Discretization of abelian topological charge on the lattice

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Introduction

- Topological terms are an important ingredient for many quantum field theories in high energy and condensed matter physics.
- Non-perturbative approaches are needed for proper description of topological terms.
- Suitable discretization of the topological charge is essential to correctly implement symmetries on the lattice.
- Topological terms generate a complex action problem that must be overcome for Monte Carlo simulations. Worldsheet representation?

Usually matter fields are coupled with compact link variables in n.n. terms $\Phi_x^* U_{x,\mu} \Phi_{x+\mu}$

$$U_{x,\mu} = e^{iA_{x,\mu}}, \quad A_{x,\mu} \in [-\pi, \pi]$$

The non-compact variables $A_{x,\mu}$ are invariant under the "shift symmetry".

$$A_{x,\mu} \to A_{x,\mu} + 2\pi k_{x,\mu}, \quad k_{x,\mu} \in \mathbb{Z}$$

We define the non-compact field strength tensor as

$$(dA)_{x,\mu\nu} \equiv A_{x+\hat{\mu},\nu} - A_{x,\nu} - A_{x+\hat{\nu},\mu} + A_{x,\mu}$$

that is not invariant under the shift

$$(dA)_{x,\mu\nu} \to (dA)_{x,\mu\nu} + 2\pi (dk)_{x,\mu\nu}, \quad (dk)_{x,\mu\nu} \in \mathbb{Z}$$

Boltzmann factor invariant under shifts

Villain-Boltzmann factor:

$$B[A] = \prod_{x,\mu<\nu} \sum_{n_{x,\mu\nu} \in Z} e^{-\frac{\beta}{2}\left((dA)_{x,\mu\nu} + 2\pi n_{x,\mu\nu}\right)^2}$$

$n_{x,\mu\nu} \in \mathbb{Z}$ eats up shift contributions $(dk)_{x,\mu\nu}$.
Due to $(d(dk))_{x,\mu\nu\rho} = 0$ it is sufficient to sum over configurations of the Villain variables $n_{x,\mu\nu}$ that obey

$$(dn)_{x,\mu\nu\rho} = 0 \qquad (1)$$

We will impose the closedness constraint (1) when constructing the topological charge.

Boltzmann factor with topological charge

We now generalize the Villain-Boltzmann factor to

$$B[A] = \sum_{\{n\}} e^{-S_G[dA+2\pi n] - i\theta Q_k[dA+2\pi n]}$$

$Q_k[dA + 2\pi n]$ is topological charge. The family index $k \in \mathbb{Z}$ labels different possible discretizations of Q.

$$\sum_{\{n\}} = \prod_{x,\mu<\nu} \cdot \sum_{n_{x,\mu\nu} \in \mathbb{Z}} \cdot \prod_{x,\mu<\nu<\rho} \delta\Big((dn)_{x,\mu\nu\rho}\Big)$$

$\prod_{x,\mu<\nu<\rho} \delta\Big((dn)_{x,\mu\nu\rho}\Big) \quad \longrightarrow \quad (dn)_{x,\mu\nu\rho} = 0$ on all 3-cubes.

Definition of the topological charge

$$Q_k[dA+2\pi n] = \frac{1}{8\pi^2} \sum_x \sum_{\mu<\nu} \sum_{\rho<\sigma} \epsilon_{\mu\nu\rho\sigma} (dA+2\pi n)_{x,\mu\nu} (dA+2\pi n)_{x-k\hat{s}-\hat{\rho}-\hat{\sigma},\rho\sigma}$$

$$\hat{s} = \hat{1} + \hat{2} + \hat{3} + \hat{4}, \quad k \in \mathbb{Z}$$

Theorem:

$$Q_k[dA + 2\pi n] = Q_k[2\pi n] \equiv Q_k[n]$$

for arbitrary $A_{x,\mu}$, as long as $(dn)_{x,\mu\nu\rho} = 0$

Thus we find that

$$Q_k[n] = \frac{1}{2} \sum_x \sum_{\mu<\nu} \sum_{\rho<\sigma} \varepsilon_{\mu\nu\rho\sigma} n_{x,\mu\nu} n_{x-k\hat{s}-\hat{\rho}-\hat{\sigma},\rho\sigma}$$

Hodge decomposition of $n_{x,\mu\nu}$

Hodge decomposition:

$$n_{x,\mu\nu} = (\delta c)_{x,\mu\nu} + (dl)_{x,\mu\nu} + h_{x,\mu\nu}$$

Since $(dn)_{x,\mu\nu\rho} = 0$ we can write $n_{x,\mu\nu}$ as

$$n_{x,\mu\nu} = (dl)_{x,\mu\nu} + h_{x,\mu\nu}$$

The theorem implies: $Q_k[n] = Q_k[dl + h] = Q_k[h]$

$Q_k[n]$ couples only to the harmonics in the Hodge decomposition of $n_{x,\mu\nu}$.

Explicit evaluation of $Q_k[n]$

The harmonic contributions can be chosen as:

$$h_{x,\mu\nu} = \omega_{\mu\nu} \sum_{i=1}^{N_\rho} \sum_{j=1}^{N_\sigma} \delta_{x,i\hat{\rho}+j\hat{\sigma}}, \quad \rho \neq \mu,\nu, \sigma \neq \mu,\nu, \rho \neq \sigma$$

$$h_{x,\mu\nu} = \begin{cases} \omega_{\mu\nu} \in \mathbb{Z} \quad \forall x \quad \text{in the } \hat{\rho} \text{ - } \hat{\sigma} \text{ plane } \perp \hat{\mu} \text{ - } \hat{\nu} \text{ through the origin} \\ 0 \qquad \text{else} \end{cases}$$

Therefore the topological term takes the form:

$$Q_k[n] = Q_k[dl + h] = Q_k[h] = \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} \quad \in \mathbb{Z} \quad \forall k$$

Result is independent of k!

Partition sum

$$Z = \int D[A] \sum_{\{n\}} \prod_{x,\mu<\nu<\rho} \delta\Big((dn)_{x,\mu\nu\rho}\Big) \cdot e^{-S_G[F]-i\theta Q_k[F]}$$

$$S_G[F] = \frac{\beta}{2} \sum_{x,\mu<\nu} (F_{x,\mu\nu})^2$$

$$Q_k[F] = \frac{1}{8\pi^2} \sum_x \sum_{\mu<\nu} \sum_{\rho<\sigma} \epsilon_{\mu\nu\rho\sigma} F_{x,\mu\nu} F_{x-\hat{s}k-\hat{\rho}-\hat{\sigma},\rho\sigma} \in \mathbb{Z}$$

with:

$$F_{x,\mu\nu} = (dA)_{x,\mu\nu} + 2\pi n_{x,\mu\nu} \qquad \int D[A] = \prod_{x,\mu} \frac{1}{2\pi} \int_{-\pi}^{\pi} dA_{x,\mu}$$

Properties of the mixed compact/non-compact discretization

- The mixed compact/non-compact discretization gives rise to the Villain action.
- We generalize the Villain action to include a topological term.
- Restricting the Villain variables to be closed $(dn)_{x,\mu\nu\rho} = 0$ we find $Q_k[F] = Q_k[n] = \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} \quad \in \mathbb{Z} \quad \forall k$.
- One may show that $Q_k[F]$ is fully self dual (see talk *C. Gattringer*).
- The freedom to choose k can be exploited to construct self dual $U(1)$ LGT with a topological term.

The simple case of $d = 2$

$$Z = \int D[A] \sum_{\{n\}} \prod_x e^{-S_G[F] - i\theta Q[F]}$$

$$S_G[F] = \frac{\beta}{2} \sum_x F_{x,12}^2 \qquad Q[F] = \frac{1}{2\pi} \sum_{x,\mu<\nu} \epsilon_{\mu\nu} F_{x,\mu\nu}$$

$$F_{x,12} = (dA)_{x,12} + 2\pi n_x$$

$$Q[F] = \sum_x n_x \quad \in \mathbb{Z}$$

Application of $d = 2$ discretization

- Using our construction the U(1) gauge Higgs model in 2-d was studied at $\theta = \pi$. It was possible to identify the critical point where charge conjugation symmetry becomes broken. $\longrightarrow$ Talk by Daniel Goeschl

  D.Goeschl, C.Gattringer, T.Sulejmanpasic N.P.B 2019

- Index theorem $Q[F] = n_- - n_+$ was explored with our discretization and shown to hold towards continuum limit.
  C.Gattringer, P.Toerek arXiv:1905.03963v1

- Plan: Explore similar questions with the 4-d formulation presented here.

Summary

1. We discretize U(1) LGT using a mixed compact/non-compact scheme $\rightarrow$ Villain action.
2. Restriction of Villain variables: closedness condition $(dn)_{x,\mu\nu\rho} = 0$.
3. We generalize Villain-Boltzman factor to include $Q_k[F]$.
4. $Q_k[F] = Q_k[n] = \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} \quad \in \mathbb{Z} \quad \forall k$.
5. A modification of the gauge action allows for a completely self dual system. (In progress)
6. Outlook: Study various 4-d physical systems with the topological term based on our construction.

Thank you for attention