Tensor network formulation of 2d gravity

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Motivation/Background

- Many approaches to quantum gravity eg string theory, loop quantum gravity, dynamical triangulations, functional renormalization group, gravity as gauge theory
- Here focus on latter. Write down a path integral. Non-renormalizable in p. theory. But what about strong coupling ? Weinberg's asymptotic safety scenario: UV fixed point at strong coupling ? ...
- Need lattice approach. Typically encounter sign problem.
- But tensor network methods can do end run around this ...

Plan

- Review Cartan/Palatini formulation of GR as a gauge theory of Lorentz group.
- Review MacDowell-Mansouri extension: GR as spontaneously broken gauge theory of (anti)de Sitter group.
- First step: two dimensional gravity.
- Translate to lattice. $SU(2)$ gauge theory.
- Tensor network formulation.
- Results, future work

Cartan formulation of gravity I

Can rewrite Einstein gravity in 1st order formalism using the frame field e_μ^a where

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$$

Clearly $g_{\mu\nu}$ invariant under local Lorentz transformations:

$$e_\mu^a(x) \rightarrow \Lambda_b^a(x) e_\mu^b(x)$$

To write down non-trivial dynamics need a covariant derivative eg.

$$D_\mu^{ab} e_\nu^b = \partial_\mu e_\nu^a + \omega_\mu^{ab} e_\nu^b$$

where ω_μ^{ab} is spin connection

$$\omega = \omega^{ab} T^{ab} \text{ with } T \in SO(3,1)$$

Cartan formulation of gravity II

Using (e_μ, ω_μ) can write down action:

$$S = \frac{1}{\ell_P^2} \int d^4x\, \epsilon^{\mu\nu\rho\lambda} \left[\epsilon_{abcd} \left(e^a_\mu e^b_\nu R^{cd}_{\lambda\rho} - \frac{1}{\ell^2} e^a_\mu e^b_\nu e^c_\lambda e^d_\rho \right) + \alpha\, T^a_{\mu\nu} T^b_{\lambda\rho} \eta_{ab} \right]$$

with $R_{\mu\nu} = [D_\mu, D_\nu]$ field strength and $T^a_{\mu\nu} = D_{[\mu}\, e^a_{\nu]}$ is called torsion. Note: no background metric !

If:

$$T = 0$$

$$\det(e^a_\mu) \neq 0 \text{ with } e^a_\mu \text{ a } 4 \times 4 \text{ matrix i.e } e^{-1} = e^\mu_a \text{ exists}$$

Can show:

$$S = \frac{1}{\ell_P^2} \int \sqrt{-g}\left(R - \frac{1}{\ell^2}\right) \text{ Einstein-Hilbert!}$$

Gravity as (almost) pure gauge theory ...

MacDowell and Mansouri showed (1979) that Cartan gravity is close to pure gauge theory of the (anti-) de Sitter group $SO(4,1)$ (SO(3,2))

For simplicity consider Euclidean theory $SO(4,1) \rightarrow SO(5)$ and identify (ω, e) as different components of $SO(5)$ connection

$$A_\mu = \omega_\mu^{ab} T^{ab} + \frac{1}{\ell} e_\mu^a T^{5a}$$

Scale ℓ input. $SO(5)$ curvature decomposed under $SO(4)$ (Lorentz) subgroup

$$F_{\mu\nu} = (R_{\mu\nu}^{ab}(\omega) - \frac{1}{\ell^2} e_{[\mu}^a e_{\nu]}^b)\, T^{ab} + D_{[\mu} e_{\nu]}^a\, T^{5a}$$

Notice that the vanishing of F automatically ensures both the Einstein equation and the vanishing torsion condition $T = 0$.

What $SO(5)$ action is needed ? And how does it break to $SO(4)$?

MacDowell-Mansouri action

$$S_M = \kappa \int d^4x \, \epsilon^{\mu\nu\lambda\rho} \epsilon_{ABCDE} \phi^E F^{AB}_{\mu\nu} F^{CD}_{\lambda\rho} \quad A = 1 \ldots 5$$

$\kappa = \frac{\ell^2}{\ell_P^2}$ – pert. renormalizable ! SO(5) hidden symmetry reduces counterterms

Here, ϕ is an additional scalar in fundamental of $SO(5)$
Assume phase where $SO(5)$ is spontaneously broken to $SO(4)$ by setting $\phi^A \phi^A = 1$.

Relation to EH ?

Set unitary gauge $\phi = \delta^{5A}$ and expand $F^{ab}_{\mu\nu} F^{cd}_{\lambda\rho} \ldots$ up to topological terms like $R^{ab}_{\mu\nu} R^{cd}_{\lambda\rho} \epsilon^{abcd} \epsilon_{\mu\nu\lambda\rho}$ find EH action !

Two dimensions

Toy model. Much is known from old days...

$SO(3) \sim SU(2)$ Euclidean de Sitter symmetry. Scalar ϕ in adjoint. Breaks to $SO(2) = U(1)$ Lorentz symmetry.

$$S = \kappa \int d^2x \, \epsilon^{\mu\nu} \mathrm{Tr} \left(\phi F_{\mu\nu} \right) \quad \text{with } \phi.\phi = 1$$

with F and ϕ in adjoint of SU(2). In unitary gauge $\phi = \sigma_3$. Again, classical EOM yield

$$R^{12}_{\mu\nu} - \frac{1}{\ell^2} e^1_{[\mu} \, e^2_{\nu]} = 0 \text{ i.e } R = \Lambda \text{ and } T^a_{\mu\nu} = 0$$

Lattice formulation

As for lattice QCD we use link fields valued in SU(2) group. Formally we write:

$$F_{\mu\nu} = \frac{1}{2i}\left(U^P - (U^P)^\dagger\right)$$

with $U^P(x)$ Wilson plaquette. This allows us to write the action as:

$$S = \kappa \sum_x \mathrm{Re}\ \mathrm{Tr}\left(i\sigma_3 U^P_{12}\right)$$

In addition we allow for a regular Wilson term. This allows us to take the limit $\beta \to \infty$ which sends the lattice spacing to zero and allows for a naive continuum limit

$$S = \sum_x \mathrm{Re}\ \mathrm{Tr}\left(\left[\beta I + i\kappa\sigma_3\right] U^P_{12}\right)$$

Character Expansion

We can rewrite this as:

$$S = \mu \sum_x \mathrm{Re}\ \mathrm{Tr}\left(MU^P\right) \text{ where } M = e^{i\sigma_3\theta} \text{ where } \mu = \sqrt{\kappa^2 + \beta^2}$$

and

$$\cos\theta = \frac{\beta}{\mu}$$

Using a character expansion each plaquette term in the action can be expressed:

$$e^{\mu \mathrm{Re}\ \mathrm{Tr}\left(U^P M\right)} = \sum_j \frac{2(2j+1)I_{2j+1}(\mu)}{\mu}\chi^j(U^P M)$$

$I_n(z)$ modified Bessel function.

Tensor network

Integrate out links using Haar measure $\rightarrow$ (2 plaquettes per link in 2d, $\int DUUU^\dagger = \delta$)

$$Z = \sum_j f_j^{N_{\text{sites}}} \quad \text{pbc}$$

where

$$f_j = \frac{2}{2j+1} U_{2j}\left(\frac{\beta}{\mu}\right) \frac{I_{2j+1}(\mu)}{\mu} \quad U_{2j} \text{ Chebyshev of 2nd kind}$$

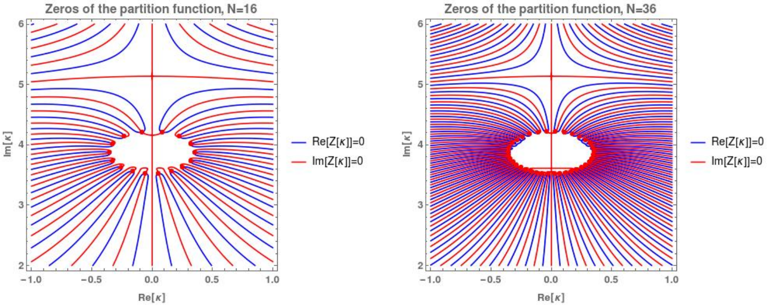
Write as tensor network $Z = \mathrm{Tr}\,(T_{ijkl}(x)..)$ with

$$T_{ijkl}(x) = \begin{cases} f_r & \text{if } i = j = k = l = 2r \\ 0 & \text{otherwise.} \end{cases}$$

where each index is associated with an adjacent plaquette

Fisher Zeroes

Phase transitions associated with zeroes of Z (Fisher zeroes) in plane of complex κ
Occur when zeroes pinch an axis in thermodynamic limit.



Clearly zeroes occur on imaginary axis as $N \to \infty$ where $I_n \to J_n$.
Sign problem for MC

Critical Behavior

Can understand behavior by truncating to $jmax = 1$ (note: $\beta = \frac{c}{N} \to 0$ at large N). Near imag axis:

$$Z = \frac{1}{\kappa^N}\left(J_1^N(\kappa) + \frac{1}{3^N} J_3^N(\kappa) + \dots\right)$$

Simple to see that

$$\begin{aligned} F = \frac{1}{N}\ln Z &= \ln\frac{J_1(\kappa)}{\kappa} \quad \text{when} \quad \frac{3J_1}{J_3} > 1 \\ &= \ln\frac{J_3(\kappa)}{\kappa} \quad \text{when} \quad \frac{3J_1}{J_3} < 1 \end{aligned}$$

In complex plane ring associated to $(\frac{3J_1(\kappa)}{J_3(\kappa)})^N = 1$

Order of transition

On imaginary axis critical point corresponds to $3J_1(\kappa_c)/J_3(\kappa_c) = 1$
Perfect agreement with Fisher zeroes.
Clearly free energy is continuous at κ_c. But its derivative is not having a discontinuity

$$\Delta \frac{\partial F}{\partial \kappa}|_{\kappa_c} = \Delta \langle \mathrm{det(e)} \rangle = \frac{\partial}{\partial \kappa} \ln(J_3(\kappa)/J_1(\kappa))|_{\kappa_c}$$

Thus transition is first order. No continuum limit

Why ?

Theory as written is topological. DT/matrix model 2d gravity requires a additional propagating dof (Liouville mode).

Additional terms

Are there additional relevant operators we left out ?
eg.

$$\int \mathrm{Tr}\ D_\mu\phi D^\mu\phi$$

Allowed by $SO(5)$. Forbidden by diffeomorphisms. But lattice breaks coordinate invariance ... therefore we should include and tune coupling.

Note:

While phase of ϕ irrelevant (unitary gauge) its magnitude is free. cf $S \sim \int R\phi + (\partial\phi)^2$ with Liouville action ...

Would like to calculate the exponent γ using TRG methods and compare to matrix models where

$$Z \sim A^{\gamma-3} e^{\mu A}$$

Conclusions/Future work

- Tensor methods allow new numerical approaches to gauge theoretic approaches to QG - avoid sign problem.
- 4D (Euclidean) de Sitter works similarly. Unitary gauge:

$$S = \kappa \sum \epsilon_{\mu\nu\rho\lambda} \mathrm{Tr} \left(\gamma_5 U^P_{\mu\nu} U^P_{\lambda\rho} \right) \quad U \in Spin(5)$$

  No exact solutions. Need new/efficient TRG ?
- Anti de Sitter gravity as tensor network ? 2D $SU(2) \rightarrow SU(1,1)$. Non compact group has both discrete and continuous reps ... subtleties. How does holography work?

Question:

Can tensor network formulations of gauge theoretic approaches to gravity improve our understanding of quantum gravity ?