase?

Expansion of H

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In $V = \frac{m_0^2}{2}\phi^2 + \ldots$ complex mass $m_0^2 \longrightarrow m_0^2 - i\epsilon\ (\epsilon > 0)$ Feynman prescription

$$\longrightarrow \exp\left[\frac{i}{2}(\phi-\phi')^2 + (-im_0^2 - \epsilon)\frac{\phi^2+\phi'^2}{8} + \ldots\right]$$

damping factor

→ compact

How to obtain Ψ Φ σ

$$H(\phi, \phi') = \sum_{m,n=0}^{\infty} \psi_m(\phi)\ X_{mn}\ \psi_n^*(\phi')$$

basis : Hermite function

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{1/2} n! 2^n}} H_n(x) e^{-x^2/2}$$

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$$X_{mn} = \int_{-\infty}^{\infty} d\phi d\phi' \psi_m^*(\phi)\ H(\phi,\phi')\ \psi_n(\phi')$$

$$\because \int_{-\infty}^{\infty} d\phi \psi_m^*(\phi) \psi_n(\phi) = \delta_{mn}$$

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SVD $$X_{mn} = \sum_{k=0}^{\infty} U_{mk} \sigma_k (V^\dagger)_{kn}$$

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SVD $\quad X_{mn} = \sum_{k=0}^{\infty} U_{mk}\sigma_k (V^\dagger)_{kn}$

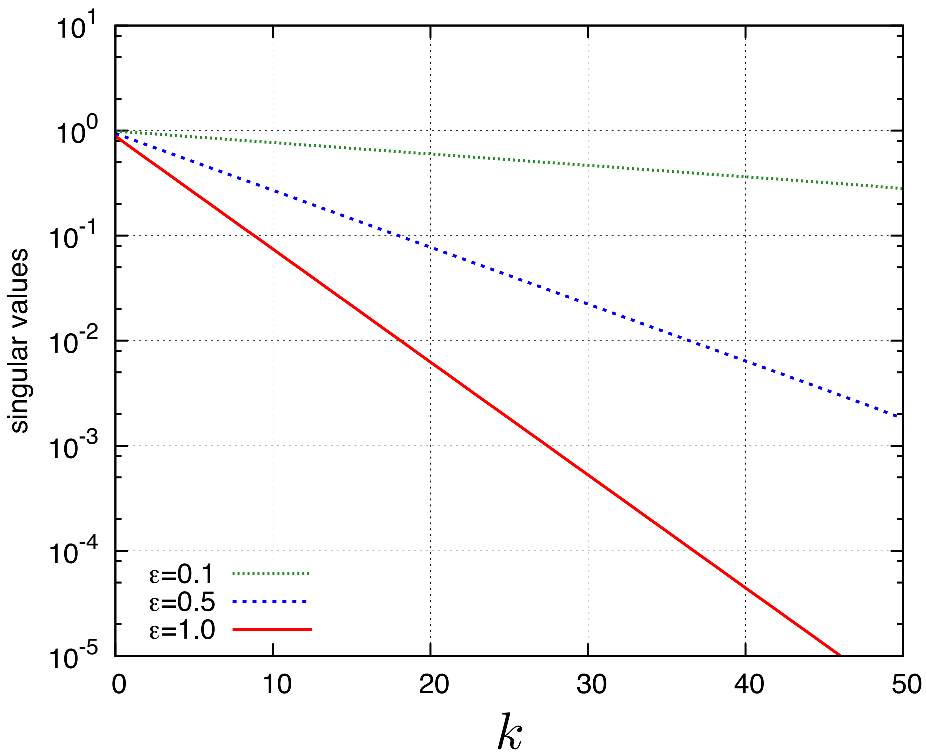
$$H(\phi,\phi') = \sum_{k=0}^{\infty} \underline{\left(\sum_{m=0}^{\infty} \psi_m(\phi) U_{mk}\right)} \sigma_k \underline{\left(\sum_{n=0}^{\infty} (V^\dagger)_{kn} \psi_n^*(\phi')\right)}$$

$$= \sum_{k=0}^{\infty} \underline{\Psi_k(\phi)} \sigma_k \underline{\Phi_k^*(\phi')}$$

Singular values

$$H_0(\phi,\phi') = \sum_{k=0}^{\infty} \Psi_k^{(0)}(\phi)\ \underline{\sigma_k^{(0)}}\ \Phi_k^{(0)*}(\phi') \quad \left(X_{mn} = \sum_{k=0}^{\infty} U_{mk}\underline{\sigma_k}(V^\dagger)_{kn} \right)$$

Free case m_0=1.0



larger ε
⇒
highly compressive

How to make tensor

For two-variable function

orthonormal basis

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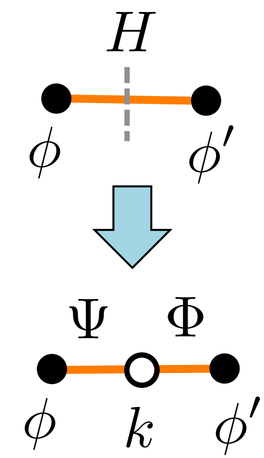
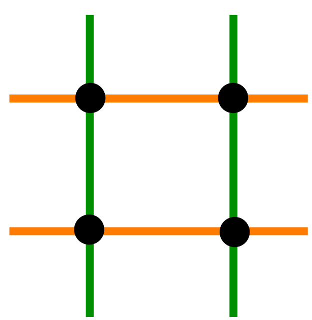
singular values



Once the orthonormal basis and singular values are obtained, tensor is formed as



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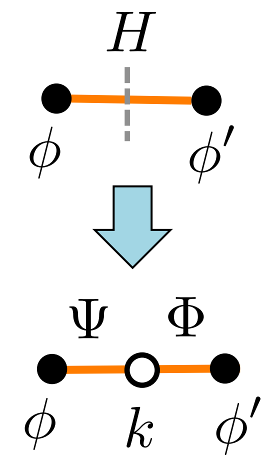

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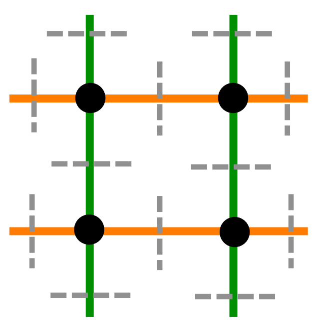

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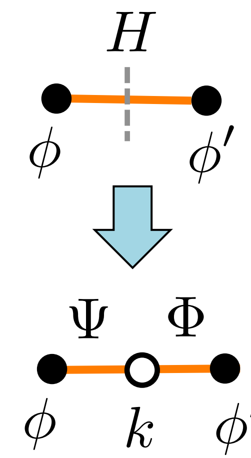

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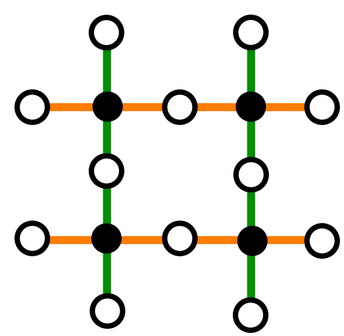

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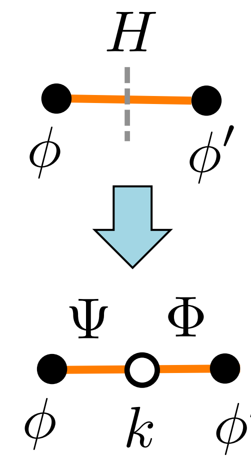

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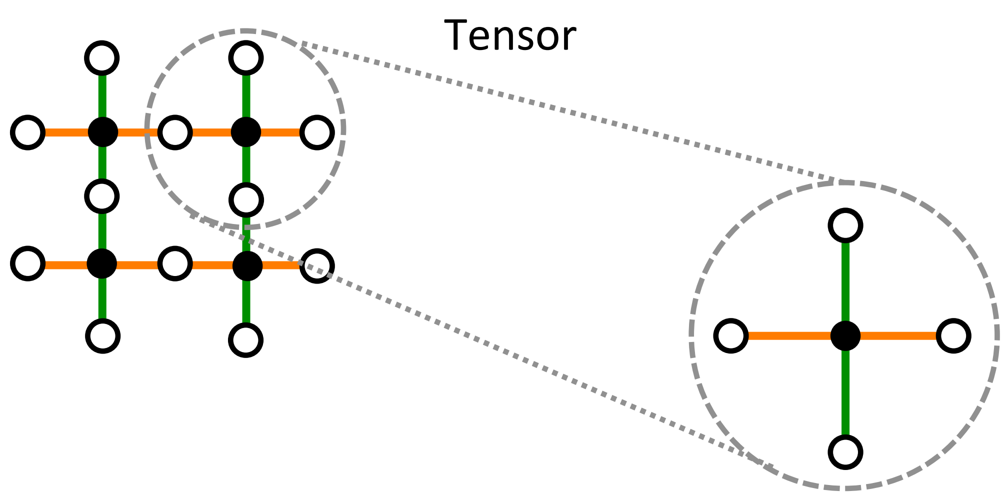

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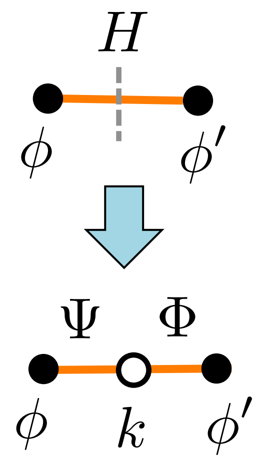
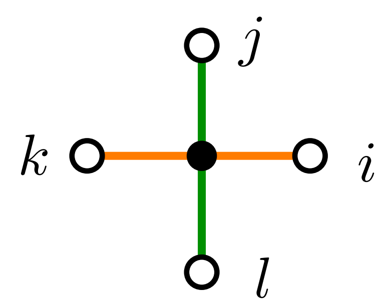


Once the orthonormal basis and singular values are obtained, tensor is formed as

$$T_{ijkl} = \sqrt{\sigma_i^{(0)}\sigma_j^{(1)}\sigma_k^{(0)}\sigma_l^{(1)}} \int_{-\infty}^{\infty} d\phi \Psi_i^{(0)} \Psi_j^{(1)} \Phi_k^{(0)*} \Phi_l^{(1)*}$$

The integral can be estimated by a recursion relation



How to make tensor

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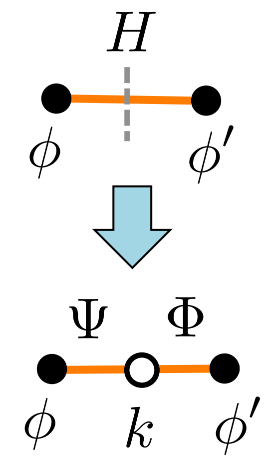

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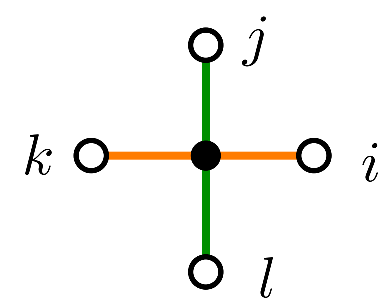
Tensor is truncated

truncation order

$$0 \le i,j,k,l \le N$$



Tensor

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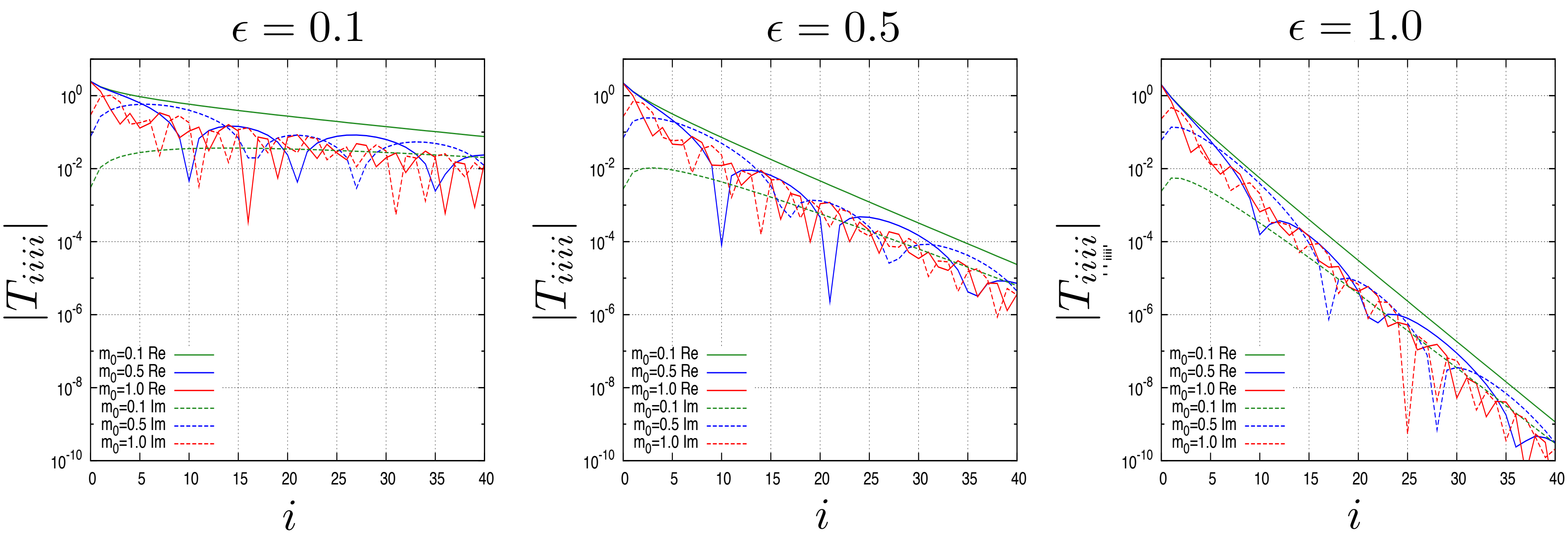
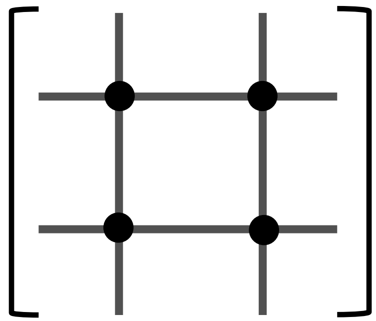
Free case



Z on 2x2 lattice for free case

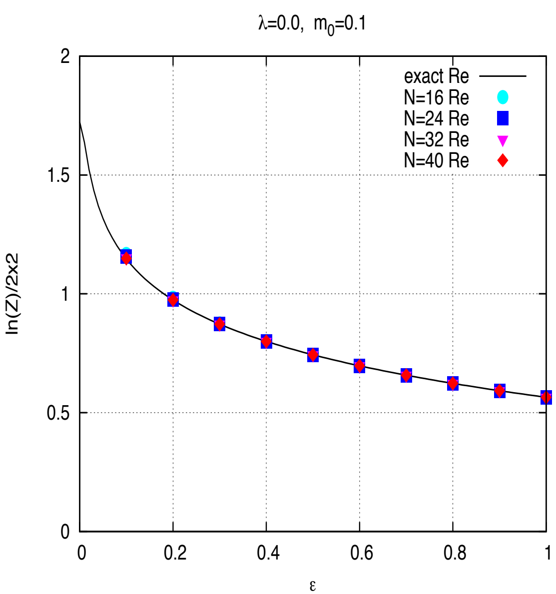
contraction for 2x2 tensor
(no coarse graining)

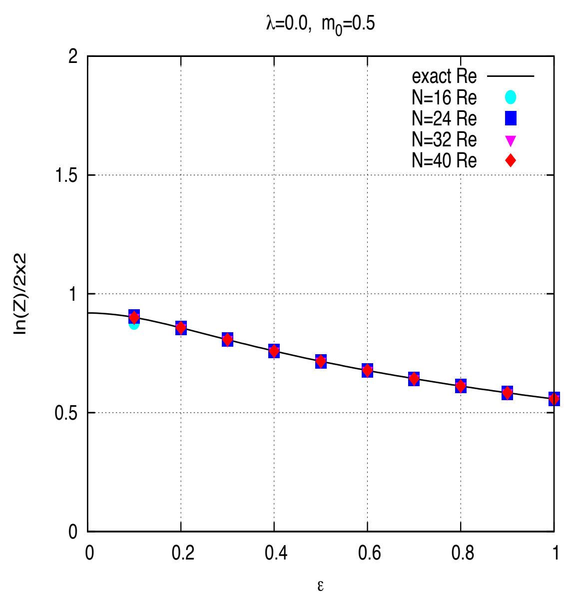
Z = Tr 

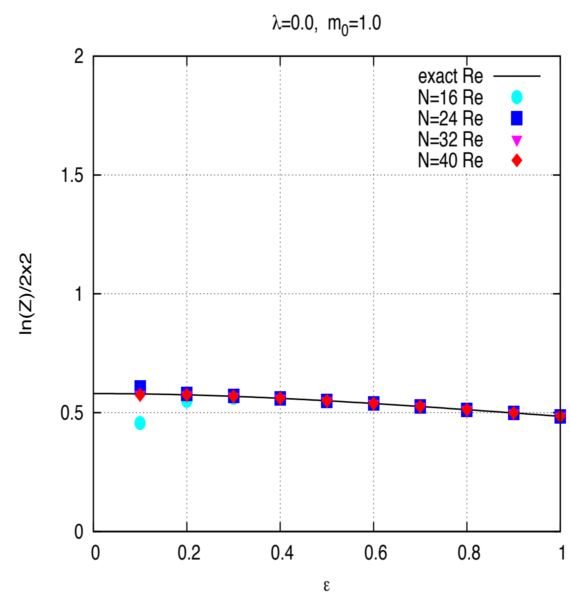
Periodic BC : not physical but …

$$T_{ijkl}$$

$$0 \leq i, j, k, l \leq N$$







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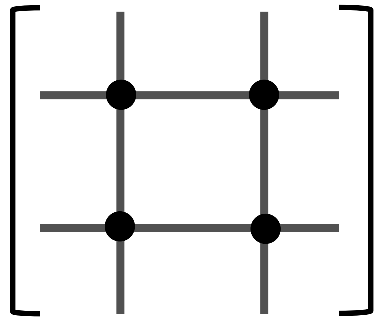
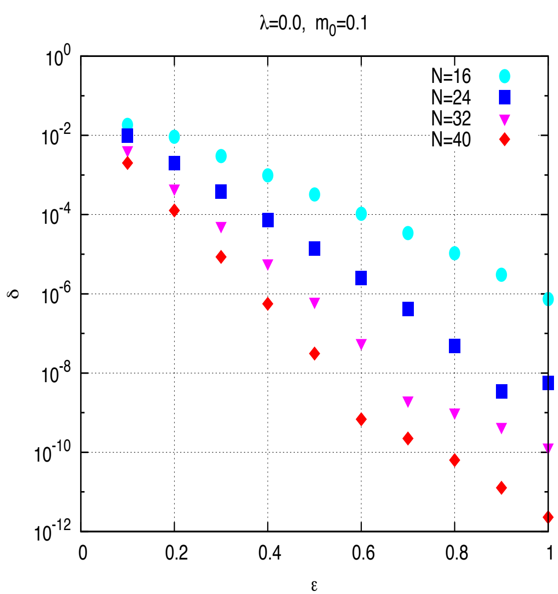
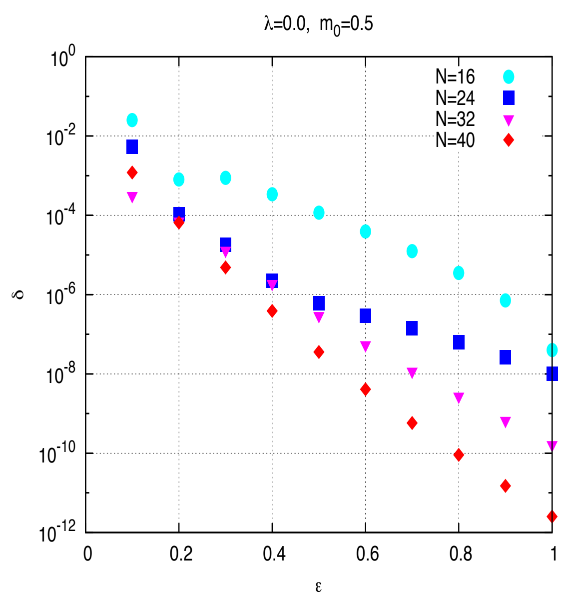
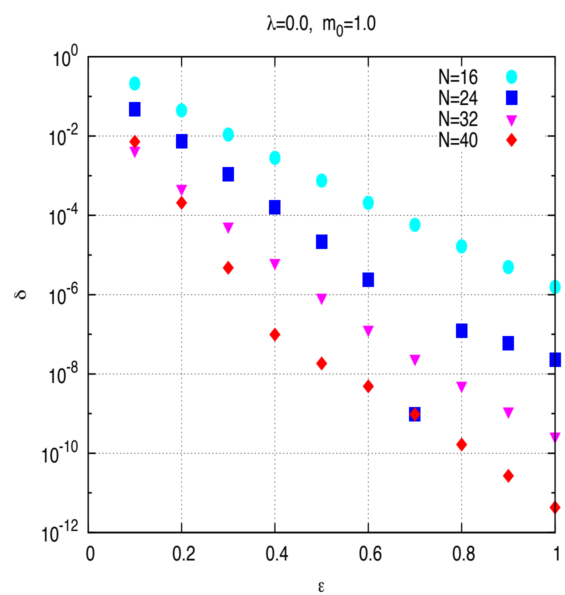
Z = Tr 

for real part of Z

$$\delta = \left| \frac{\ln Z - \ln Z_{\mathrm{exact}}}{\ln Z_{\mathrm{exact}}} \right|$$

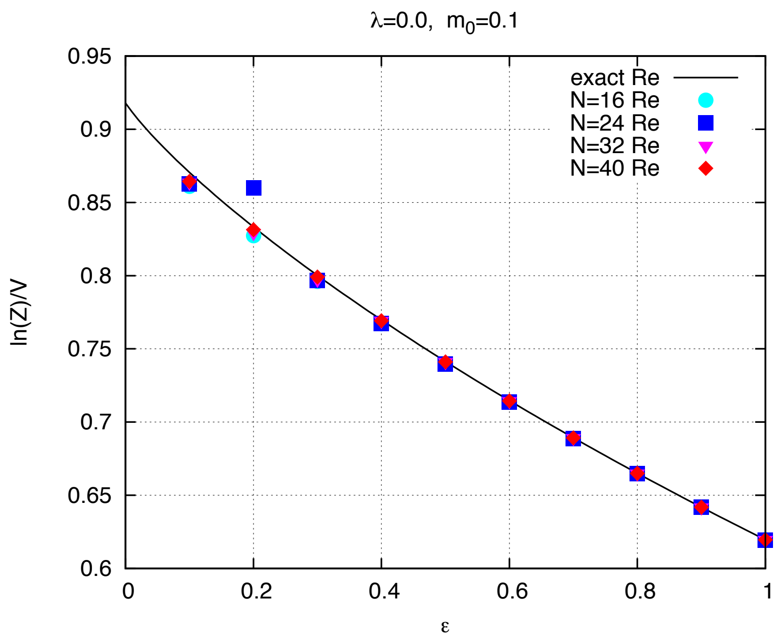


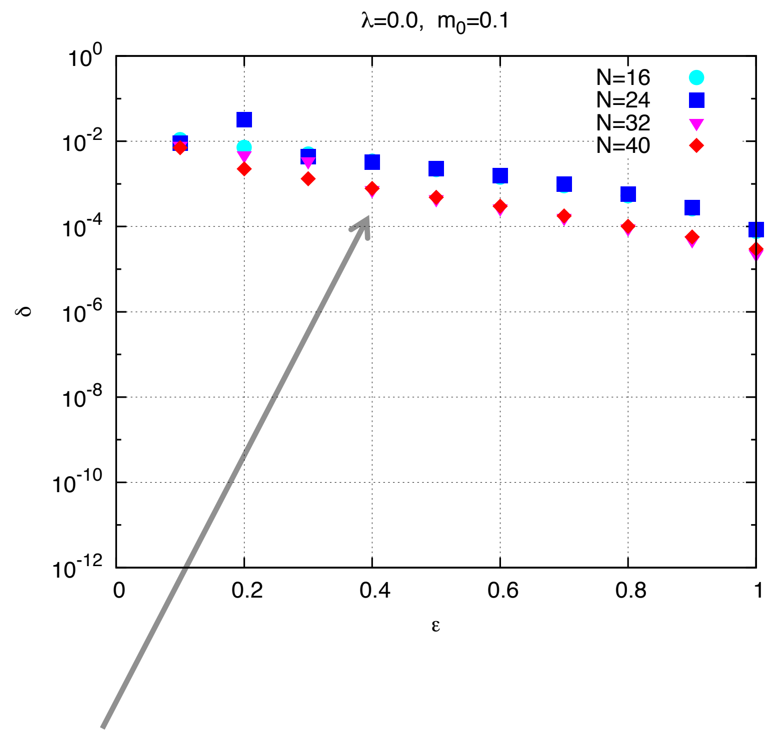




Larger volume in free case

Using TRG as coarse-graining $V = (1024)^2$



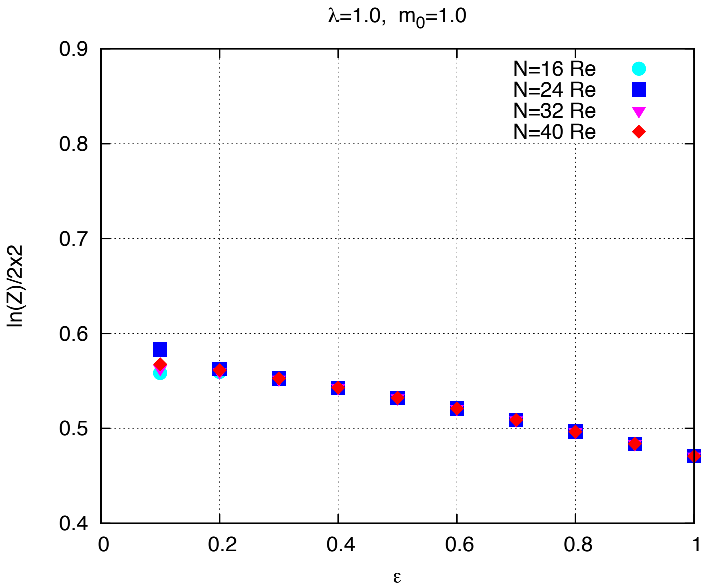


need improved algorithm: TNR, Loop-TNR, GILT?

Interacting case λ=1

preliminary

$$V = 2^2$$



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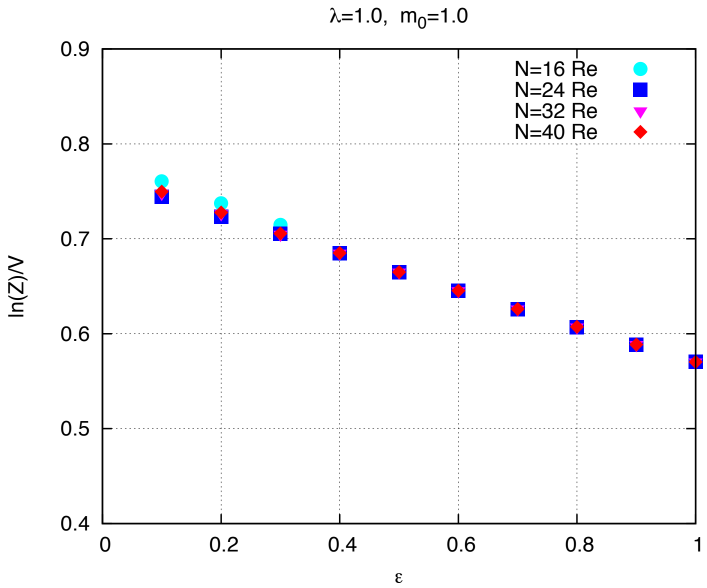

Summary

- Tensor network representation for scalar field theory with Minkowskian metric is derived
- Orthonormal basis function (Hermite function) plays an important role (SVD & avoid the sign problem)
- Feynman prescription ($m_0{}^2 - i\varepsilon$) provides a damping factor (compact operator)
- No sign problem but there is a problem of information compressibility (bad hierarchy of singular values)
- Need $\varepsilon \to 0$ extrapolation
- For larger volume and interacting case, the accuracy tends to be worse $\Rightarrow$ need improvement

Future

- Improvement of initial tensor using idea of TNR, GILT, etc
- Tilted time axis $t \longrightarrow te^{-i\xi/2}$ (instead of ${m_0}^2 - i\varepsilon$)
- Schwinger-Keldysh, Out of equilibrium
- Real-time correlator, Spectral function, Transport coefficients
- Other models including fermions and gauge fields
- Higher dimensional system (Hard!!!)

How to obtain Ψ Φ σ

$$\exp\left[-i\phi\phi' + (-im_0^2 - \epsilon)\frac{\phi^2 + \phi'^2}{8}\right] e^{\frac{i}{2}(\phi^2+\phi'^2)}$$

For free case

up to 2π factor

Remember 1-dim QM

$$e^{ixp} = \langle x|p\rangle = \sum_{n=0}^{\infty}\langle x|n\rangle\langle n|p\rangle$$

$$= \sum_{n=0}^{\infty}\psi_n(x)\tilde{\psi}_n^*(p)$$

$$\tilde{\psi}_n^*(p) = i^n\psi_n^*(p)$$

if basis is Hermite function

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{1/2}n!2^n}}H_n(x)e^{-x^2/2}$$

$$= \sum_{n=0}^{\infty}\psi_n(x)\ i^n\ \psi_n(p)$$

Truncation is not allowed

How to obtain Ψ Φ σ

For free case

$$\exp\left[-i\phi\phi' + (-im_0^2 - \epsilon)\frac{\phi^2 + \phi'^2}{8}\right]$$

$\mathrm{Re}[\beta] > 0$

up to 2π factor

$$e^{ixp}\underline{e^{-\beta(x^2+p^2)}} = \sum_{n=0}^{\infty}\left(\underline{e^{-\beta x^2}\psi_n(x)}\right)\ i^n\ \left(e^{-\beta p^2}\psi_n(p)\right)$$

damping factor

$$\sum_{m=0}^{\infty}\underline{G_{nm}}\psi_m(x)$$

$$G_{nm} = \int_{-\infty}^{\infty} dx e^{-\beta x^2}\psi_m(x)\psi_n(x)$$

How to obtain Ψ Φ σ

$$\exp\left[-i\phi\phi' + (-im_0^2 - \epsilon)\frac{\phi^2 + \phi'^2}{8}\right]$$

For free case

$\mathrm{Re}[\beta] > 0$

up to 2π factor

$$e^{ixp}e^{-\beta(x^2+p^2)} = \sum_{n=0}^{\infty}\left(e^{-\beta x^2}\psi_n(x)\right)\ i^n\ \left(e^{-\beta p^2}\psi_n(p)\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}G_{nm}\psi_m(x)\right)i^n\left(\sum_{k=0}^{\infty}G_{nk}\psi_k(p)\right)$$

$$= \sum_{m,k=0}^{\infty}\psi_m(x)\left(\sum_{n=0}^{\infty}i^n G_{nm}G_{nk}\right)\psi_k(p)$$

$$X_{mk} = \sum_{a=0}^{\infty}U_{ma}\sigma_a(V^\dagger)_{ak}$$

How to obtain Ψ Φ σ

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up to 2π factor

$$e^{ixp}e^{-\beta(x^2+p^2)} = \sum_{n=0}^{\infty}\left(e^{-\beta x^2}\psi_n(x)\right)\ i^n\ \left(e^{-\beta p^2}\psi_n(p)\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} G_{nm}\psi_m(x)\right) i^n \left(\sum_{k=0}^{\infty} G_{nk}\psi_k(p)\right)$$

$$= \sum_{m,k=0}^{\infty} \psi_m(x) \left(\sum_{n=0}^{\infty} i^n G_{nm}G_{nk}\right) \psi_k(p)$$

$$= \sum_{a=0}^{\infty}\left(\sum_{m=0}^{\infty} \psi_m(x)U_{ma}\right) \sigma_a \left(\sum_{k=0}^{\infty}(V^\dagger)_{ak}\psi_k(p)\right)$$

$$= \sum_{a=0}^{\infty} \Psi_a(x)\sigma_a\Phi_a^*(p)$$

How to obtain Ψ Φ σ

$$\exp\left[-i\phi\phi' + (-im_0^2 - \epsilon)\frac{\phi^2 + \phi'^2}{8}\right] = \sum_{a=0}^{\infty} \Psi_a(\phi)\sigma_a\Phi_a^*(\phi')$$

① $$G_{nm} = \int_{-\infty}^{\infty} dx e^{-(im_0^2+\epsilon)x^2/8}\psi_m(x)\psi_n(x)$$

② $$X_{mk} = \sum_{n=0}^{\infty} i^n G_{nm} G_{nk}$$

range of m, n, k is truncated at $K(\gg N)$

③ $$X_{mk} \approx \sum_{a=0}^{N} U_{ma}\sigma_a(V^\dagger)_{ak}$$

(SVD)

truncated at N

→ tensor $\approx N^4$

④ $$\begin{cases} \Psi = U\psi \\ \Phi = V\psi \end{cases}$$

ψ Hermite function

Numerical results

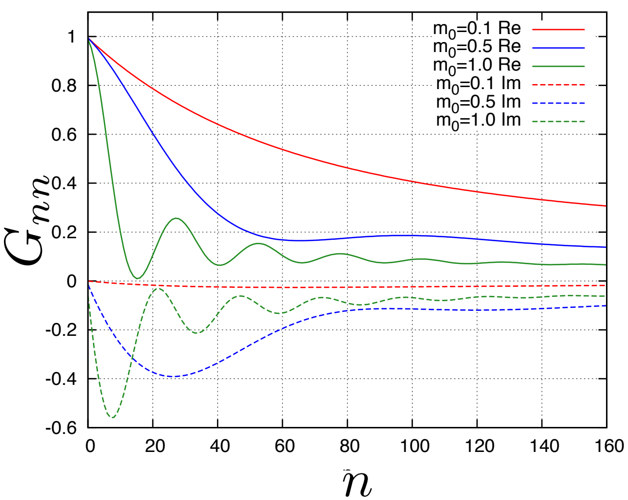
$$G_{nm} = \int_{-\infty}^{\infty} dx e^{-(im_0^2+\epsilon)x^2/8} \psi_m(x)\psi_n(x)$$

no sign problem

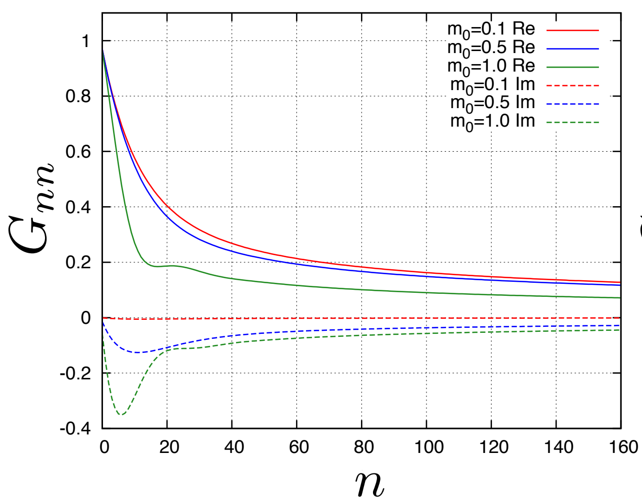
$$\beta = im_0^2 + \epsilon$$

$$G_{m+1,n+1} = \frac{1}{(1+\beta)\sqrt{(m+1)(n+1)}} \Big[G_{mn} + (1-\beta)\sqrt{mn}\, G_{m-1,n-1} - \beta\sqrt{(m+1)n}\, G_{m+1,n-1} - \beta\sqrt{m(n+1)}\, G_{m-1,n+1} \Big].$$

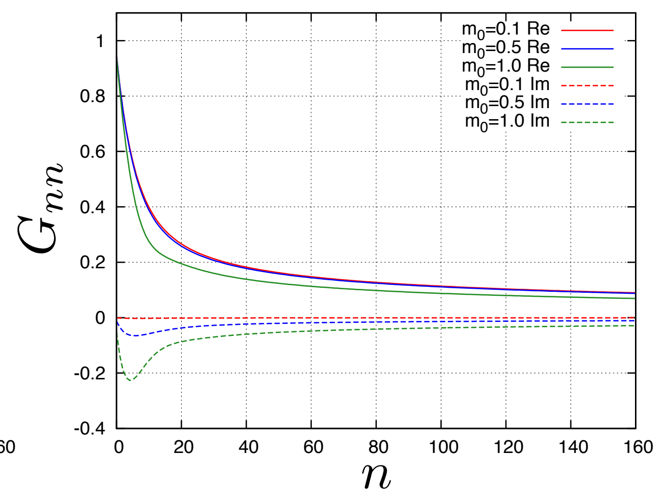
$\epsilon = 0.1$



$\epsilon = 0.5$



$\epsilon = 1.0$



Path integral in PBC

$$\mathcal{Z} = \int [d\phi] e^{iS} = \sqrt{\frac{(-2\pi i)^V}{\prod_p K(p)}}$$

$\lambda = 0$

$m \in \mathbb{C}$

$$K(p) = -(2\sin(p_0/2))^2 + (2\sin(p_1/2))^2 + m^2$$

$$p_\mu = \frac{2\pi}{L_\mu} n_\mu, \qquad n_\mu = 0, 1, 2, ..., L_\mu - 1.$$

For 2x2 lattice

$$\mathcal{Z}^{(2\times 2)} = (2\pi)^2 \sqrt{\frac{1}{m^4(m^2+4)(m^2-4)}}$$

singular points

$$m = 0,\ \pm 2,\ \pm 2i$$