Logarithmic corrections to a^2 scaling in lattice Yang Mills theory

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HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

Motivation

$$\delta\Sigma(a) = \Sigma(a) - \Sigma(0)$$

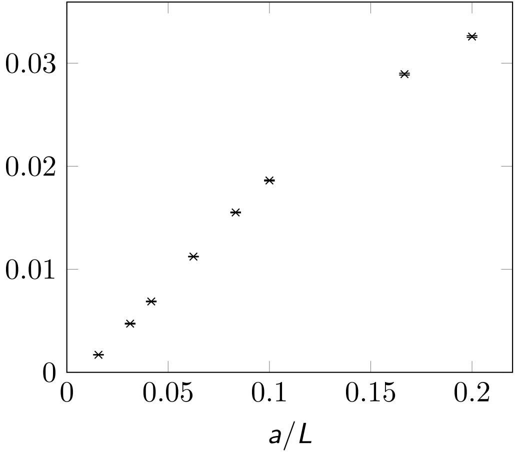

Figure: Deviation of the step scaling function from its continuum counterpart, as an example, in the 2-dimensional O(3) model [Balog et al., 2009, 2010].

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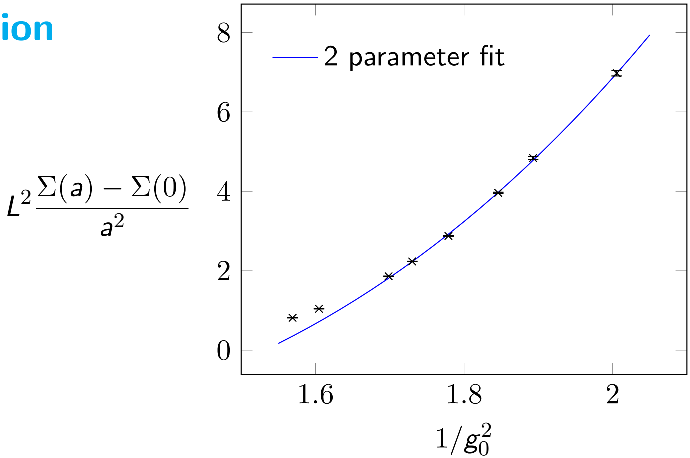

Figure: Deviation of the step scaling function from its continuum counterpart, as an example, in the 2-dimensional O(3) model [Balog et al., 2009, 2010].

O(3) model: "Worst case" example [Balog et al., 2009, 2010]

$$\delta\Sigma = \text{const.}\, a^2 \left[(g_0^2)^{-3} - 1.1386 (g_0^2)^{-2} + \mathrm{O}((g_0^2)^{-1})\right]$$

Symanzik effective theory I

Idea: Parametrise lattice artifacts originating from the lattice action [and for a field Φ] by a minimal basis of operators living in a continuous Symanzik effective theory [Symanzik, 1980, 1981, 1983a,b]

$$\mathscr{L}_{\mathrm{eff}} = \mathscr{L} + a^2 \delta\mathscr{L} + \mathrm{O}(a^4)\,, \quad (1)$$

$$\Phi_{\mathrm{eff}} = \Phi + a^2 \delta\Phi + \mathrm{O}(a^4)\,, \quad (2)$$

where a is the lattice spacing and

$$\delta\mathscr{L} = \sum_i b_i \mathcal{O}_i\,, \quad (3)$$

$$\delta\Phi = \sum_i c_i \Phi_i\,, \quad (4)$$

with free coefficients b_i and c_i.

Symanzik effective theory II

Occurring operators $\mathcal{O}_i$ and Φ_i must comply with symmetries, i.e. for the lattice pure gauge action

- Local SU(N) gauge symmetry,
- $\mathcal{C}$-, $\mathcal{P}$- and $\mathcal{T}$-symmetry,
- discrete rotation and translation invariance (at least in infinite volume)
  $\Rightarrow$ broken O(4) symmetry.

Require minimal basis for physical energies and matrix elements ("on-shell")
$\Rightarrow$ use EOMs to reduce set of operators [Lüscher and Weisz, 1985; Georgi, 1991].

Symanzik effective theory III

The only operators $\mathcal{O}_i$ left are [Lüscher and Weisz, 1985]

$$\mathcal{O}_0 = \frac{1}{g_0^2}\mathrm{tr}(D_\mu F_{\nu\rho} D_\mu F_{\nu\rho}), \qquad \mathcal{O}_1 = \frac{1}{g_0^2}\sum_\mu \mathrm{tr}(D_\mu F_{\mu\rho} D_\mu F_{\mu\rho}),$$
$$\mathcal{O}_2 = \frac{1}{g_0^2}\mathrm{tr}(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}) \overset{\text{EOM}}{=} 0\,. \tag{5}$$

We then find for a field Φ in analogy to [Balog et al., 2009, 2010]

$$\langle\Phi\rangle_{\mathrm{latt}(a)} = \langle\Phi\rangle_{\mathrm{cont}}\left\{1 + a^2\left(\sum_i c_i \delta_i^\Phi(a) - \sum_{j=0}^{1} b_j \delta_j^{\mathcal{O}}(a)\right) + \mathrm{O}(a^4)\right\}. \tag{6}$$

Symanzik effective theory IV

Leading lattice artifacts

$$\frac{\langle\Phi\rangle_{\mathrm{latt}(a)}}{\langle\Phi\rangle_{\mathrm{cont}}} = 1 + a^2\left(\sum_i \bar{c}_i \delta_i^{\Phi}(a) - \sum_{j=0}^{1} \bar{b}_j \delta_j^{\mathcal{O}}(a)\right) \times [1 + \mathrm{O}(\alpha(1/a))] + \mathrm{O}(a^4)$$

with tree-level coefficients $\bar{c}_i$ and $\bar{b}_j$.

Need to understand (leading) lattice spacing dependence of

$$\delta_j^{\mathcal{O}}(a) = \int \mathrm{d}^4x \left[\frac{\langle\Phi\mathcal{O}_j(x)\rangle_{\mathrm{cont;R}}}{\langle\Phi\rangle_{\mathrm{cont;R}}} - \langle\mathcal{O}_j(x)\rangle_{\mathrm{cont;R}}\right]_{\mu=1/a},$$

$$\delta_i^{\Phi}(a) = \left.\frac{\langle\Phi_i\rangle_{\mathrm{cont;R}}}{\langle\Phi\rangle_{\mathrm{cont;R}}}\right|_{\mu=1/a}.$$

$\Rightarrow$ Renormalisation Group

Symanzik effective theory V

Leading lattice artifacts

$$\frac{\langle\Phi\rangle_{\mathrm{latt}(a)}}{\langle\Phi\rangle_{\mathrm{cont}}} = 1 + a^2\left(\sum_i \bar{c}_i\delta_i^{\Phi}(a) - \sum_{j=0}^{1}\bar{b}_j\delta_j^{\mathcal{O}}(a)\right)\times[1+\mathrm{O}(\alpha(1/a))] + \mathrm{O}(a^4)$$

Remarks:

- Tree-level coefficients $\bar{c}_i$ and $\bar{b}_j$ can be obtained from classical expansion in the lattice spacing.
- We limit ourselves to the leading behaviour as $a \searrow 0$, i.e. we do not require 1-loop coefficients. However, if tree-level coefficient is zero 1-loop might be needed to obtain leading logarithms.
- We consider only the case $c_i = 0$ to all orders, i.e. no additional artifacts from observables.

Renormalisation Group I

Use Renormalisation Group Equations (RGEs) to determine renormalisation scale dependence

$$\mu^2 \frac{\mathrm{d}\delta_i^{\mathcal{O}}(\mu)}{\mathrm{d}\mu^2} = \gamma_{ij}\delta_j^{\mathcal{O}}(\mu), \qquad \beta(\alpha) = \mu^2 \frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\mu^2} = -\alpha^2 \sum_{n\geq 0} \beta_n \alpha^n, \tag{7}$$

where γ is the anomalous dimension matrix

$$\gamma_{ij} = \mu^2 \frac{\mathrm{d}\ln(Z)_{ij}}{\mathrm{d}\mu^2} = (\gamma_0)_{ij}\alpha + \mathrm{O}(\alpha^2), \qquad \mathcal{O}_{i;\mathrm{R}} = Z_{ij}\mathcal{O}_{j;0}\,. \tag{8}$$

renormalisation scheme independent

Renormalisation Group II

We choose a basis such that $\gamma_0 = \mathrm{diag}\{(\gamma_0)_1, \ldots, (\gamma_0)_n\}$ and introduce the Renormalisation Group Invariant (RGI)

$$D_i^{\mathcal{O}}(\Lambda) = \lim_{\mu\to\infty} \left[2\beta_0\alpha(\mu)\right]^{\hat{\gamma}_i} \delta_i^{\mathcal{O}}(\mu)\,, \qquad \hat{\gamma}_i = \frac{(\gamma_0)_i}{\beta_0}\,, \tag{9}$$

with RGI scale Λ. This allows us to rewrite

$$\delta_i^{\mathcal{O}}(\mu) = (2\beta_0\alpha(\mu))^{-\hat{\gamma}_i} \,\mathrm{Pexp}\left[\int_0^{\alpha(\mu)} \mathrm{d}x \left\{\frac{\gamma(x)}{\beta(x)} + \frac{\gamma_0}{\beta_0 x}\right\}\right]_{ij} D_j^{\mathcal{O}}(\Lambda) \tag{10}$$

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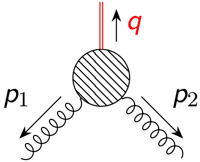
$$\begin{aligned}\delta_i^{\mathcal{O}}(\mu) =& (2\beta_0\alpha(\mu))^{-\hat{\gamma}_i}\, \mathrm{Pexp}\left[\int_0^{\alpha(\mu)} \mathrm{d}x \left\{\frac{\gamma(x)}{\beta(x)} + \frac{\gamma_0}{\beta_0 x}\right\}\right]_{ij} D_j^{\mathcal{O}}(\Lambda) \\ =& (2\beta_0\alpha(\mu))^{-\hat{\gamma}_i}\, D_i^{\mathcal{O}}(\Lambda) + \mathrm{O}\left([\alpha(\mu)]^{1-\hat{\gamma}_i}\right)\,.\end{aligned} \tag{10}$$

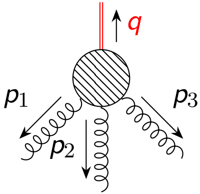
Note: The renormalisation scale dependence is only in the prefactor of the RGI with leading power determined by $\hat{\gamma}_i$.

Computing leading anomalous dimensions I

Renormalise operator basis at 1-loop by computing connected Green's functions with operator insertion in continuum theory

$$\left\langle \tilde{A}_{1;\mathrm{R}}(p_1)\ldots\tilde{A}_{n;\mathrm{R}}(p_n)\tilde{\mathcal{O}}_{i;\mathrm{R}}(q)\right\rangle_{\mathrm{con}} = \\ = Z_A^n Z_{ij}\left\langle \tilde{A}_{1;0}(p_1)\ldots\tilde{A}_{n;0}(p_n)\tilde{\mathcal{O}}_{j;0}(q)\right\rangle_{\mathrm{con}}$$





with fundamental gauge fields $\tilde{A}$ (gauge fixed), and momenta p_k, q.

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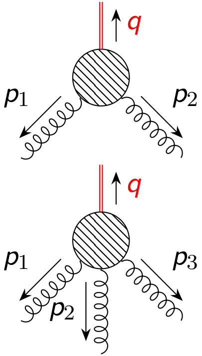

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Tools: QGRAF, FORM

To obtain the anomalous dimension we extract only the UV-pole contributions following e.g. the procedure from [Misiak and Münz, 1995; Chetyrkin et al., 1998].

Computing leading anomalous dimensions II

Common approach

- “on-shell” momenta avoid gauge-variant contributions from gauge-fixing.
- If $q \neq 0$ total divergence operators contribute, but 2- and 3-point functions are accessible for renormalisation.

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Background field method

['t Hooft, 1975; Abbott, 1981, 1982; Lüscher and Weisz, 1995]

- Only gauge-invariant operators relevant for renormalisation.
- Can keep p_k arbitrary and $q = 0$, but EOM-vanishing operators contribute (unless on-shell).

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Obtain relevant part of mixing matrix via

needed additionally

$$\begin{pmatrix} \mathcal{O} \\ Q \end{pmatrix}_{\mathrm{R}} = \begin{pmatrix} Z_{\mathcal{O}\mathcal{O}} & Z_{\mathcal{O}Q} \\ 0 & Z_{QQ} \end{pmatrix} \begin{pmatrix} \mathcal{O} \\ Q \end{pmatrix}_0 \qquad (11)$$

with class of "redundant" operators Q.

Lattice artifacts originating from the action

We find for the minimal basis

$$\begin{pmatrix}\mathcal{O}_0\\ \mathcal{O}_1\end{pmatrix}_{\mathrm{R}} = \begin{pmatrix}1+\frac{7C_{\mathsf{A}}}{3\epsilon}\frac{\alpha}{4\pi} & 0\\ -\frac{7C_{\mathsf{A}}}{15\epsilon}\frac{\alpha}{4\pi} & 1+\frac{21C_{\mathsf{A}}}{5\epsilon}\frac{\alpha}{4\pi}\end{pmatrix}\begin{pmatrix}\mathcal{O}_0\\ \mathcal{O}_1\end{pmatrix}_{0}. \qquad (12)$$

Lattice artifacts originating from the action

We find for the minimal basis due to O(4) symmetry

$$\begin{pmatrix} \mathcal{O}_0 \\ \mathcal{O}_1 \end{pmatrix}_{\mathrm{R}} = \begin{pmatrix} 1 + \frac{7C_{\mathsf{A}}}{3\epsilon}\frac{\alpha}{4\pi} & 0 \\ -\frac{7C_{\mathsf{A}}}{15\epsilon}\frac{\alpha}{4\pi} & 1 + \frac{21C_{\mathsf{A}}}{5\epsilon}\frac{\alpha}{4\pi} \end{pmatrix} \begin{pmatrix} \mathcal{O}_0 \\ \mathcal{O}_1 \end{pmatrix}_0 . \qquad (12)$$

Thus the diagonal basis and leading anomalous dimensions are

$$\mathcal{B}_0 = \mathcal{O}_0 \qquad \mathcal{B}_1 = \mathcal{O}_1 - \frac{1}{4}\mathcal{O}_0 , \qquad (13)$$

$$-\hat{\gamma}_0 = \frac{7C_{\mathsf{A}}}{12\pi\beta_0} = \frac{7}{11} \approx 0.636 , \qquad -\hat{\gamma}_1 = \frac{21C_{\mathsf{A}}}{20\pi\beta_0} = \frac{63}{55} \approx 1.145 .$$

[Narison and Tarrach, 1983; Alonso et al., 2014]

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Leading lattice artifacts from the action

$$\frac{\langle\Phi\rangle_{\mathrm{latt}(a)}}{\langle\Phi\rangle_{\mathrm{cont}}} = 1 - a^2\left(\sum_{j=0}^{1}\tilde{b}_j[\alpha(1/a)]^{-\hat{\gamma}_j}D_j^{\mathcal{B}}(\Lambda)\right)\times[1+\mathrm{O}(\alpha(1/a))]+\mathrm{O}(a^4)$$

$\Rightarrow$ Leading anomalous dimensions improve the convergence as $a \searrow 0$.

Conclusion

- No $a^2[\alpha(1/a)]^{-3}$ behaviour like for the O(3) model nor naive a^2, but

$$\boxed{a^2\left([\alpha(1/a)]^{0.636} + d_1[\alpha(1/a)]^{1.145} + d_2[\alpha(1/a)]^{1.636} + \ldots\right)}$$

  is the leading behaviour.
- Leading anomalous dimensions of contributions from the lattice Yang Mills action improve convergence as $a \searrow 0$.
- Short-cuts in perturbation theory can ease computational effort.
- Full $\left(\bigtimes_{j=1}^{N_\mathrm{f}} \mathrm{U}(1)_\mathrm{V} \text{ flavour symmetric}\right)$ lattice QCD is currently in progress (well advanced).
- Gradient flow observables require additional operators on the 4D-boundary, also in pure gauge theory.

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