Logarithmic corrections to $a^2$ scaling in lattice Yang Mills theory

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Motivation

\[ \delta \Sigma(a) = \Sigma(a) - \Sigma(0) \]

**Figure:** Deviation of the step scaling function from its continuum counterpart, as an example, in the 2-dimensional O(3) model [Balog et al., 2009, 2010].
Motivation

\[ L^2 \frac{\Sigma(a) - \Sigma(0)}{a^2} \]

Figure: Deviation of the step scaling function from its continuum counterpart, as an example, in the 2-dimensional O(3) model [Balog et al., 2009, 2010].

**O(3) model: “Worst case” example [Balog et al., 2009, 2010]**

\[ \delta \Sigma = \text{const.} a^2 \left[ (g_0^2)^{-3} - 1.1386 (g_0^2)^{-2} + O((g_0^2)^{-1}) \right] \]
Idea: Parametrise lattice artifacts originating from the lattice action [and for a field \( \Phi \)] by a minimal basis of operators living in a continuous Symanzik effective theory [Symanzik, 1980, 1981, 1983a,b]

\[
\mathcal{L}_{\text{eff}} = \mathcal{L} + a^2 \delta \mathcal{L} + \mathcal{O}(a^4), \tag{1}
\]

\[
\Phi_{\text{eff}} = \Phi + a^2 \delta \Phi + \mathcal{O}(a^4), \tag{2}
\]

where \( a \) is the lattice spacing and

\[
\delta \mathcal{L} = \sum_i b_i \mathcal{O}_i, \tag{3}
\]

\[
\delta \Phi = \sum_i c_i \Phi_i, \tag{4}
\]

with free coefficients \( b_i \) and \( c_i \).
Occurring operators $O_i$ and $\Phi_i$ must comply with symmetries, i.e. for the lattice pure gauge action

- Local SU($N$) gauge symmetry,
- $\mathcal{C}$-, $\mathcal{P}$- and $\mathcal{T}$-symmetry,
- discrete rotation and translation invariance (at least in infinite volume)

$\Rightarrow$ broken O(4) symmetry.

Require minimal basis for physical energies and matrix elements ("on-shell")

$\Rightarrow$ use EOMs to reduce set of operators [Lüscher and Weisz, 1985; Georgi, 1991].
The only operators $O_i$ left are [Lüscher and Weisz, 1985]

\[
O_0 = \frac{1}{g_0^2} \text{tr}(D_\mu F_{\nu\rho} D_\mu F_{\nu\rho}), \quad O_1 = \frac{1}{g_0^2} \sum_\mu \text{tr}(D_\mu F_{\mu\rho} D_\mu F_{\mu\rho}),
\]

\[O_2 = \frac{1}{g_0^2} \text{tr}(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}) \overset{\text{EOM}}{=} 0. \tag{5}\]

We then find for a field $\Phi$ in analogy to [Balog et al., 2009, 2010]

\[
\langle \Phi \rangle_{\text{latt}(a)} = \langle \Phi \rangle_{\text{cont}} \left\{ 1 + a^2 \left( \sum_i c_i \delta_i^\Phi(a) - \sum_{j=0}^1 b_j \delta_j^O(a) \right) + O(a^4) \right\}. \tag{6}\]
Symanzik effective theory IV

Leading lattice artifacts

\[
\frac{\langle \Phi \rangle_{\text{latt}(a)}}{\langle \Phi \rangle_{\text{cont}}} = 1 + a^2 \left( \sum_i \bar{c}_i \delta^\Phi_i(a) - \sum_{j=0}^1 \bar{b}_j \delta^{\mathcal{O}}_j(a) \right) \times \left[1 + O(\alpha(1/a))\right] + O(a^4)
\]

with tree-level coefficients \(\bar{c}_i\) and \(\bar{b}_j\).

Need to understand (leading) lattice spacing dependence of

\[
\delta^\mathcal{O}_j(a) = \int d^4x \left[ \frac{\langle \Phi \mathcal{O}_j(x) \rangle_{\text{cont};R}}{\langle \Phi \rangle_{\text{cont};R}} - \langle \mathcal{O}_j(x) \rangle_{\text{cont};R} \right]_{\mu=1/a},
\]

\[
\delta^\Phi_i(a) = \frac{\langle \Phi_i \rangle_{\text{cont};R}}{\langle \Phi \rangle_{\text{cont};R}} \bigg|_{\mu=1/a}.
\]

⇒ Renormalisation Group
Symanzik effective theory V

**Leading lattice artifacts**

\[
\frac{\langle \Phi \rangle_{\text{latt}(a)}}{\langle \Phi \rangle_{\text{cont}}} = 1 + a^2 \left( \sum_i \bar{c}_i \delta_i^\Phi(a) - \sum_{j=0}^{1} \bar{b}_j \delta^\circ_j(a) \right) \times \left[ 1 + O(\alpha(1/a)) \right] + O(a^4)
\]

**Remarks:**

> Tree-level coefficients $\bar{c}_i$ and $\bar{b}_j$ can be obtained from classical expansion in the lattice spacing.

> We limit ourselves to the leading behaviour as $a \downarrow 0$, i.e. we do not require 1-loop coefficients. However, if tree-level coefficient is zero 1-loop might be needed to obtain leading logarithms.

> We consider only the case $c_i = 0$ to all orders, i.e. no additional artifacts from observables.
Renormalisation Group I

Use Renormalisation Group Equations (RGEs) to determine renormalisation scale dependence

\[
\mu^2 \frac{d\delta_i^O(\mu)}{d\mu^2} = \gamma_{ij} \delta_j^O(\mu), \quad \beta(\alpha) = \mu^2 \frac{d\alpha(\mu)}{d\mu^2} = -\alpha^2 \sum_{n \geq 0} \beta_n \alpha^n, \tag{7}
\]

where \( \gamma \) is the anomalous dimension matrix

\[
\gamma_{ij} = \mu^2 \frac{d \ln (Z)_{ij}}{d\mu^2} = (\gamma_0)_{ij} \alpha + O(\alpha^2), \quad O_{i;R} = Z_{ij} O_{j;0}. \tag{8}
\]

renormalisation scheme independent
Renormalisation Group II

We choose a basis such that \( \gamma_0 = \text{diag}\{(\gamma_0)_1, \ldots, (\gamma_0)_n\} \) and introduce the Renormalisation Group Invariant (RGI)

\[
D_i^O(\Lambda) = \lim_{\mu \to \infty} [2\beta_0 \alpha(\mu)]^{\hat{\gamma}_i} \delta_i^O(\mu), \quad \hat{\gamma}_i = \frac{(\gamma_0)_i}{\beta_0},
\]

with RGI scale \( \Lambda \). This allows us to rewrite

\[
\delta_i^O(\mu) = (2\beta_0 \alpha(\mu))^{-\hat{\gamma}_i} \text{Pexp} \left[ \int_0^{\alpha(\mu)} \left\{ \frac{\gamma(x)}{\beta(x)} + \frac{\gamma_0}{\beta_0 x} \right\} \right] D_j^O(\Lambda) \quad (10)
\]
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\]

with RGI scale \( \Lambda \). This allows us to rewrite

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\delta_i^O(\mu) = (2\beta_0 \alpha(\mu))^{-\hat{\gamma}_i} \text{Pexp} \left[ \alpha(\mu) \int_0^\infty dx \left\{ \frac{\gamma(x)}{\beta(x)} + \frac{\gamma_0}{\beta_0 x} \right\} \right] D_j^O(\Lambda) \tag{10}
\]

\[
= (2\beta_0 \alpha(\mu))^{-\hat{\gamma}_i} D_i^O(\Lambda) + O \left( [\alpha(\mu)]^{1-\hat{\gamma}_i} \right).
\]

**Note:** The renormalisation scale dependence is only in the prefactor of the RGI with leading power determined by \( \hat{\gamma}_i \).
Computing leading anomalous dimensions I

Renormalise operator basis at 1-loop by computing connected Green’s functions with operator insertion in continuum theory

\[
\left\langle \tilde{A}_{1;R}(p_1) \ldots \tilde{A}_{n;R}(p_n) \tilde{O}_{i;R}(q) \right\rangle_{\text{con}} = \frac{1}{Z_A Z_{ij}} \left\langle \tilde{A}_{1;0}(p_1) \ldots \tilde{A}_{n;0}(p_n) \tilde{O}_{j;0}(q) \right\rangle_{\text{con}}
\]

with fundamental gauge fields \( \tilde{A} \) (gauge fixed), and momenta \( p_k, q \).
Computing leading anomalous dimensions I

Renormalise operator basis at 1-loop by computing connected Green’s functions with operator insertion in continuum theory

\[ \left\langle \tilde{A}_1; R(p_1) \ldots \tilde{A}_n; R(p_n) \tilde{O}_{i; R}(q) \right\rangle_{\text{con}} = \]

\[ = Z^n_A Z_{ij} \left\langle \tilde{A}_{1;0}(p_1) \ldots \tilde{A}_{n;0}(p_n) \tilde{O}_{j;0}(q) \right\rangle_{\text{con}} \]

with fundamental gauge fields \( \tilde{A} \) (gauge fixed), and momenta \( p_k, q \).

**Tools:** QGRAF, FORM

To obtain the anomalous dimension we extract only the UV-pole contributions following e.g. the procedure from [Misiak and Münz, 1995; Chetyrkin et al., 1998].
Computing leading anomalous dimensions II

Common approach

> “on-shell” momenta avoid
gauge-variant contributions
from gauge-fixing.

> If $q \neq 0$ total divergence
operators contribute, but 2- and
3-point functions are accessible
for renormalisation.
Computing leading anomalous dimensions II

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Background field method

[’t Hooft, 1975; Abbott, 1981, 1982; Lüscher and Weisz, 1995]

> Only gauge-invariant operators
relevant for renormalisation.

> Can keep $p_k$ arbitrary and
$q = 0$, but EOM-vanishing
operators contribute (unless
on-shell).
Computing leading anomalous dimensions II

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> Can keep \( p_k \) arbitrary and \( q = 0 \), but EOM-vanishing operators contribute (unless on-shell).

Obtain relevant part of mixing matrix via

\[
\begin{pmatrix}
\mathcal{O} \\
\mathcal{Q}
\end{pmatrix}_R =
\begin{pmatrix}
Z_{\mathcal{O}\mathcal{O}} & Z_{\mathcal{O}\mathcal{Q}} \\
0 & Z_{\mathcal{Q}\mathcal{Q}}
\end{pmatrix}
\begin{pmatrix}
\mathcal{O} \\
\mathcal{Q}
\end{pmatrix}_0
\]

needed additionally

with class of “redundant” operators \( Q \).
Lattice artifacts originating from the action

We find for the minimal basis

\[
\begin{pmatrix}
O_0 \\
O_1
\end{pmatrix}_{R} = \begin{pmatrix}
1 + \frac{7C_A}{3\epsilon} \frac{\alpha}{4\pi} & 0 \\
-\frac{7C_A}{15\epsilon} \frac{\alpha}{4\pi} & 1 + \frac{21C_A}{5\epsilon} \frac{\alpha}{4\pi}
\end{pmatrix} \begin{pmatrix}
O_0 \\
O_1
\end{pmatrix}_0.
\]

(12)
**Lattice artifacts originating from the action**

We find for the minimal basis due to $O(4)$ symmetry

\[
\begin{pmatrix}
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\end{pmatrix}_R = \begin{pmatrix}
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\end{pmatrix} \begin{pmatrix}
O_0 \\
O_1
\end{pmatrix}_0.
\] (12)

Thus the diagonal basis and leading anomalous dimensions are

\[
B_0 = O_0, \quad B_1 = O_1 - \frac{1}{4} O_0,
\] (13)

\[
-\hat{\gamma}_0 = \frac{7C_A}{12\pi \beta_0} = \frac{7}{11} \approx 0.636, \quad -\hat{\gamma}_1 = \frac{21C_A}{20\pi \beta_0} = \frac{63}{55} \approx 1.145.
\]

[Narison and Tarrach, 1983; Alonso et al., 2014]
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[Narison and Tarrach, 1983; Alonso et al., 2014]

Leading lattice artifacts from the action

\[
\frac{\langle \Phi \rangle_{\text{latt}(a)}}{\langle \Phi \rangle_{\text{cont}}} = 1 - a^2 \left( \sum_{j=0}^{1} \tilde{b}_j [\alpha(1/a)]^{-\hat{\gamma}_j} D_j^B(\Lambda) \right) \times [1 + O(\alpha(1/a))] + O(a^4)
\]

$\Rightarrow$ Leading anomalous dimensions improve the convergence as $a \downarrow 0$. 

Conclusion

- No $a^2[\alpha(1/a)]^{-3}$ behaviour like for the O(3) model nor naive $a^2$, but

$$a^2 \left( [\alpha(1/a)]^{0.636} + d_1[\alpha(1/a)]^{1.145} + d_2[\alpha(1/a)]^{1.636} + \ldots \right)$$

is the leading behaviour.

- Leading anomalous dimensions of contributions from the lattice Yang Mills action improve convergence as $a \downarrow 0$.

- Short-cuts in perturbation theory can ease computational effort.

- Full $\left( \prod_{N_f}^{i=1} U(1)_V \text{ flavour symmetric} \right)$ lattice QCD is currently in progress (well advanced).

- Gradient flow observables require additional operators on the 4D-boundary, also in pure gauge theory.


References II


References III


References IV