New developments for worldline and worldsheet representations of lattice field theories

$\Rightarrow$ Construction of self-dual U(1) lattice gauge theory with a topological term

Christof Gattringer, Maria Anosova, Tin Sulejmanpasic

C. Gattringer, D. Göschl, T. Sulejmanpasic, Nucl. Phys. B935 (2018) [arXiv:1807.07793]
T. Sulejmanpasic, C. Gattringer, Nucl. Phys. B943 (2019) [arXiv:1901.02637]
M. Anosova, C. Gattringer, T. Sulejmanpasic, work in preparation

Introductory comments

- Many lattice fields theories can be exactly rewritten using worldlines and worldsheets as the new degrees of freedom.
- The WL/WS description highlights geometrical aspects of a theory and often leads to a solution of complex action problems.
- Successful examples of scalar- and gauge-Higgs models at finite chemical potential.
- More recently: Dualization of 2-d systems with topological term.

See the talk of Daniel Göschl, Friday 16:30

In this talk we analyze U(1) lattice gauge theory with a topological term in 4-d and show that for a suitable discretization the theory becomes self-dual.

Setting for U(1) lattice gauge theory with topological terms

Villain formulation:

$$Z \;=\; \int D[A] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_{x,\mu<\nu} F_{x,\mu\nu}F_{x,\mu\nu}}$$

$$(dA)_{x,\mu\nu} \;=\; A_{x+\hat{\mu},\nu} - A_{x,\nu} - A_{x+\hat{\nu},\mu} + A_{x,\mu}$$

$$F_{x,\mu\nu} \;=\; (dA)_{x,\mu\nu} + 2\pi n_{x,\mu\nu}$$

$$\sum_{\{n\}} = \prod_{x,\mu<\nu} \sum_{n_{x,\mu\nu}}$$

- Link variables $U_{x,\mu} \;=\; e^{iA_{x,\mu}}$ are invariant under $A_{x,\mu} \to A_{x,\mu} + 2\pi\, k_{x,\mu}$
- Exterior derivatives transform as $(dA)_{x,\mu\nu} \to (dA)_{x,\mu\nu} + 2\pi\, (dk)_{x,\mu\nu}$
- Summation over the Villain variables $n_{x,\mu\nu}$ eats up the shifts $(dk)_{x,\mu\nu}$
- Exterior derivative is nilpotent $d^2 = 0$ and we can impose the closedness constraint

$$(dn)_{x,\mu\nu\rho} \;=\; 0 \qquad \forall \quad \text{cubes} \quad (x,\mu\nu\rho)$$

Generalization and topological terms

Generalized Villain formulation with constraints

$$Z \;=\; \int D[A] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_{x,\mu<\nu} F_{x,\mu\nu} F_{x,\mu\nu}} \prod_{\substack{x \\ \mu<\nu<\rho}} \delta\Big((dn)_{x,\mu\nu\rho}\Big) \quad , \qquad F_{x,\mu\nu} \;=\; (dA)_{x,\mu\nu} + 2\pi\, n_{x,\mu\nu}$$

Constraining the Villain variables allows one to introduce a new type of topological term

$$Z \;=\; \int D[A] \sum_{\{n\}} e^{-\, S_\beta[F] \,-\, i\,\theta\, Q[F]} \prod_{\substack{x \\ \mu<\nu<\rho}} \delta\Big((dn)_{x,\mu\nu\rho}\Big)$$

with

$$Q[F] \;=\; \frac{1}{32\pi^2} \sum_x F_{x,\mu\nu}\, \epsilon_{\mu\nu\rho\sigma} F_{x-\hat\rho-\hat\sigma-k\hat s,\rho\sigma} \quad , \qquad \hat s = \hat 1 + \hat 2 + \hat 3 + \hat 4 \;, \; k \in \mathbb{Z}$$

Because of the constraints one can show that: (see next talk by Maria Anosova)

- $Q[F]$ is an integer independent of the parameter k
- $Q[F]$ is determined by the harmonics in the Hodge decomposition of $n_{x,\mu\nu}$

Quadratic form and further generalization to self duality

We write action and topological term as quadratic form:

$$Z = \int D[A] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_{x,\mu<\nu \atop y,\rho<\sigma} F_{x,\mu\nu} K_{x,\mu\nu|y,\rho\sigma} F_{y,\rho\sigma}} \prod_{x \atop \mu<\nu<\rho} \delta\Big((dn)_{x,\mu\nu\rho}\Big) \quad , \quad F_{x,\mu\nu} \; = \; (dA + 2\pi n)_{x,\mu\nu}$$

The kernel $K_{x,\mu\nu|y,\rho\sigma}$ generalizes the action to allow for self duality:

$$K_{\,x,\mu\nu|y,\rho\sigma} \; = \; \sum_{k=0}^{\infty} c_k \, \gamma^{\,2k} \left[\delta_{\mu\rho}\, \delta_{\nu\sigma}\, \delta^{(4)}_{x-k\hat{s},y} \; + \; i\,\gamma\,\epsilon_{\mu\nu\rho\sigma}\; \delta^{(4)}_{x-\hat{\rho}-\hat{\sigma}-k\hat{s},y} \right]$$

$$c_k = (-1)^k \, (2k-1)!! \, / \, (2k)!!$$

$$\theta \; = \; \beta\, 4\pi^2 \sum_{k=0}^{\infty} c_k \, \gamma^{\,2k} \; = \; \beta\, 4\pi^2 \, \frac{\gamma}{\sqrt{1+\gamma^2}}$$

Expansion coefficients $c_k \, \gamma^{\,2k}$ decay exponentially for $\beta > \mid \theta \mid / 2\pi^2$ and the action is local.

$$K^{\,-1}_{\,x,\mu\nu|y,\rho\sigma} \; = \; \sum_{k=0}^{\infty} c_k \, \gamma^{\,2k} \left[\delta_{\mu\rho}\, \delta_{\nu\sigma}\, \delta^{(4)}_{x-k\hat{s},y} \; - \; i\,\gamma\,\epsilon_{\mu\nu\rho\sigma}\; \delta^{(4)}_{x-\hat{\rho}-\hat{\sigma}-k\hat{s},y} \right]$$

Rewrite the constraints with cube-based variables

Use integral representation of the Kronecker deltas in the constraints:

$$\prod_{\substack{x \\ \mu<\nu<\rho}} \delta\Big((dn)_{x,\mu\nu\rho}\Big) = \prod_{\substack{x \\ \mu<\nu<\rho}} \int_{-\pi}^{\pi} \frac{d\alpha_{x,\mu\nu\rho}}{2\pi}\, e^{i\alpha_{x,\mu\nu\rho}(dn)_{x,\mu\nu\rho}}$$

$$= \int D[\alpha]\, e^{\frac{i}{2\pi}\sum \alpha_{x,\mu\nu\rho}(d\,F)_{x,\mu\nu\rho}} = \int D[\alpha]\, e^{-\frac{i}{2\pi}\sum_{x,\mu<\nu}(\partial\,\alpha)_{x,\mu\nu}F_{x,\mu\nu}}$$

Partition sum is converted into a Gaussian integral ...

$$Z = \int D[\alpha] \int D[A] \sum_{\{n\}} e^{-\frac{\beta}{2}\sum_{\substack{x,\mu<\nu \\ y,\rho<\sigma}} (dA+2\pi n)_{x,\mu\nu}\, K_{x,\mu\nu|y,\rho\sigma}\, (dA+2\pi n)_{y,\rho\sigma} - \frac{i}{2\pi}\sum_{x,\mu<\nu}(\partial\,\alpha)_{x,\mu\nu}(dA+2\pi n)_{x,\mu\nu}}$$

... which can be solved with a generalized Poisson resummation formula $\Rightarrow$

$$Z = C \int D[\alpha] \sum_{\{p\}} e^{-\frac{\tilde{\beta}}{2}\sum_{\substack{x,\mu<\nu \\ y,\rho<\sigma}} (\partial\alpha+2\pi p)_{x,\mu\nu}\, K^{-1}_{x,\mu\nu|y,\rho\sigma}\, (\partial\alpha+2\pi p)_{y,\rho\sigma}} \int D[A]\, e^{-i\sum_{x,\mu<\nu}(dA)_{x,\mu\nu}p_{x,\mu\nu}}$$

$$\tilde{\beta} = \frac{1}{4\pi^2\,\beta} \quad , \qquad p_{x,\mu\nu} \in \mathbb{Z}$$

Gauge field integration generates worldsheet constraints

Gauge field integration generates constraints for the plaquette occupation numbers $p_{x,\mu\nu}$

$$\int D[A] \, e^{-i \sum_{x,\mu<\nu} (d\, A)_{x,\mu\nu} \, p_{x,\mu\nu}} \; = \; \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi} \, e^{\, i \, A_{x,\mu} (\partial\, p)_{x,\mu}} \; = \; \prod_{x,\mu} \delta\Big((\partial\, p)_{x,\mu} \Big)$$

The generalized zero divergence condition $(\partial\, p)_{x,\mu} = 0$ forces the total flux $(\partial\, p)_{x,\mu}$ from all plaquette occupation numbers at a link (x,μ) to vanish $\Rightarrow$ $p_{x,\mu\nu}$ form closed worldsheets.

Partition sum in worldsheet representation

$$Z \; = \; C \int D[\alpha] \sum_{\{p\}} e^{-\frac{\tilde{\beta}}{2} \sum_{\substack{x,\mu<\nu \\ y,\rho<\sigma}} (\partial\alpha + 2\pi p)_{x,\mu\nu} \, K^{\,-1}_{\,x,\mu\nu|y,\rho\sigma} \, (\partial\alpha + 2\pi p)_{y,\rho\sigma}} \; \prod_{x,\mu} \delta\Big((\partial\, p)_{x,\mu} \Big)$$

$K^{\,-1}_{\,x,\mu\nu|y,\rho\sigma} \sim K_{\,x,\mu\nu|y,\rho\sigma}$ and $p_{x,\mu\nu} \sim n_{x,\mu\nu}$ $\Rightarrow$ structural similarity to original formulation.

But $d \; \leftrightarrow \; \partial$!!

Switch to the dual lattice

Introduce variables on the dual lattice: $\tilde{p}_{\tilde{x},\mu\nu} \in \mathbb{Z}$ and $\tilde{\alpha}_{\tilde{x},\mu} \in [-\pi, \pi]$

$$p_{x,\mu\nu} = \sum_{\rho<\sigma} \epsilon_{\mu\nu\rho\sigma}\, \tilde{p}_{\tilde{x}-\hat{\rho}-\hat{\sigma},\rho\sigma} \quad , \qquad \alpha_{x,\mu\nu\rho} = \sum_{\sigma} \epsilon_{\mu\nu\rho\sigma}\, \tilde{\alpha}_{\tilde{x}-\hat{\sigma},\sigma}$$

Conversion of discretized differential operators into their duals

$$(\partial\, p)_{x,\mu} = \sum_{\nu<\rho<\sigma} \epsilon_{\mu\nu\rho\sigma}\, (d\, \tilde{p})_{\tilde{x}-\hat{\nu}-\hat{\rho}-\hat{\sigma},\nu\rho\sigma} \ , \quad (\partial\, \alpha + 2\pi p)_{x,\mu\nu} = \sum_{\rho<\sigma} \epsilon_{\mu\nu\rho\sigma}\, (d\, \tilde{\alpha} + 2\pi \tilde{p})_{\tilde{x}-\hat{\rho}-\hat{\sigma},\rho\sigma}$$

- Zero divergence condition $\partial\, p = 0$ is converted to closedness condition $d\, \tilde{p} = 0$
- Boundary operator $\partial\, \alpha$ is converted to exterior derivative $d\, \tilde{\alpha}$

$\Rightarrow$ Self duality established !!

Final result

Original representation

$$Z \;=\; \int D[A] \sum_{\{n\}} e^{-\,S_{\beta}[F]\,-\,i\,\theta\,Q[F]} \prod_{\substack{x \\ \mu<\nu<\rho}} \delta\Big((dn)_{x,\mu\nu\rho}\Big)\;, \quad F_{x,\mu\nu} \;=\; (dA)_{x,\mu\nu} + 2\pi\, n_{x,\mu\nu}$$

Dual form

$$Z \;=\; C\int D[\tilde{\alpha}] \sum_{\{\tilde{p}\}} e^{-\,S_{\tilde{\beta}}[\tilde{F}]\,-\,i\,\tilde{\theta}\,Q[\tilde{F}]} \prod_{\substack{x \\ \mu<\nu<\rho}} \delta\Big((d\tilde{p})_{x,\mu\nu\rho}\Big)\;, \quad \tilde{F}_{x,\mu\nu} \;=\; (d\tilde{\alpha})_{x,\mu\nu} + 2\pi\, \tilde{p}_{x,\mu\nu}$$

Dual couplings

$$\tilde{\beta} \;=\; \frac{1}{4\pi^2\beta} \quad , \qquad \tilde{\theta} \;=\; -\,\frac{\theta}{4\pi^2\beta^2} \quad , \qquad C \;=\; \frac{1}{(2\pi\beta)^{3V}}$$

Consistency relations

$$\tilde{\tilde{\beta}} \;=\; \beta \quad , \qquad \tilde{\tilde{\theta}} \;=\; \theta \quad , \qquad C\,\tilde{C} \;=\; 1$$

Summary and outlook

- We constructed self-dual U(1) lattice gauge theory with a topological term in 4-d
- Several crucial steps:
  - Use Villain discretization with closedness constraint for Villain variables
  - Closedness constraint allows for new form of topological term (talk by M. Anosova)
  - Write constraints as auxiliary integrals $\Rightarrow$ Gaussian integral
  - Solve Gaussian integral with generalized Poisson summation
  - Switch to dual lattice for converting lattice differential operators
- Ongoing work:
  - Explore implications of the theta term and self-duality: Critical behavior, Witten effect on the lattice ....
  - Partial solution of complex action problem? Generalization to non-Abelian theories?
  - Couple matter fields (talk by D. Göschl, Friday 16:30)