Domain-wall fermion and Atiyah-Patodi-Singer index

Hidenori Fukaya (Osaka U.)

HF,. T Onogi, S. Yamaguchi PRD96(2017) no.12, 125004 [arXiv:1710.03379]

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[arXiv: 19xx.xxxxx]

Physicist-friendly APS index

In Lattice 2017, we proposed

“A physicist-friendly reformulation of the Atiyah-Patodi-Singer index theorem.”

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Recently, we invited 3 mathematicians and succeeded in a mathematical proof.

F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita, in progress

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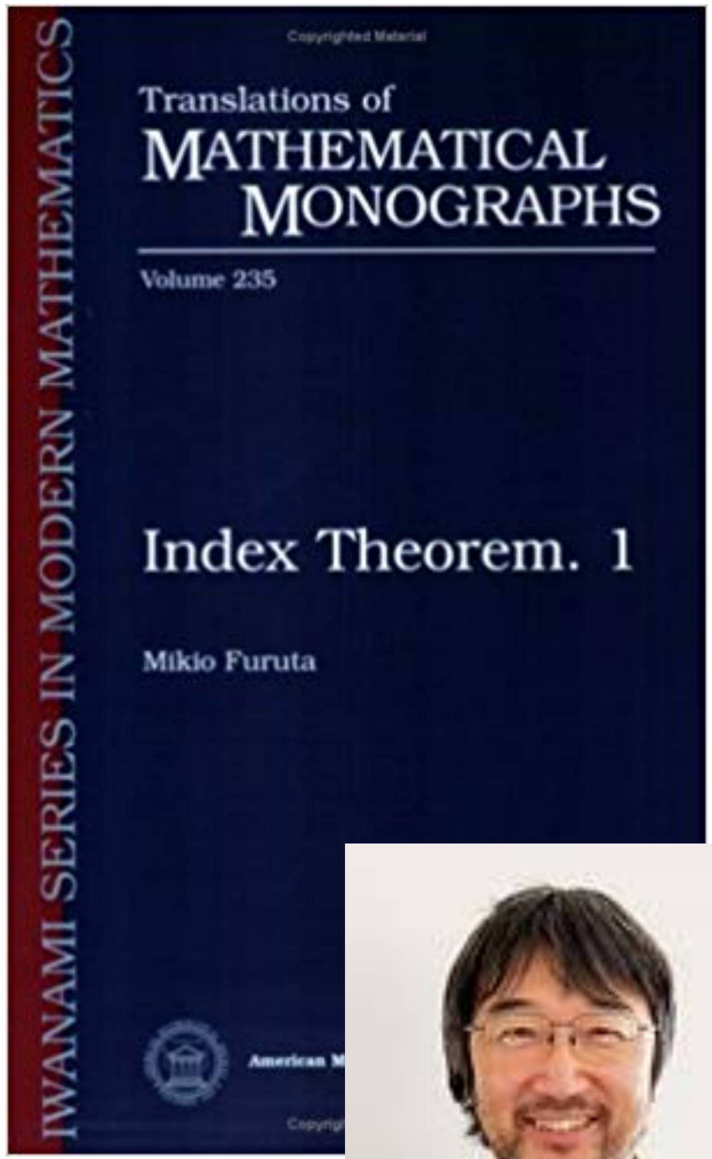


Atiyah-Patodi-Singer index theorem

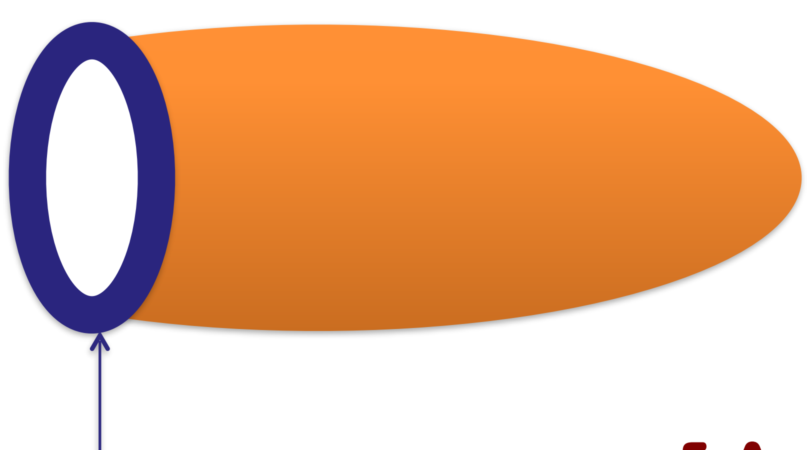
index on a manifold with boundary,

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\gamma_5 e^{D^2_{4\mathrm{D}}/\Lambda^2} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

integer

non-integer

non-integer



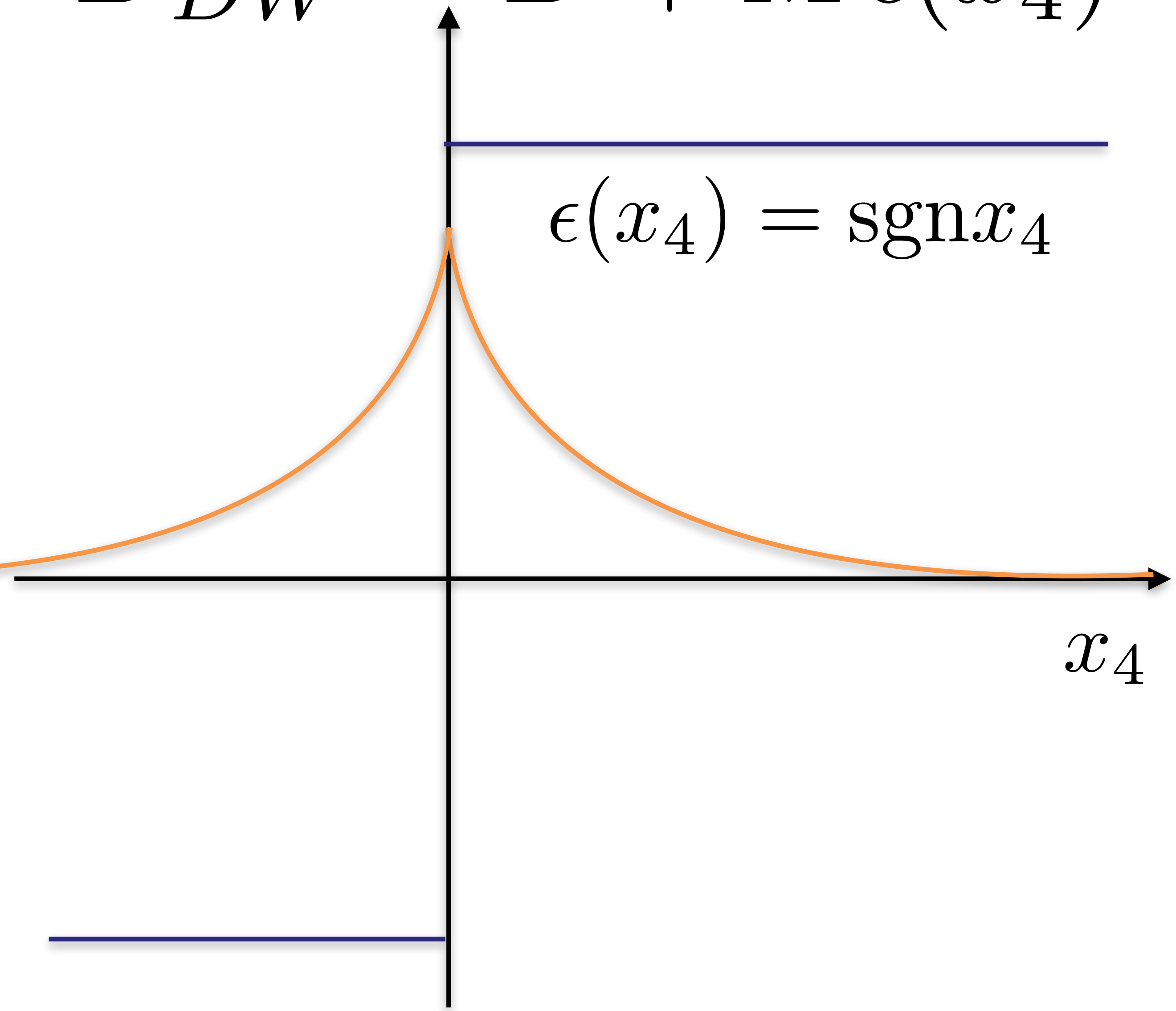
$x_4 = 0$

$$\eta(iD^{3\mathrm{D}}) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg} = \sum_{\lambda}^{reg}\mathrm{sgn}\lambda$$

[Atiyah-Patodi-Singer 1975]

4-dim. domain-wall fermion in continuum

$$D_{DW} = D + M\epsilon(x_4)$$



Massless Dirac fermion localized at 3-dim edge.

No gauge anomaly, but T(or parity) anomaly.

good model for topological insulator.

APS index in topological insulator

APS index is a key to understand bulk-edge correspondence in symmetry protected topological insulator [Witten 2015]:

fermion path integrals

$$Z_{\rm edge} \propto \exp(-i\pi\eta(iD^{\rm 3D})/2)$$

T-anomalous

$$Z_{\rm bulk} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}]\right)$$

T-anomalous

$$Z_{\rm edge}Z_{\rm bulk} \propto (-1)^{\mathfrak{I}} \Rightarrow$$

T is protected !

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}{\rm tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\rm 3D})}{2}$$

[Related works: Metlitski 15, Seiberg-Witten 16, Tachikawa-Yonekura 16, Freed-Hopkins 16, Witten 16, Yonekura 16…]

What puzzled us

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1. APS boundary condition is non-local, while that of topological matter is local.

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4. No physicist-friendly literature [except for Alvarez-Gaume et al. 1985 (but boundary condition is obscure.)]

What puzzled us

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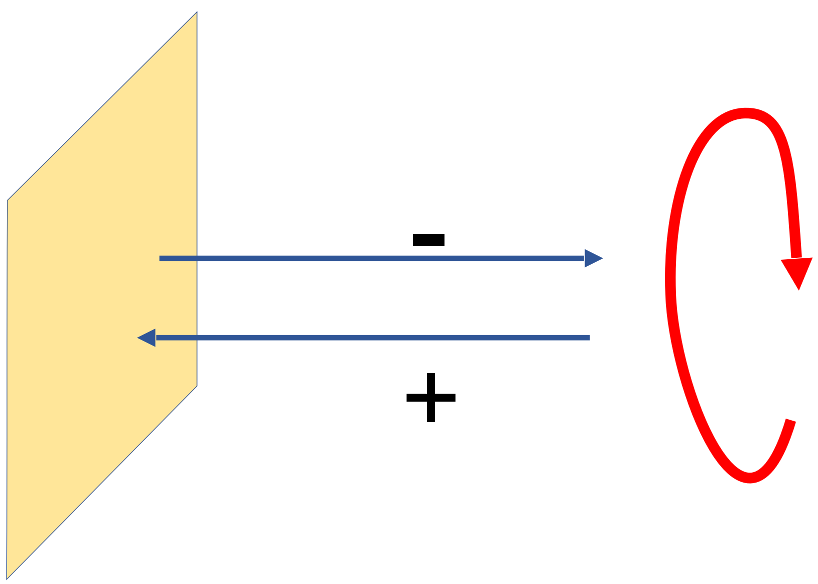
→ We launched a study group reading original APS paper and it took 3 months to translate it into “physics language”, and we proposed an alternative expression.

Difficulty with boundary

If we impose **local** and **Lorentz (rotation)** invariant boundary condition, + and – chirality sectors do not decouple any more.



angular momentum is conserved

n_+, n_- and the index do not make sense.

Atiyah-Patodi-Singer boundary condition

[Atiyah, Patodi, Singer 75]

Gives up the locality and rotational symmetry but keeps the chirality.

Eg. 4 dim $x^4 \geq 0$ $A_4 = 0$ gauge

$$D = \gamma^4 \partial_4 + \gamma^i D_i = \gamma^4 (\partial_4 + \underbrace{\gamma^4 \gamma^i D_i}_{A})$$

They imposed a non-local b.c.

$$(A + |A|)\psi|_{x^4=0} = 0$$

$$[\gamma_5, A] = 0.$$

$$\text{index} = n_+ - n_-$$

Beautiful!

But physicist-unfriendly.

Locality >> chirality for physicists

Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.

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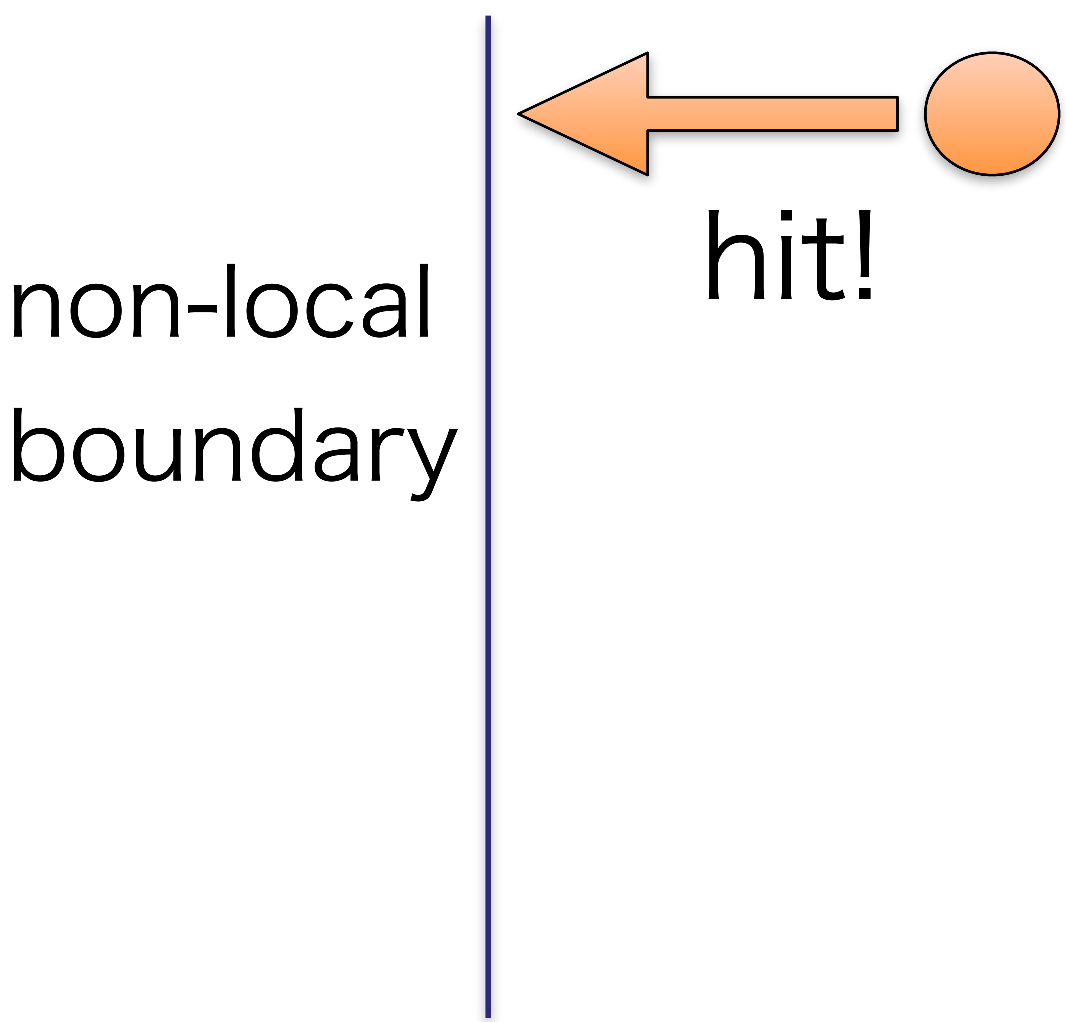

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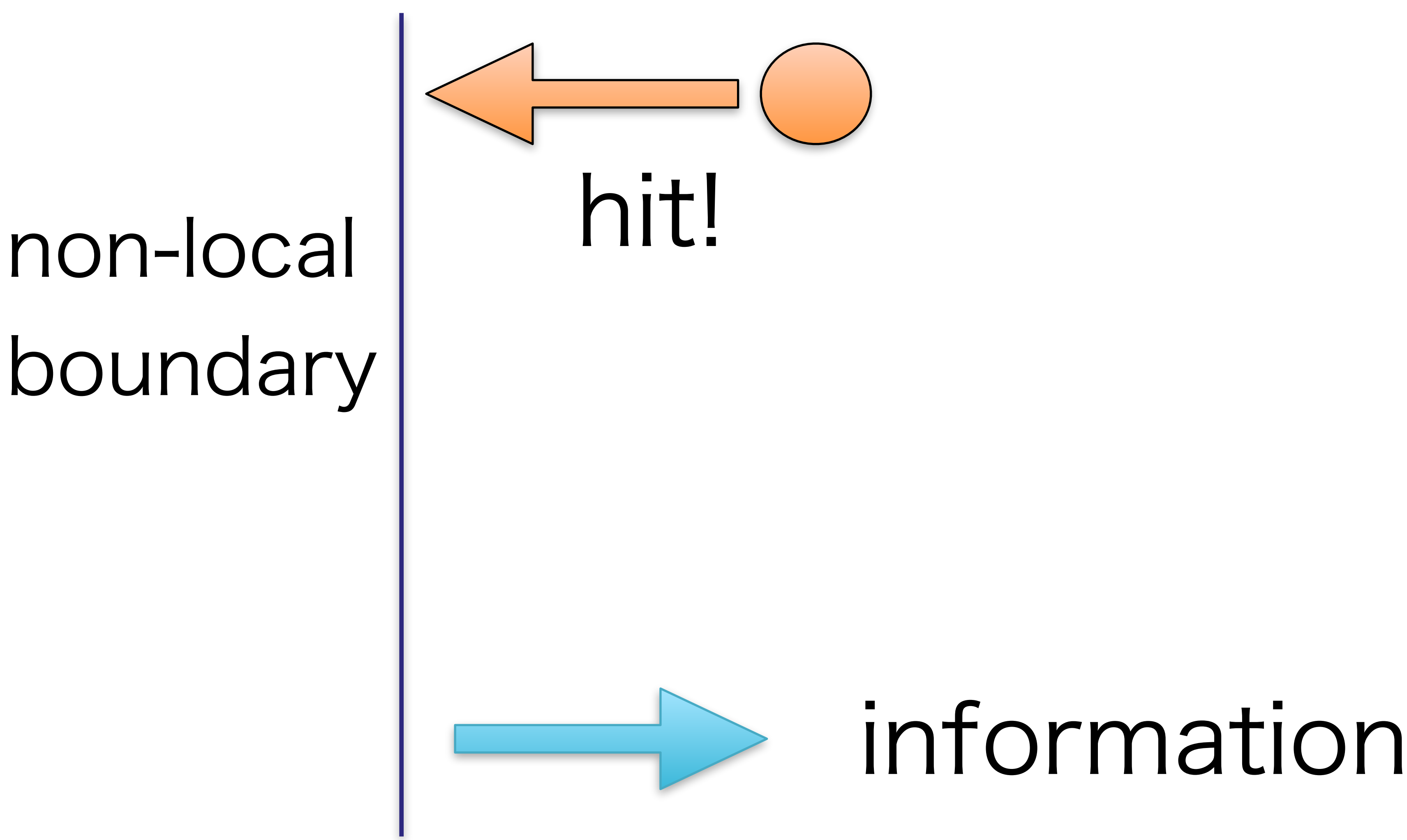

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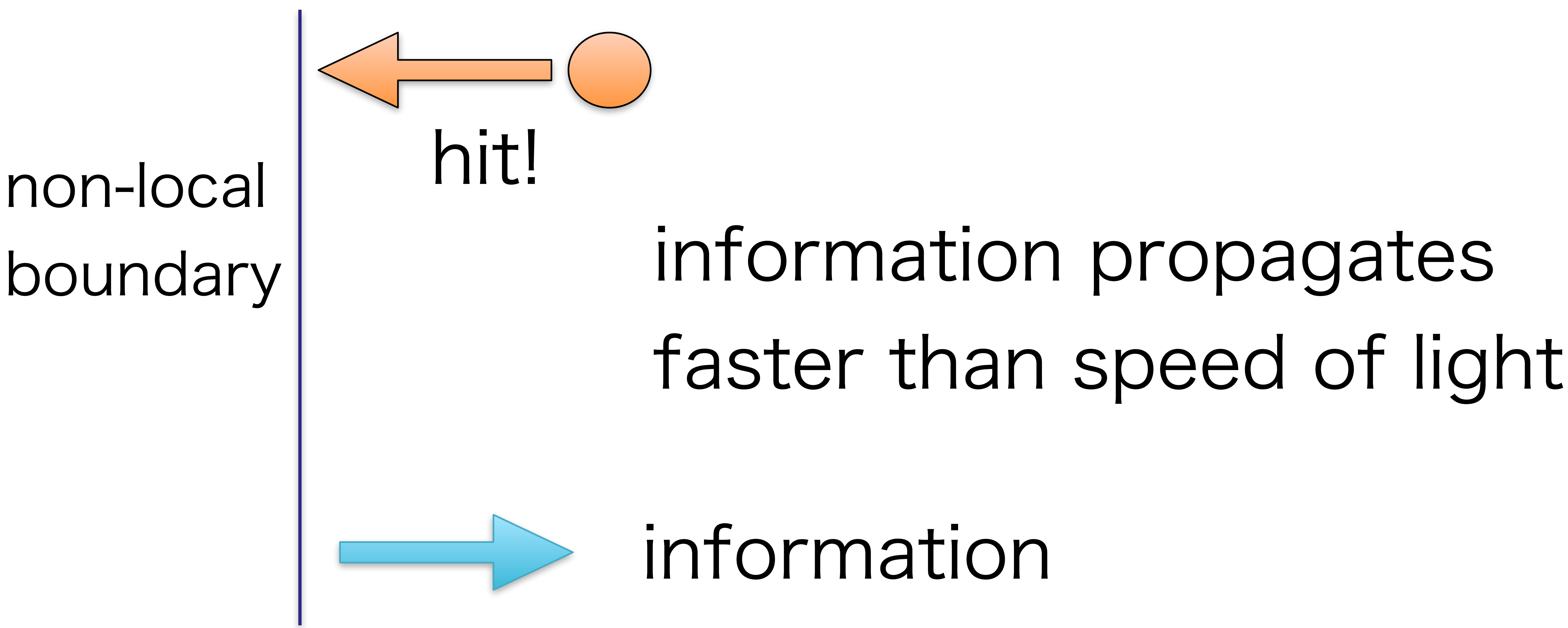

Locality >> chirality for physicists

Locality (=causality) is essential.

We cannot accept APS condition even if it is beautiful.

→ need to give up chirality and consider L/R mixing

(massive case)

$$\cancel{n_+ - n_-} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\,\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

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Can we still make a fermionic integer (even if it is ugly)?

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Can we still make a fermionic integer (even if it is ugly)?

Our answer is "Yes, we can".

Contents

✔ 1. Introduction

We have to consider massive fermions.

2. Eta-invariant of domain-wall Dirac operator

[F, Onogi, Yamaguchi 2017]

3. Mathematical proof

[F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita in progress]

4. Discussion

5. Summary

Atiyah-Singer(AS) index from massive Dirac operator

$$H = \gamma_5(D+M)$$

Zero-modes of D = still eigenstates of H:

$$H\phi_0 = \gamma_5 M\phi_0 = \pm M\phi_0.$$

Non-zero modes make ± pairs

$$H\phi_i = \lambda_i\phi_i \quad HD\phi_i = -DH\phi_i = -\lambda_i D\phi_i$$

$$\eta(H) = \sum_i \mathrm{sgn}\lambda_i$$

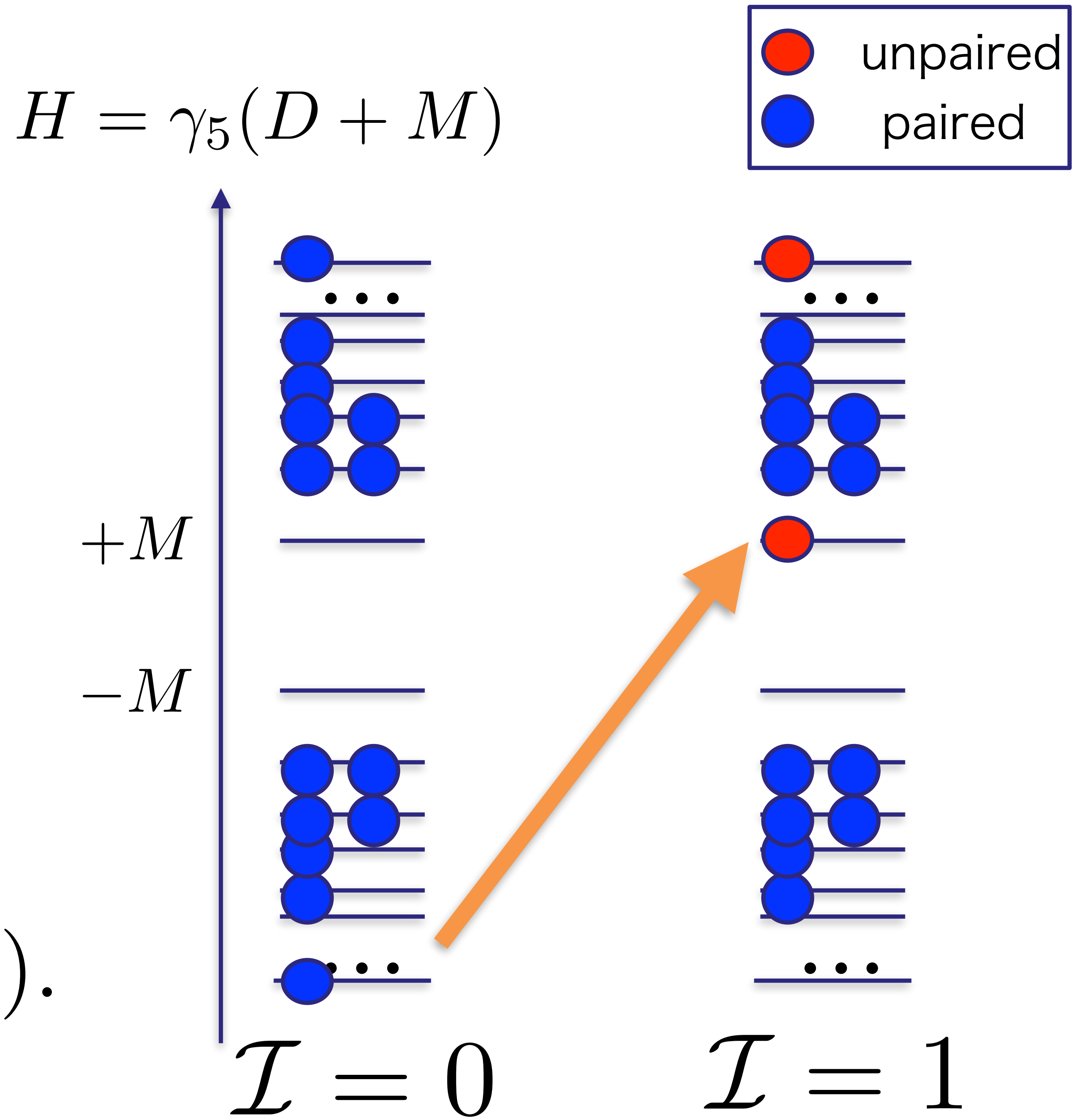
$$= \#\text{ of } +M - \#\text{ of } -M$$ = AS index!

$\eta(H)$ always jumps by 2.

To increase + modes,
we have to borrow
one from - (UV) modes.

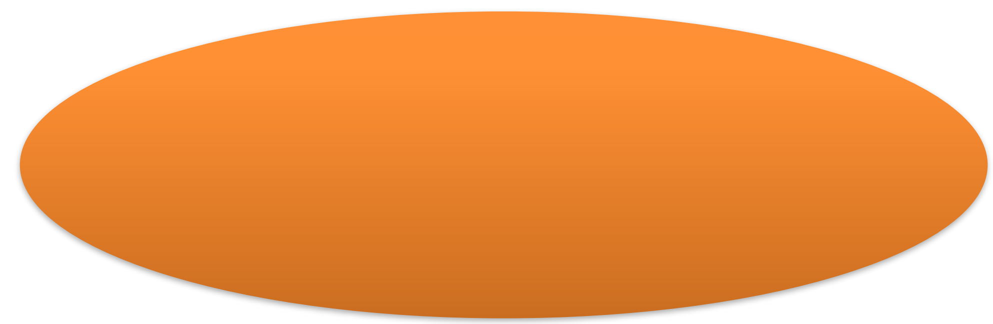
Good regularizations
(e.g. Pauli-Villars, lattice)
respect this fact.

$$\Rightarrow \quad \mathrm{Index}(D) = \frac{1}{2}\eta(H).$$

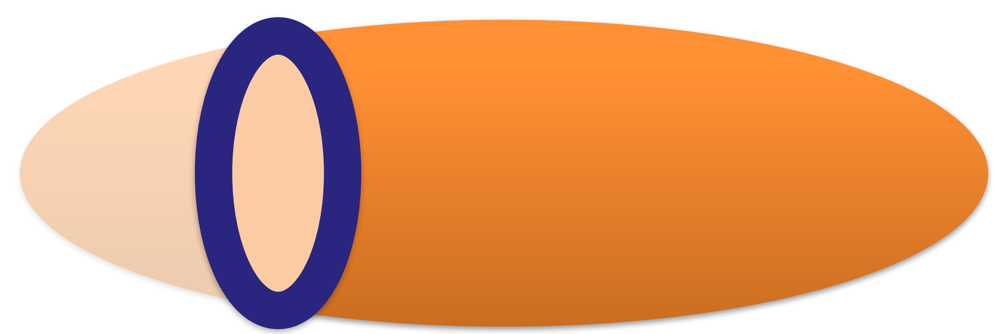


“new” APS index [F-Onogi-Yamaguchi 2017]

$\frac{1}{2}\eta(\gamma_5(D+M))^{reg}$ = AS index



$$\frac{1}{2}\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}$$



$$\epsilon(x_4) = \mathrm{sgn} x_4$$

$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

which can be shown with Fujikawa method.

Trace computation done with free domain-wall fermion eigenmode set

$$\eta(H_{DW}) = \lim_{s\to 0} \mathrm{Tr}\frac{H_{DW}}{(\sqrt{H_{DW}^2})^{1+s}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt t^{-1/2}\mathrm{Tr} H_{DW} e^{-tH_{DW}^2}$$

$$H_{DW} = \gamma_5(D + M\varepsilon(x_4))$$

or solutions to Schrodinger equation with δ-function-like potential:

$$H_{DW}^2 = -\partial_\mu^2 + M^2 - 2M\gamma_4\delta(x_4)$$

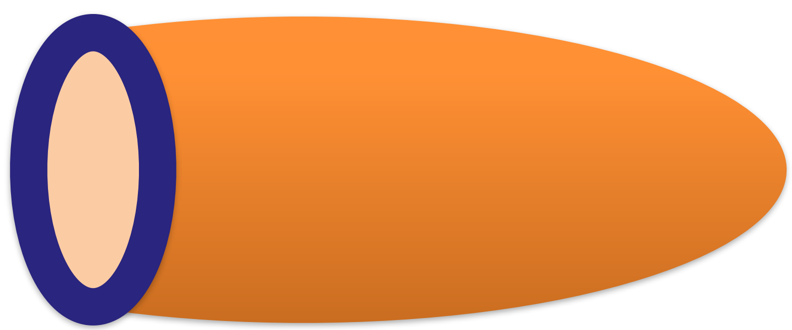
$$\varphi_{\pm,o}^{\omega}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right),$$

$$\varphi_{\pm,e}^{\omega}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega \mp M)e^{i\omega|x_4|} + (i\omega \pm M)e^{-i\omega|x_4|}\right),$$

$\varphi_{+,e}^{\text{edge}}(x_4) = \sqrt{M}e^{-M|x_4|},$ ➡ Edge mode appears !

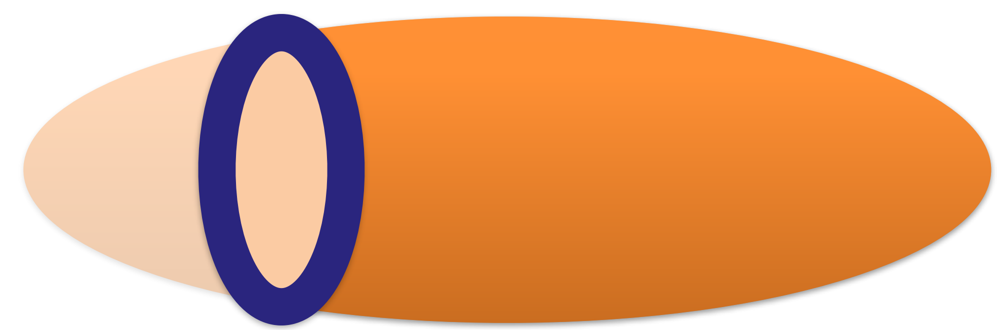
Domain-wall fermion is physicist-friendly

(similar to topological insulators).



APS

1. massless Dirac
   (even in bulk)
2. non-local boundary cond.
   (depending on gauge fields)
3. SO(2) rotational sym. on boundary is lost.
4. no edge mode appears.
5. manifold + boundary



Domain-wall fermion

1. massive Dirac in bulk (massless mode at edge)
2. local boundary cond.
3. SO(2) rotational sym. on boundary is kept.
4. Edge mode describes eta-invariant.
5. closed manifold + domain-wall

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[F, Onogi, Yamaguchi 2017]

$\mathfrak{I} = \eta(\gamma_5(D + M\epsilon(x_4)))^{reg}/2$ coincides with the APS index.

3. Mathematical proof

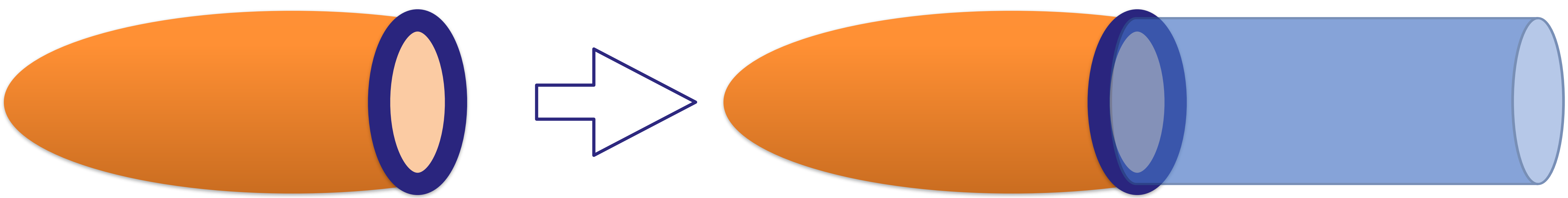
[F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita in progress]

4. Discussion

5. Summary

Theorem 1:
APS index = index with infinite cylinder

In original APS paper, they showed

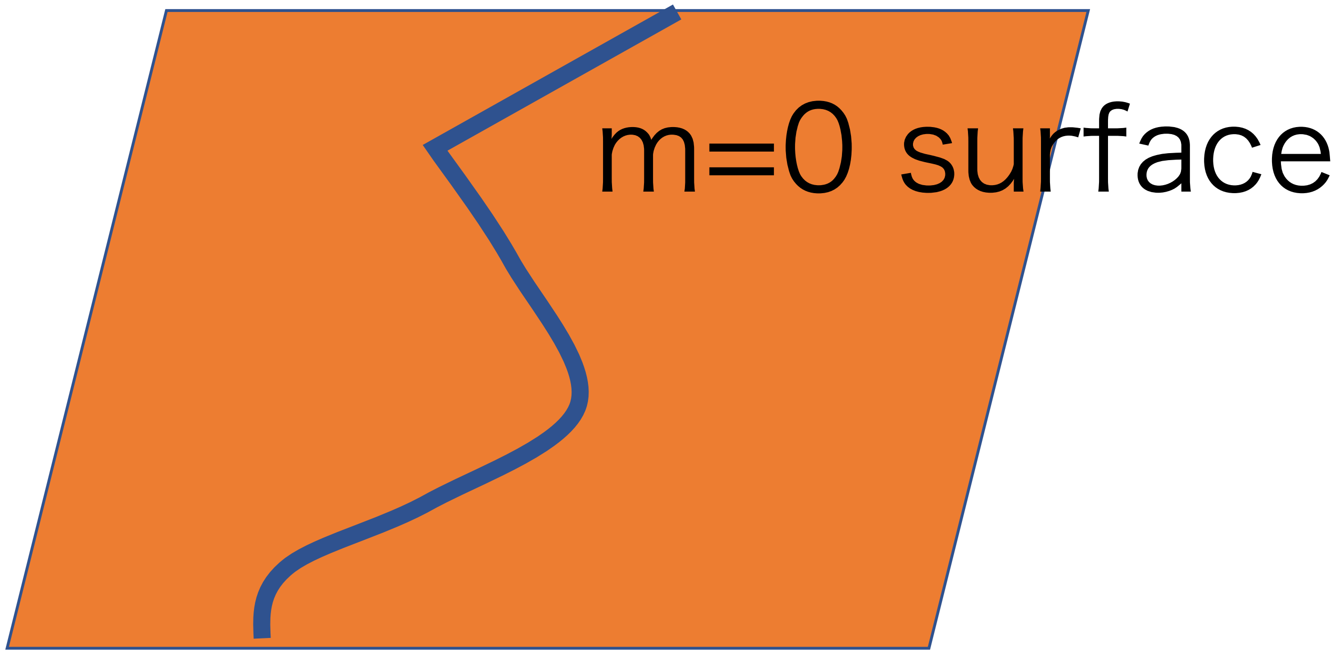


Index w/ APS b.c. = Index with infinite cylinder attached to the original boundary (w.r.t. square integrable modes).

* On cylinder, gauge fields are constant in the extra-direction.

Theorem 2:
Localization (& product formula)

By giving position-dependent "mass", we can localize the zero modes to "massless" lower-dimensional surface and the index is given by the product:



$$Ind(\gamma_s(D^d + \partial_s + i\gamma_s M(s))) =$$
$$Ind(D^d) \times Ind(\gamma_s \partial_s + M(s))$$

= generalization of domain-wall fermion

Theorem 3:
In odd-dim, APS index = boundary eta-invariant

$$\int F \wedge F \wedge \cdots$$

exists only in even-dim.

$$Ind(D_{\mathrm{APS}}^{odd-dim}) = \frac{1}{2}\left[\eta(D^{\mathrm{boundary1}}) - \eta(D^{\mathrm{boundary2}})\right]$$

5-dimensional Dirac operator

we consider

$$D^{5\mathrm{D}} = \begin{pmatrix} 0 & \partial_5 + \gamma_5(D^{4\mathrm{D}} + m(x_5, x_4)) \\ -\partial_5 + \gamma_5(D^{4\mathrm{D}} + m(x_5, x_4)) & 0 \end{pmatrix}$$

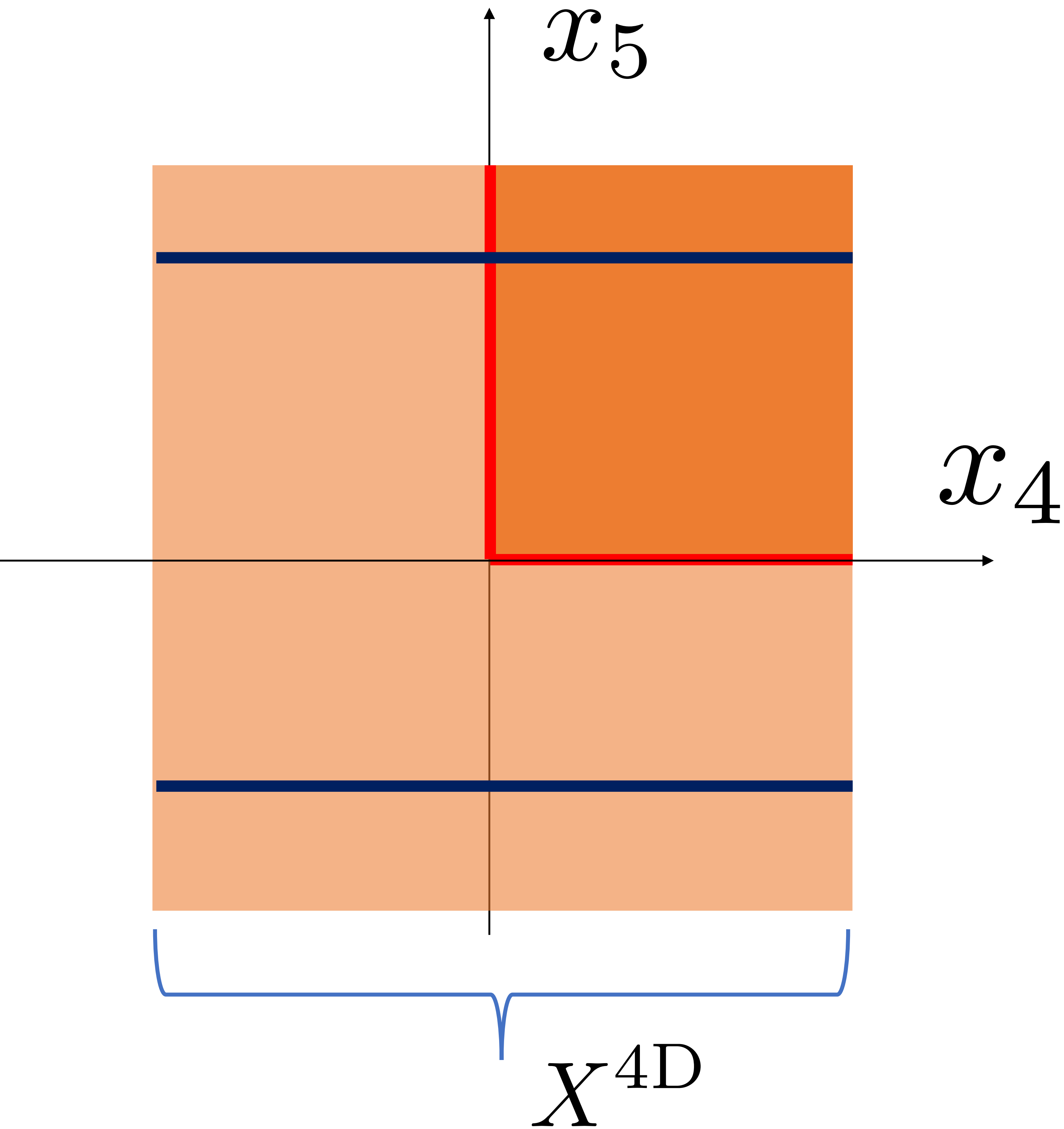
where

$$m(x_5, x_4) = \begin{cases} M & \text{for } x_4 > 0 \text{ \& } x_5 > 0 \\ -M_2 & \text{otherwise} \end{cases}$$

and A_μ is

independent of x_5.

On $X^{4D} \times R$,



we compute

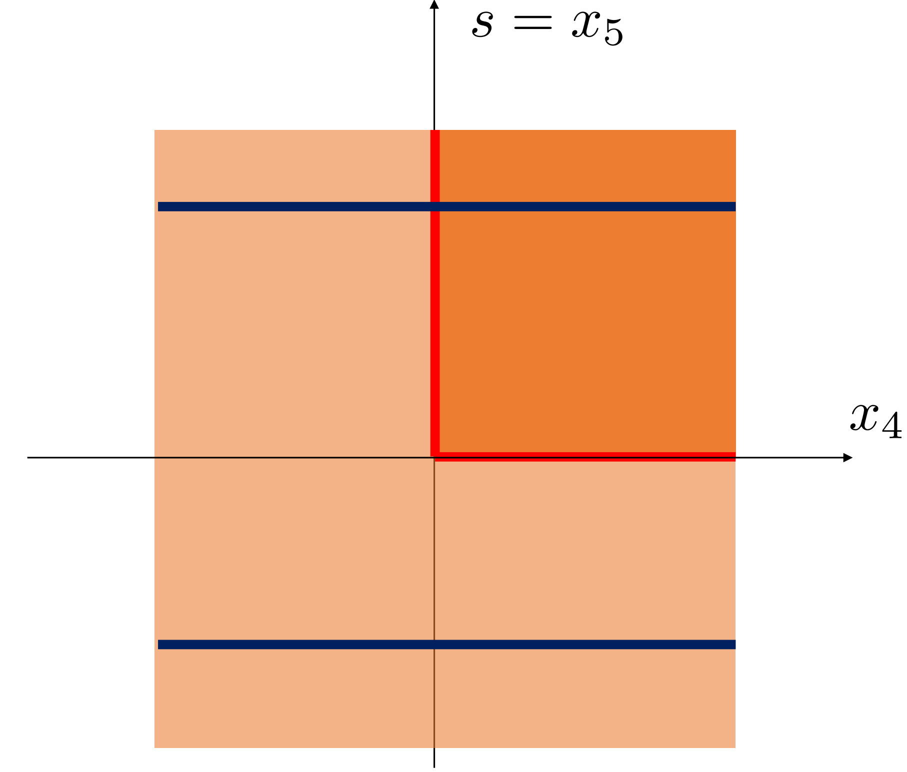
$$Ind(D^{5\mathrm{D}})$$

in two different ways:

1. localization

2. eta-inv. at $s = \pm 1.$

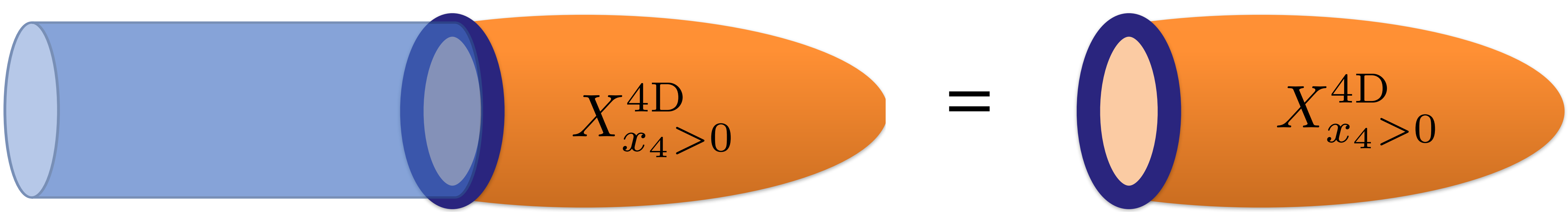
Localization



Theorem 2 tells us

$$Ind(D^{5\mathrm{D}})|_{M, M_2 \to \infty} = Ind(D^{4\mathrm{D}}_{m=0\mathrm{surface}}) \times \underbrace{Ind D^{1\mathrm{D}}_{normal}}_{=1}$$

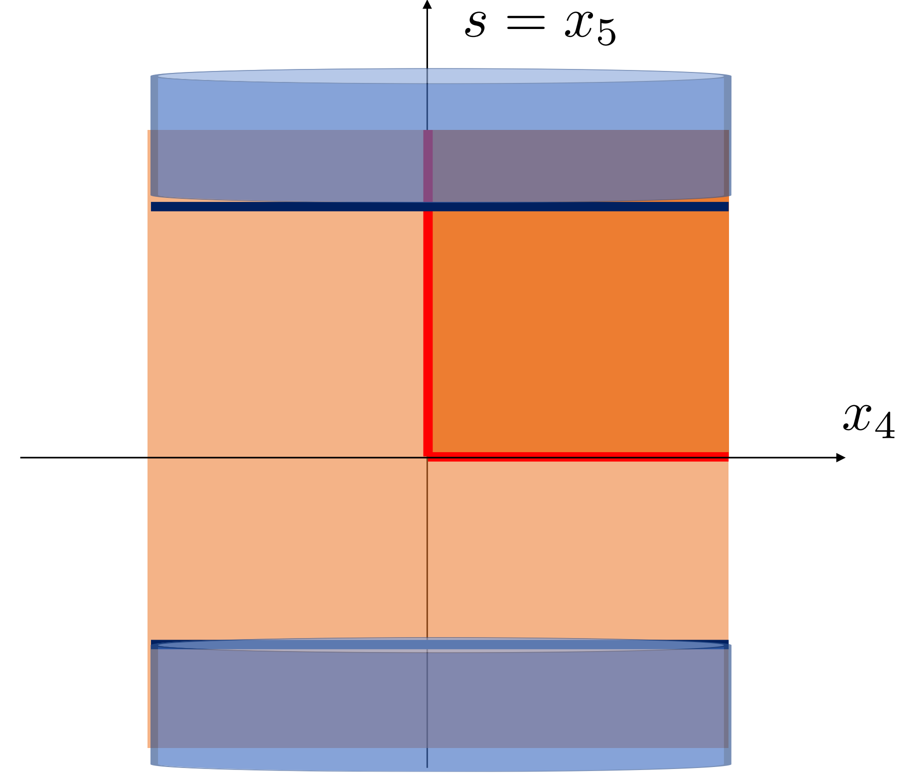
and on the massless surface



theorem 1 indicates

$$Ind(D^{4\mathrm{D}}_{m=0\mathrm{surface}}) = Ind(D^{X^{4\mathrm{D}}_{x_4>0}}_{\mathrm{APS}})$$

Boundary eta invariants



Theorem 1 tells us

$$Ind(D^{\rm 5D}) = Ind(D^{\rm 5D}_{\rm APS\ b.c.\mathit{at s=\pm 1}})$$

and from theorem 3, we obtain

$$Ind(D^{\rm 5D}_{\rm APS\ b.c.\mathit{at s=\pm 1}}) = \frac{1}{2}\left[\eta(D^{\rm 4D}_{s=1}) - \eta(D^{\rm 4D}_{s=-1})\right]$$

$$= \frac{1}{2}\left[\eta(\gamma_5(D^{\rm 4D} + M\epsilon(x_4)) - \eta(\gamma_5(D^{\rm 4D} - M_2)\right] = \frac{1}{2}\eta^{PV reg.}(\gamma_5(D^{\rm 4D} + M\epsilon(x_4))$$

therefore, $Ind(D^{5D}) = Ind(D_{\rm APS}) = \dfrac{1}{2}\eta(H_{DW})$

Q.E.D.

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✔ 3. Mathematical proof

[F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita in progress]

$Ind(D_{\rm APS})$ and $\eta(\gamma_5(D + M\epsilon(x_4)))^{reg}/2$ =the same 5D index!

4. Discussion

5. Summary

Massive fermion : chiral symmetry is NOT important.

The lattice fermion "knew" this fact:

$$Ind(D_{ov}) = \frac{1}{2}\mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) \quad D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right)$$

$$= -\frac{1}{2}\mathrm{Tr}\frac{H_W}{\sqrt{H_W^2}} = -\frac{1}{2}\eta(\gamma_5(D_W - M))!$$

If the original AS index were given by

$$-\frac{1}{2}\eta(\gamma_5(D - M))$$

we should have known the lattice index theorem much before Hasenfratz 1998 or Neuberguer 1998.

Massless vs. massive

index theorems with massless Dirac

	continuum	lattice
AS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$	$\mathrm{Tr}\gamma^5(1-aD_{ov}/2)$
APS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$ w/ APS b.c.	not known.

index theorems with massive Dirac

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	

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Next talk by N. Kawai

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✔ 4. Discussion

eta-invariant gives a united view of index theorems.

5. Summary

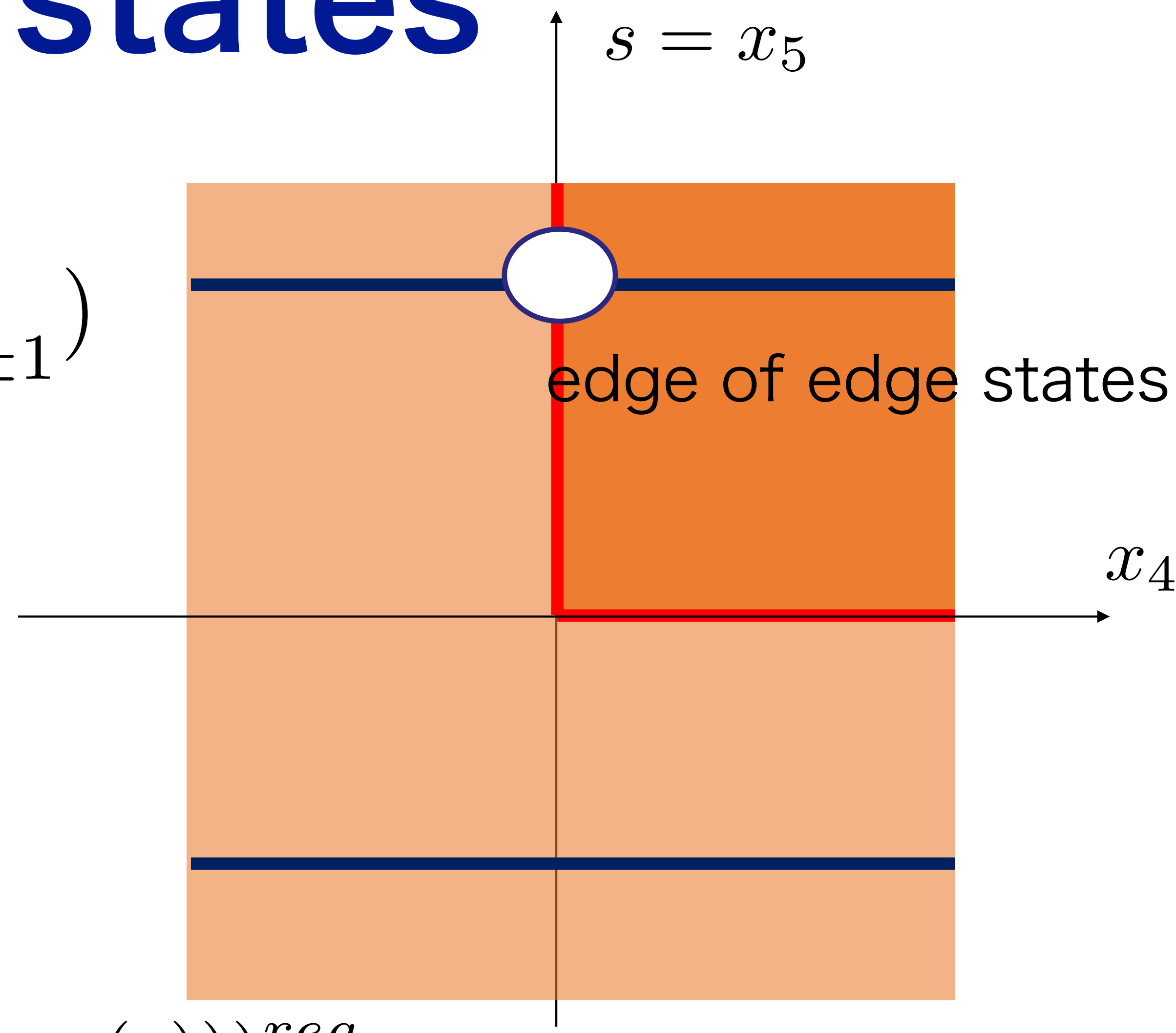
Summary

$$Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW})$$

1. APS index describes bulk-edge correspondence of topological insulators.
2. APS (as well as AS) index can be re-defined by the eta-inv. of massive domain-wall operator.
3. We have given a mathematical proof for general cases by the 5D index.
4. eta-invariant of massive operator gives a unified view of index theorems (including their lattice version).

Backup slides

Recursion of eta-invariants and edge of edge states



We have seen

$$Ind(D^{5\mathrm{D}}) = Ind(D^{5\mathrm{D}}_{\mathrm{APS\ b.c.}ats=\pm 1})$$

$$= \quad Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H^{reg}_{DW})$$

We can recursively use equivalence to show

$$Ind(D^{5\mathrm{D}}_{\mathrm{APS\ b.c.}ats=\pm 1}) = \frac{1}{2}\eta(\gamma_7(D^{5D} + \mu(s)))^{reg.}.$$

then, the original 3D-edge state becomes

edge of edge state in 5D dimension.

[F, Onogi, Yamamoto, Yamamura 2016, Hashimoto, Kimura Wu 2016]

Example : 1+1d bulk + 0+1d edge Majorana fermion coupled to gravity

APS index tells

$$Z \propto \exp\left(2\pi i \frac{n}{8}\right)$$

consistent with Z_8 classification of Kitaev's interacting Majorana chain.

Eta invariant = Chern Simons term + integer (non-local effect)

$$\frac{\eta(iD^{3\mathrm{D}})}{2} = \frac{CS}{2\pi} + \text{integer}$$

$$CS \equiv \frac{1}{4\pi}\int_Y d^3x \;\mathrm{tr}_c\left[\epsilon_{\nu\rho\sigma}\left(A^\nu\partial^\rho A^\sigma + \frac{2i}{3}A^\nu A^\rho A^\sigma\right)\right],$$

= surface term.

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$