Discretization of abelian topological charge on the lattice

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Introduction

- **Topological terms** are an important ingredient for many quantum field theories in high energy and condensed matter physics.

- **Non-perturbative approaches are needed** for proper description of topological terms.

- Suitable discretization of the topological charge is essential to correctly implement symmetries on the lattice.

- Topological terms generate a complex action problem that must be overcome for Monte Carlo simulations. *Worldsheet representation?
Usually matter fields are coupled with compact link variables in n.n. terms
\[ \Phi_x^* U_{x,\mu} \Phi_{x+\mu} \]
\[ U_{x,\mu} = e^{i A_{x,\mu}}, \quad A_{x,\mu} \in [-\pi, \pi] \]

The non-compact variables \( A_{x,\mu} \) are invariant under the "shift symmetry".
\[ A_{x,\mu} \rightarrow A_{x,\mu} + 2\pi k_{x,\mu}, \quad k_{x,\mu} \in \mathbb{Z} \]

We define the non-compact field strength tensor as
\[ (dA)_{x,\mu\nu} \equiv A_{x+\hat{\mu},\nu} - A_{x,\nu} - A_{x+\hat{\nu},\mu} + A_{x,\mu} \]
that is not invariant under the shift
\[ (dA)_{x,\mu\nu} \rightarrow (dA)_{x,\mu\nu} + 2\pi (dk)_{x,\mu\nu}, \quad (dk)_{x,\mu\nu} \in \mathbb{Z} \]
**Boltzmann factor invariant under shifts**

Villain-Boltzmann factor:

\[
B[A] = \prod_{x, \mu < \nu} \sum_{n_{x, \mu \nu} \in \mathbb{Z}} e^{-\frac{\beta}{2} \left( (dA)_{x, \mu \nu} + 2\pi n_{x, \mu \nu} \right)^2}
\]

\(n_{x, \mu \nu} \in \mathbb{Z}\) eats up shift contributions \((dk)_{x, \mu \nu}\).

Due to \((d(dk))_{x, \mu \nu \rho} = 0\) it is sufficient to sum over configurations of the Villain variables \(n_{x, \mu \nu}\) that obey

\[
(dn)_{x, \mu \nu \rho} = 0 \quad (1)
\]

We will impose the closedness constraint \((1)\) when constructing the topological charge.
Boltzmann factor with topological charge

We now generalize the Villain-Boltzmann factor to

\[ B[A] = \sum_{\{n\}} e^{-S_G[dA + 2\pi n] - i\theta Q_k[dA + 2\pi n]} \]

\( Q_k[dA + 2\pi n] \) is topological charge. The family index \( k \in \mathbb{Z} \) labels different possible discretizations of \( Q \).

\[ \sum_{\{n\}} = \prod_{x,\mu<\nu} \sum_{n_{x,\mu\nu} \in \mathbb{Z}} \prod_{x,\mu<\nu<\rho} \delta\left((dn)_{x,\mu\nu\rho}\right) \]

\[ \prod_{x,\mu<\nu<\rho} \delta\left((dn)_{x,\mu\nu\rho}\right) \rightarrow (dn)_{x,\mu\nu\rho} = 0 \text{ on all 3-cubes.} \]
Definition of the topological charge

\[ Q_k[dA+2\pi n] = \frac{1}{8\pi^2} \sum_x \sum_{\mu<\nu} \sum_{\rho<\sigma} \epsilon_{\mu\nu\rho\sigma} (dA+2\pi n)_{x,\mu\nu} (dA+2\pi n)_{x-k\hat{s}-\hat{\rho}-\hat{\sigma},\rho\sigma} \]

\[ \hat{s} = \hat{1} + \hat{2} + \hat{3} + \hat{4}, \quad k \in \mathbb{Z} \]

Theorem:

\[ Q_k[dA + 2\pi n] = Q_k[2\pi n] \equiv Q_k[n] \]

for arbitrary \( A_{x,\mu} \), as long as \( (dn)_{x,\mu\nu\rho} = 0 \)

Thus we find that

\[ Q_k[n] = \frac{1}{2} \sum_x \sum_{\mu<\nu} \sum_{\rho<\sigma} \epsilon_{\mu\nu\rho\sigma} n_{x,\mu\nu} n_{x-k\hat{s}-\hat{\rho}-\hat{\sigma},\rho\sigma} \]
Hodge decomposition of $n_{x,\mu\nu}$

Hodge decomposition:

$$n_{x,\mu\nu} = (\delta c)_{x,\mu\nu} + (dl)_{x,\mu\nu} + h_{x,\mu\nu}$$

Since $(dn)_{x,\mu\nu\rho} = 0$ we can write $n_{x,\mu\nu}$ as

$$n_{x,\mu\nu} = (dl)_{x,\mu\nu} + h_{x,\mu\nu}$$

The theorem implies: $Q_k[n] = Q_k[dl + h] = Q_k[h]$

$Q_k[n]$ couples only to the harmonics in the Hodge decomposition of $n_{x,\mu\nu}$. 
Explicit evaluation of \( Q_k[n] \)

The harmonic contributions can be chosen as:

\[
h_{x,\mu\nu} = \omega_{\mu\nu} \sum_{i=1}^{N_\rho} \sum_{j=1}^{N_\sigma} \delta_{x,\hat{\rho} + j\hat{\sigma}}, \quad \rho \neq \mu, \nu, \sigma \neq \mu, \nu, \rho \neq \sigma
\]

\[
h_{x,\mu\nu} = \begin{cases} 
\omega_{\mu\nu} \in \mathbb{Z} & \forall x \text{ in the } \hat{\rho} - \hat{\sigma} \text{ plane } \perp \hat{\mu} - \hat{\nu} \text{ through the origin} \\
0 & \text{else}
\end{cases}
\]

Therefore the topological term takes the form:

\[
Q_k[n] = Q_k[dl + h] = Q_k[h] = \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} \in \mathbb{Z} \quad \forall k
\]

Result is independent of \( k \)!
Partition sum

\[ Z = \int D[A] \sum_{\{n\}} \prod_{x,\mu<\nu<\rho} \delta\left((dn)_x,\mu\nu\rho\right) \cdot e^{-S_G[F] - i\theta Q_k[F]} \]

\[ S_G[F] = \frac{\beta}{2} \sum_{x,\mu<\nu} (F_{x,\mu\nu})^2 \]

\[ Q_k[F] = \frac{1}{8\pi^2} \sum_x \sum_{\mu<\nu} \sum_{\rho<\sigma} \epsilon_{\mu\nu\rho\sigma} F_{x,\mu\nu} F_{x-\hat{s}k-\hat{\rho}-\hat{\sigma},\rho\sigma} \in \mathbb{Z} \]

with:

\[ F_{x,\mu\nu} = (dA)_x,\mu\nu + 2\pi n_{x,\mu\nu} \quad \int D[A] = \prod_{x,\mu} \frac{1}{2\pi} \int_{-\pi}^{\pi} dA_{x,\mu} \]
Properties of the mixed compact/non-compact discretization

- The mixed compact/non-compact discretization gives rise to the Villain action.
- We generalize the Villain action to include a topological term.
- Restricting the Villain variables to be closed \((dn)_{\mu\nu\rho} = 0\) we find
  \[Q_k[F] = Q_k[n] = \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} \in \mathbb{Z} \quad \forall k.\]
- One may show that \(Q_k[F]\) is fully self dual (see talk C. Gattringer).
- The freedom to choose \(k\) can be exploited to construct self dual \(U(1)\) LGT with a topological term.
The simple case of $d = 2$

$$Z = \int D[A] \sum_{\{n\}} \prod_x e^{-S_G[F] - i\theta Q[F]}$$

$$S_G[F] = \frac{\beta}{2} \sum_x F^2_{x,12} \quad Q[F] = \frac{1}{2\pi} \sum_{x,\mu<\nu} \epsilon_{\mu\nu} F_{x,\mu\nu}$$

$$F_{x,12} = (dA)_{x,12} + 2\pi n_x$$

$$Q[F] = \sum_x n_x \quad \in \mathbb{Z}$$
Application of $d = 2$ discretization

- Using our construction the $U(1)$ gauge Higgs model in 2-d was studied at $\theta = \pi$. It was possible to identify the critical point where charge conjugation symmetry becomes broken. → Talk by Daniel Goeschl


- Index theorem $Q[F] = n_- - n_+$ was explored with our discretization and shown to hold towards continuum limit.

*C.Gattringer, P.Toerek* arXiv:1905.03963v1

- Plan: Explore similar questions with the 4-d formulation presented here.
Summary

1. We discretize U(1) LGT using a mixed compact/non-compact scheme → Villain action.

2. Restriction of Villain variables: closedness condition \((dn)_{x,\mu\nu\rho} = 0\).

3. We generalize Villain-Boltzman factor to include \(Q_k[F]\).

4. \(Q_k[F] = Q_k[n] = \omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} \in \mathbb{Z} \ \forall k\).

5. A modification of the gauge action allows for a completely self dual system. (In progress)

6. Outlook: Study various 4-d physical systems with the topological term based on our construction.
Thank you for attention