

Tensor network study of two dimensional complex ϕ^4 theory at finite density

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Complex ϕ^4 theory (relativistic bose gas) at finite density

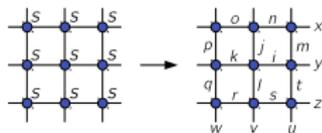
$$S = \int d^2x \left\{ |\partial_\nu \phi|^2 + (m^2 - \mu^2) |\phi|^2 + \mu (\phi^* \partial_2 \phi - \partial_2 \phi^* \phi) + \lambda |\phi|^4 \right\}.$$

- ϕ : complex scalar field; $\phi = (\phi_1, \phi_2)$
 - m, λ, μ : mass, coupling constant, chemical potential
- Typical model suffering from the severe sign problem
 - Many previous studies, e.g. [Endres 2007; Aarts 2009; Gattringer and Kloiber 2013]
 - ⇒ Good practice table
 - We analyze this model using TRG, a sign problem free method.

Tensor renormalization group (TRG) [Levin and Nave 2007]

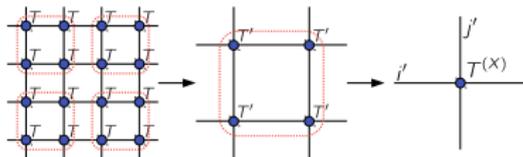
1 Tensor network representation of Z

$$Z = \sum_{\{s\}} e^{-\beta H[s]} \rightarrow \sum_{\dots, i, j, k, l, \dots} \dots T_{ijkl} T_{ymit} T_{xunm} \dots$$



2 Coarse-graining of the tensor network

$$TT \dots T \xrightarrow[\text{X times}]{\text{Coarse-graining}} T^{(X)}$$



3 Contraction of the "effective tensor" $T^{(X)}$

$$Z \approx \sum_{i', j'} T_{i'j'i'j'}^{(X)}$$

$$Z \approx \sum_{i', j'} \text{Tr} T_{i'j'i'j'}^{(X)}$$

- ✓ Deterministic data compression
- ✓ **Completely free of the sign problem**
- ✓ Systematic error is controlled by one parameter (the size of $T^{(X)}$).

Partition function of the 2D complex ϕ^4 model

$$Z = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-S},$$
$$S = \sum_n \left[(4 + m^2) |\phi_n|^2 + \lambda |\phi_n|^4 - \sum_{\nu=1}^2 (e^{\mu\delta_{\nu,2}} \phi_n^* \phi_{n+\hat{\nu}} + e^{-\mu\delta_{\nu,2}} \phi_{n+\hat{\nu}}^* \phi_n) \right]$$

- ϕ : complex scalar field; $\phi = (\phi_1, \phi_2)$
- m, λ, μ : mass, coupling constant, chemical potential
- $n, \hat{\nu}$: lattice coordinate, unit vector along ν -direction

How to construct a tensor network representation of Z :

- 1 Expanding the Boltzmann factor
- 2 Integrating out the original d.o.f. (the scalar field ϕ)

TN rep. of scalar theory

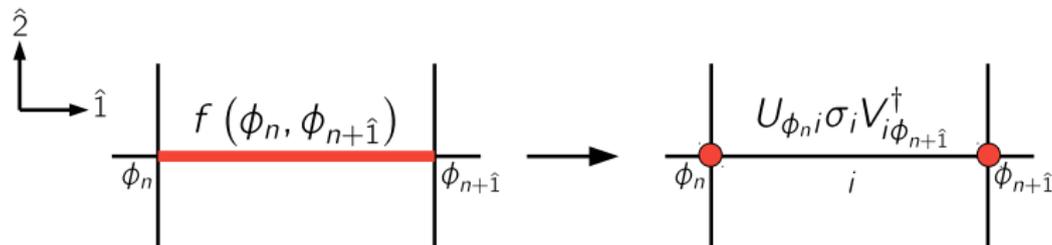
- Since S takes nearest-neighbor form,

$$e^{-S} = \prod_n \prod_\nu f_\nu(\phi_n, \phi_{n+\hat{\nu}}),$$

$$f_\nu(\phi_1, \phi_2) = \exp \left\{ -\frac{1}{4} (4 + m^2) (|\phi_1|^2 + |\phi_2|^2) - \frac{\lambda}{4} (|\phi_1|^4 + |\phi_2|^4) + e^{\mu\delta_{\nu,2}} \phi_1^* \phi_2 + e^{-\mu\delta_{\nu,2}} \phi_1 \phi_2^* \right\}.$$

- If f can be decomposed using discrete d.o.f,

$$f(\phi_n, \phi_{n+\hat{\mu}}) = \sum_i U_{\phi_n i} \sigma_i V_{i \phi_{n+\hat{\mu}}}^\dagger$$



TN rep. of scalar theory

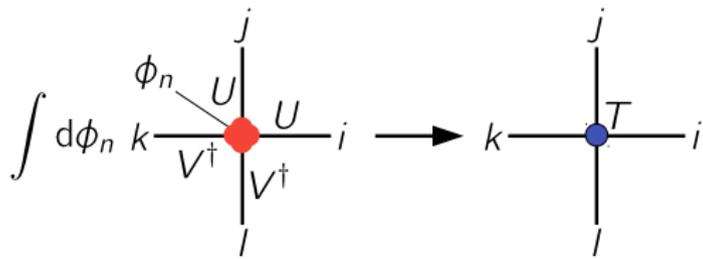
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- If f can be decomposed using discrete d.o.f,

$$T_{ijkl} = \int d\phi_n \sqrt{\sigma_i \sigma_j \sigma_k \sigma_l} U_{\phi_n i} U_{\phi_n j} V_{k\phi_n}^{\dagger} V_{l\phi_n}^{\dagger}$$



Gaussian quadrature rule

$$\int_{-\infty}^{\infty} d\phi e^{-\phi^2} g(\phi) \approx \sum_{\alpha=1}^K w_{\alpha} g(x_{\alpha})$$

- $g(\phi)$: (well-behaved) arbitrary function
- K : degree of the Hermite polynomial
- x_{α} : α -th zero point of the Hermite polynomial
- w_{α} : α -th weight of the GH quadrature

Spectral decomposition of local Boltzmann weight

Replace the integral of ϕ using the GH quadrature
(Interpolate f using the roots of the Hermite polynomial)

$$\int d\phi_{n,1} d\phi_{n,2} \prod_{\nu=1}^2 f_{\nu}((\phi_{n-\hat{\nu},1}, \phi_{n-\hat{\nu},2}), (\phi_{n,1}, \phi_{n,2}))$$

$$\cdot f_{\nu}((\phi_{n,1}, \phi_{n,2}), (\phi_{n+\hat{\nu},1}, \phi_{n+\hat{\nu},2}))$$

$$\approx \sum_{\alpha_1, \alpha_2=1}^K w_{\alpha_1} w_{\alpha_2} e^{x_{\alpha_1}^2 + x_{\alpha_2}^2} \prod_{\nu=1}^2 f_{\nu}((\phi_{n-\hat{\nu},1}, \phi_{n-\hat{\nu},2}), (x_{\alpha_1}, x_{\alpha_2}))$$

$$\cdot f_{\nu}((x_{\alpha_1}, x_{\alpha_2}), (\phi_{n+\hat{\nu},1}, \phi_{n+\hat{\nu},2})),$$

- K : degree of the Hermite polynomial
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$$\int d\phi_{n,1} d\phi_{n,2} \prod_{\nu=1}^2 f_{\nu}((\phi_{n-\hat{\nu},1}, \phi_{n-\hat{\nu},2}), (\phi_{n,1}, \phi_{n,2})) \cdot f_{\nu}((\phi_{n,1}, \phi_{n,2}), (\phi_{n+\hat{\nu},1}, \phi_{n+\hat{\nu},2}))$$

$$\approx \sum_{\alpha_1, \alpha_2=1}^K w_{\alpha_1} w_{\alpha_2} e^{x_{\alpha_1}^2 + x_{\alpha_2}^2} \prod_{\nu=1}^2 f_{\nu}((\phi_{n-\hat{\nu},1}, \phi_{n-\hat{\nu},2}), (x_{\alpha_1}, x_{\alpha_2})) \cdot f_{\nu}((x_{\alpha_1}, x_{\alpha_2}), (\phi_{n+\hat{\nu},1}, \phi_{n+\hat{\nu},2})),$$

$$f_{\nu}(\phi_1, \phi_2) \rightarrow f((x_{\alpha_1}, x_{\alpha_2}), (x_{\beta_1}, x_{\beta_2})) = \sum_{m=1}^{K^2} U_{(x_{\alpha_1}, x_{\alpha_2})m} \sigma_m V_{m(x_{\beta_1}, x_{\beta_2})}^{\dagger}$$

- f becomes a matrix and can be numerically decomposed by singular values!

TN rep.

$$Z = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-S} \approx \sum_{i,j,k,l,\dots} T_{ijkl} T_{mnio} \dots$$

- Z is written as \int of field $\phi \rightarrow \sum$ of tensor indices i, j, k, \dots !
- TN rep. of $\langle \phi \rangle$ and correlators can be similarly constructed.

Coarse-graining of the tensor network

Eckart–Young theorem [Eckart and Young 1936] states that the SVD is the best low rank approximation of a matrix.

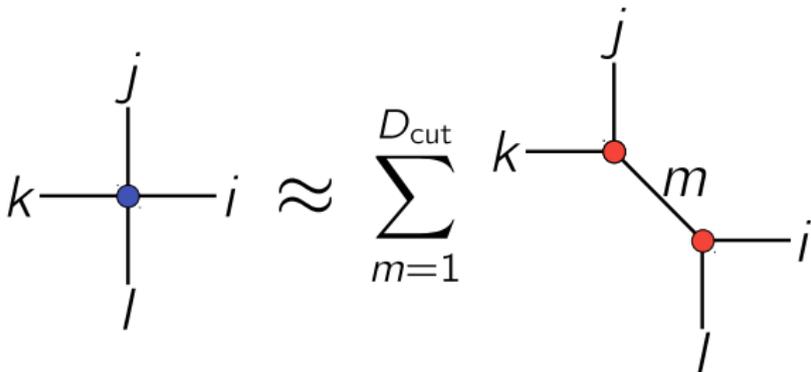
$$\begin{aligned} T_{ijkl} = M_{(jk)(li)} &= \sum_{m=1}^{D^2} U_{(jk)m} \sigma_m V_{m(li)}^\dagger \\ &\approx \sum_{m=1}^{D_{\text{cut}}} U_{(jk)m} \sigma_m V_{m(li)}^\dagger \quad \leftarrow \text{the best approx!} \end{aligned}$$

- $1 \leq i, j, k, l \leq D$
- $D_{\text{cut}} < D^2$
- $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{D^2} \geq 0$

Coarse-graining of the tensor network

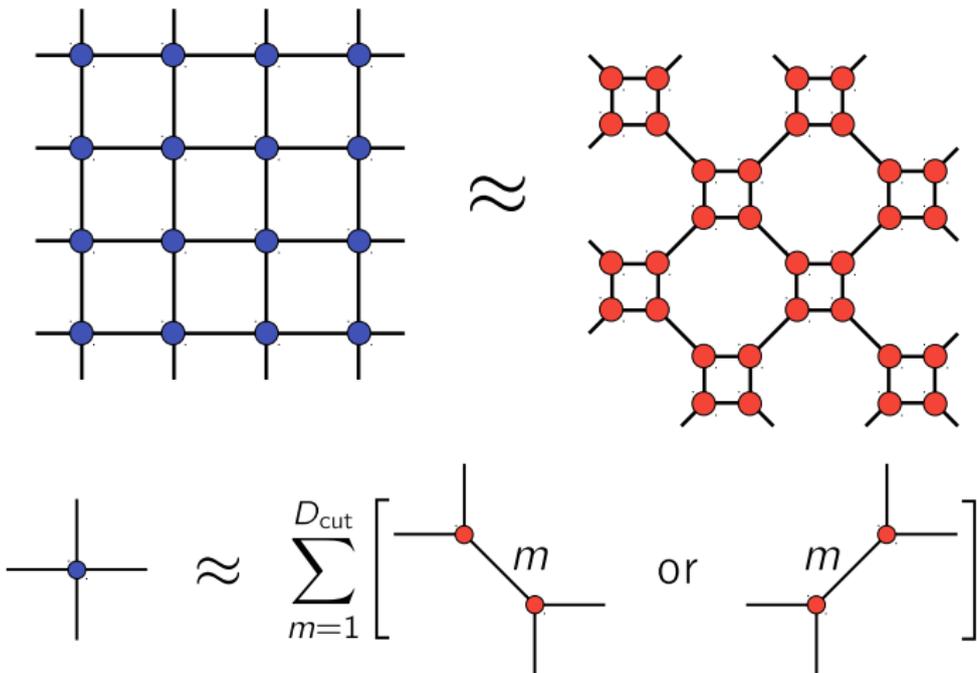
Eckart–Young theorem [Eckart and Young 1936] states that the SVD is the best low rank approximation of a matrix.

$$T_{ijkl} = M_{(jk)(li)} \approx \sum_{m=1}^{D_{\text{cut}}} U_{(jk)m} \sigma_m V_{m(li)}^\dagger$$



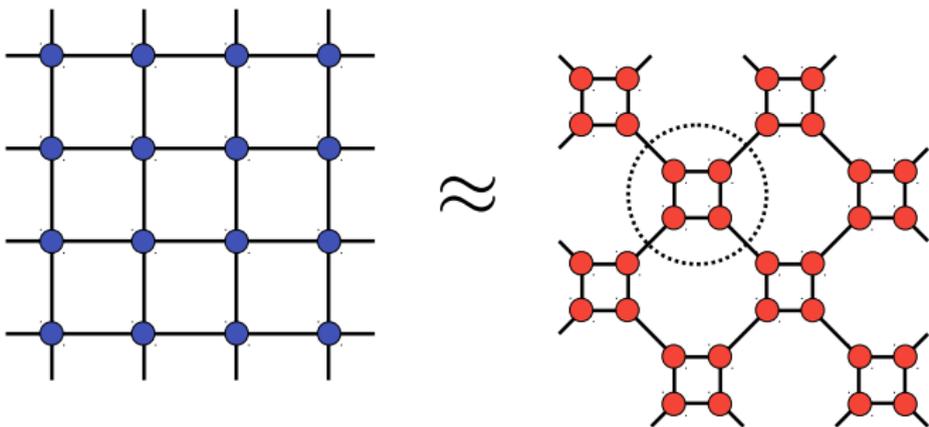
Coarse-graining of the tensor network

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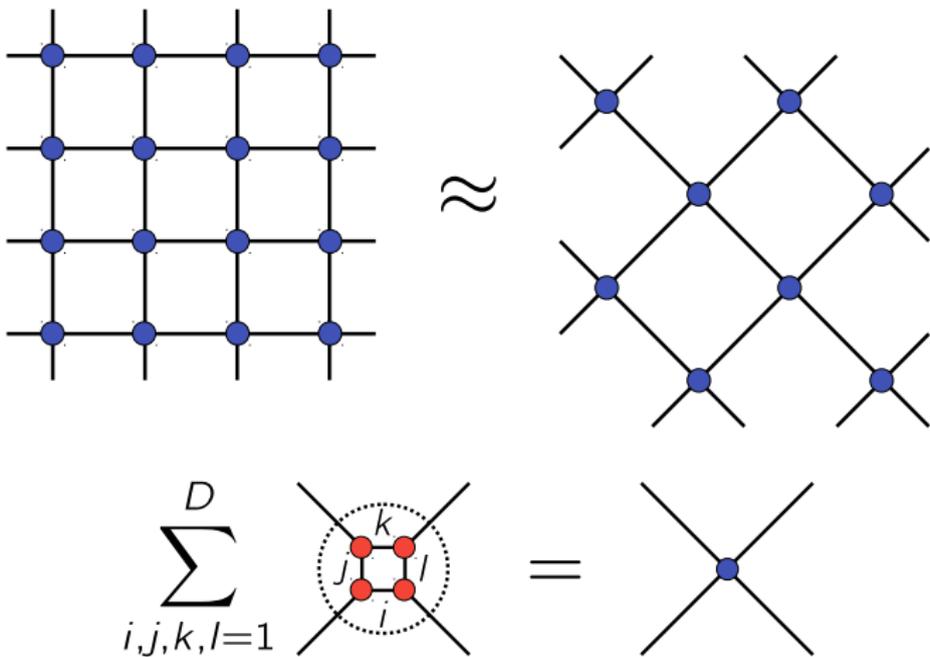


$$\sum_{i,j,k,l=1}^D \text{ (diagram of a 2x2 red node grid) } = \text{ (diagram of a single blue node) }$$

The diagram shows the contraction of a 2x2 grid of red nodes. Each red node is connected to its four neighbors. The four external legs of the 2x2 grid are labeled i , j , k , and l . This contraction is shown to be equivalent to a single blue node with four legs.

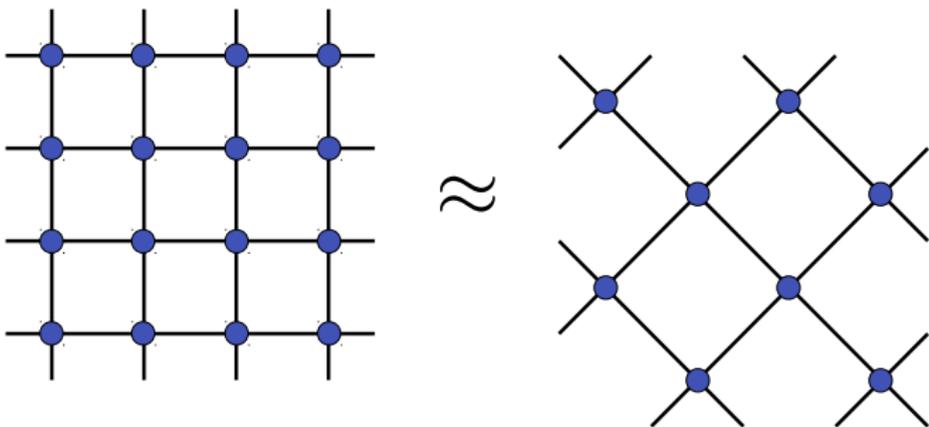
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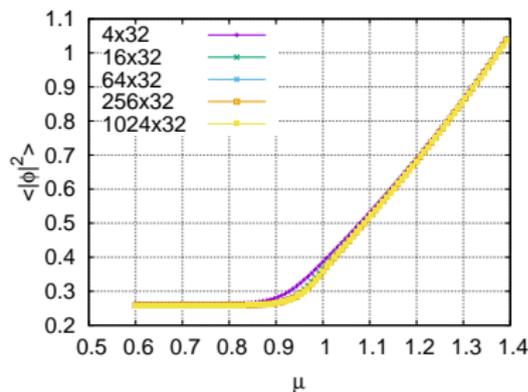
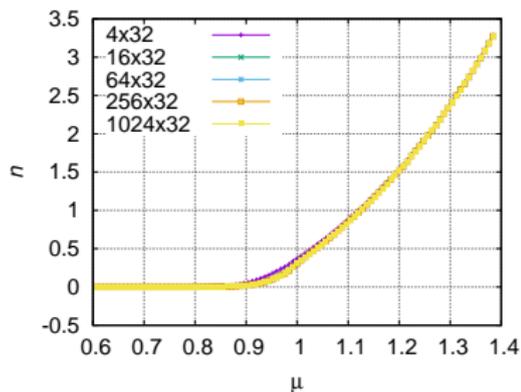
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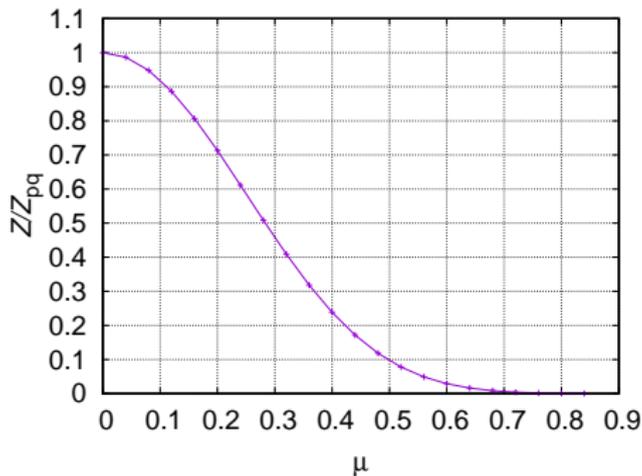
- $\#$ of lattice sites is reduced by 1/2 through a single cycle.
- TN is uniform. \Rightarrow One needs to just repeat local procedures.
- Total cost $\propto \ln(\text{Vol.})$.

Silver Blaze phenomenon



- left) particle number density,
right) expectation value of squared absolute value of field
- $m^2 = 0.01$, $\lambda = 1$, $K^2 = 4096$, $D = 64$.

Expectation value of phase



- $Z_{pq} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-\text{Re}(S)}$.
- $m^2 = 0.01$, $\lambda = 1$, $K^2 = 4096$, $D = 64$, $V = 1024 \times 32$.
- In the large μ region, the ratio deviate from 1, and the sign problem is severe.

Sign problem free method vs. sign problem free formulation

Using

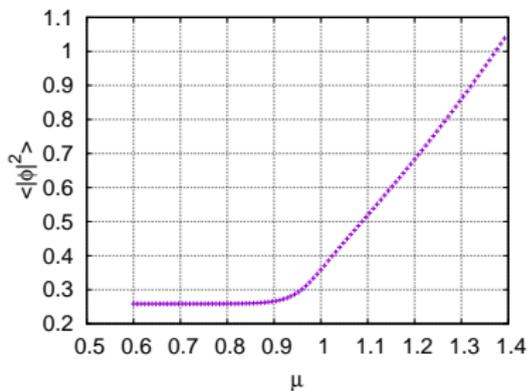
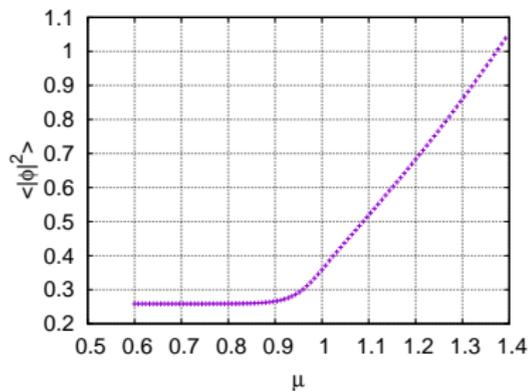
- the polar coordinate expression: $\phi_{n,1} = r_n \cos \theta_n$, $\phi_{n,2} = r_n \sin \theta_n$,
- the character expansion: $e^{x \cos z} = \sum_{l=-\infty}^{\infty} I_l(x) e^{ilz}$ for $x \in \mathbb{R}$, $z \in \mathbb{C}$,

Z can be expressed in a sign problem free version:

$$Z = \left(\prod_n \sum_{l_{n,1}, l_{n,2}=-\infty}^{\infty} \right) \left(\prod_n \int_0^{\infty} dr_n \right) \prod_n 2\pi r_n \prod_{\nu=1}^2 e^{-\frac{1}{4}(4+m^2)(r_n^2+r_{n+\nu}^2) - \frac{\lambda}{4}(r_n^4+r_{n+\nu}^4)} \\ \cdot I_{l_{n,\nu}}(2r_n r_{n+\nu}) e^{l_{n,\nu} \mu \delta_{\nu,2}} \delta_{(l_{n,1}+l_{n,2}-l_{n-1,1}-l_{n-2,2}),0}$$

- I_l : the modified Bessel function of first kind (**non negative**)
- [Endres 2007] (*cf.* loop formulation, world line representation).

Sign problem free method vs. sign problem free formulation



- left) naive application of TRG,
right) application to the sign problem free formulation [Endres 2007] (cf. loop formulation, world line representation).
- $m^2 = 0.01$, $\lambda = 1$, $K^2 = 4096$, $D = 64$, $V = 1024 \times 32$.
- truncation order of CE (in right panel) = 128.
- Good agreement in the large μ region (severe sign problem region)

Summary and outlook

- TN formulation of the 2D complex ϕ^4 theory
- Silver Blaze phenomenon is observed
- Result by naive TRG is consistent with that of sign problem free formulation

In future

- Phase diagram
- Check the order of phase transition

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- [2] C. Eckart and G. Young, “The approximation of one matrix by another of lower rank”, *Psychometrika* **1**, 211 (1936).
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- [4] C. Gattringer and T. Kloiber, “Lattice study of the Silver Blaze phenomenon for a charged scalar ϕ^4 field”, *Nucl. Phys.* **B869**, 56 , [arXiv:1206.2954 \[hep-lat\]](#).
- [5] M. Levin and C. P. Nave, “Tensor renormalization group approach to 2D classical lattice models”, *Phys. Rev. Lett.* **99**, 120601 , [arXiv:cond-mat/0611687 \[cond-mat.stat-mech\]](#).