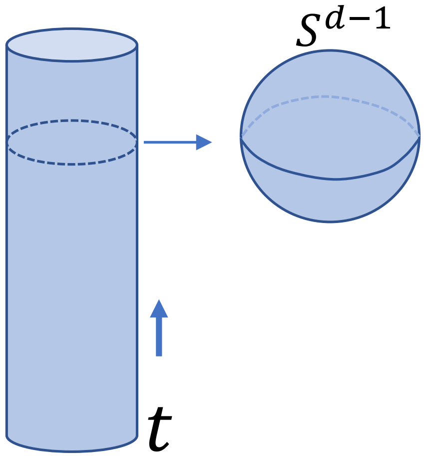
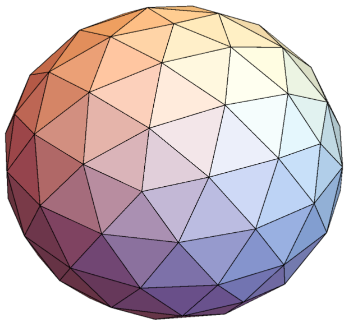
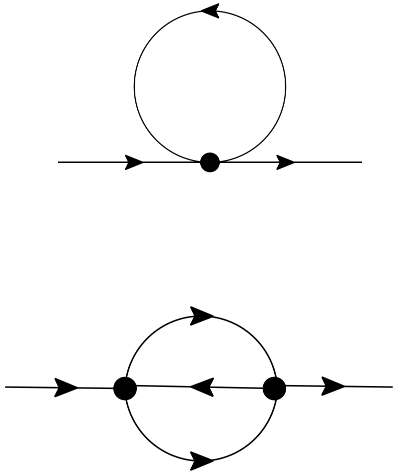
Accessing 3D CFTs in Radial Quantization on the Lattice







Andrew Gasbarro

University of Bern

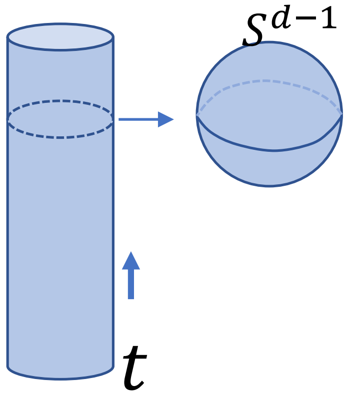
1.) 10.1103/PhysRevD.98.014502
arXiv:1803.08512

2.) 10.1103/PhysRevD.95.114510
arXiv:1610.08587

with Richard Brower (Boston), George Fleming (Yale), Dean Howard (Boston),
Timothy Raben (Michigan State),Chung-I Tan (Brown), Evan Weinberg (Nvidia)

Lattice 2019, Wuhan, 20/6/2019

The problem: Lattice implementation of *radial quantization*



$$ds^2_{R^d} = dr^2 + r^2 d\Omega^2 = \Omega_W^2(t,\Omega)\left(dt^2 + R^2 d\Omega^2\right) = \Omega_W^2 ds^2_{R\times S^{d-1}}$$

$r = R_0 e^{t/R}$ $\Omega_W = r/R$

CFT insensitive to local rescaling of metric
Weyl Factor cancels in homogeneous ratios

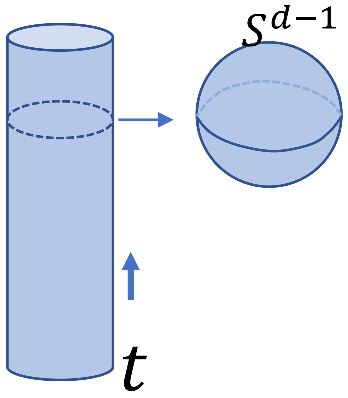
- A critical theory (CFT) mapped to $R \times S^{d-1}$ behaves like a gapped theory with $\xi_\phi = R/\Delta_\phi$. I.e.

$$\langle\phi(\vec{x}_1)\phi(\vec{x}_2)\rangle_{R^d} = \frac{1}{[\vec{x}_1 - \vec{x}_2]^{2\Delta_\phi}} \rightarrow \langle\phi(t_1,\vec{n}_1)\phi(t_2,\vec{n}_2)\rangle_{R\times S^{d-1}} = \frac{1}{[2\cosh((t_1-t_2)/R) - 2\cos\theta_{12}]^{\Delta_\phi}}$$

$$\langle\phi(t_1,\vec{n}_1)\phi(t_2,\vec{n}_2)\rangle_{R\times S^{d-1}} \xrightarrow{|t_2-t_1|\gg R} e^{-\Delta_\phi|t_2-t_1|/R}$$

$$\boxed{\xi_\phi^{-1} = \Delta_\phi/R}$$

The problem: Lattice implementation of *radial quantization*



- “For [this result] to be useful for numerically estimating the scaling dimension, it is necessary to approximate the continuum by a sequence of lattices. It is most convenient to choose these lattices to be regular. For $d \neq 2$, however, the space is curved and only a finite number of regular lattices may be embedded in the space. For S^2 for example these correspond to the platonic solids….”
- “Whether this will provide a useful numerical approach to critical exponents remains to be seen”
- – John Cardy (1985)

J. L. Cardy, J. Phys. A 18, L757 (1985)
Domb and Lebowitz “Phase Transitions and Critical Phenomena”

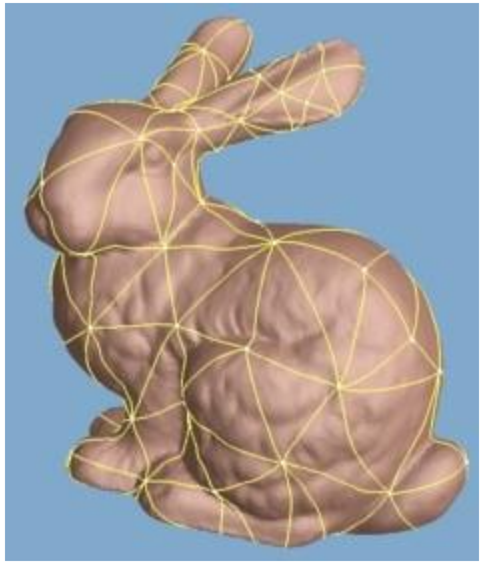
Outline:

i. Our Solution: "**quantum finite elements**"
ii. Scalar ϕ^4 on the sphere, S^2 ; 2D Ising or c=1/2 minimal CFT
iii. Scalar ϕ^4 on the cylinder, $R \times S^2$; accessing the 3D Ising CFT
iv. Further Applications & Future Directions

i.) **Our solution:** Quantum Finite Elements

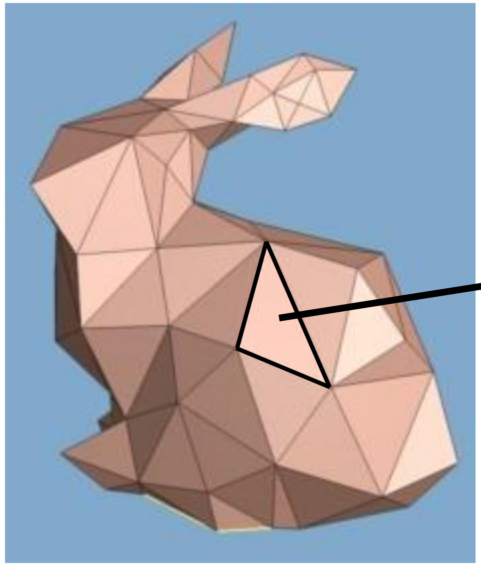
Target Manifold, M

Topology $M \to M_\sigma$



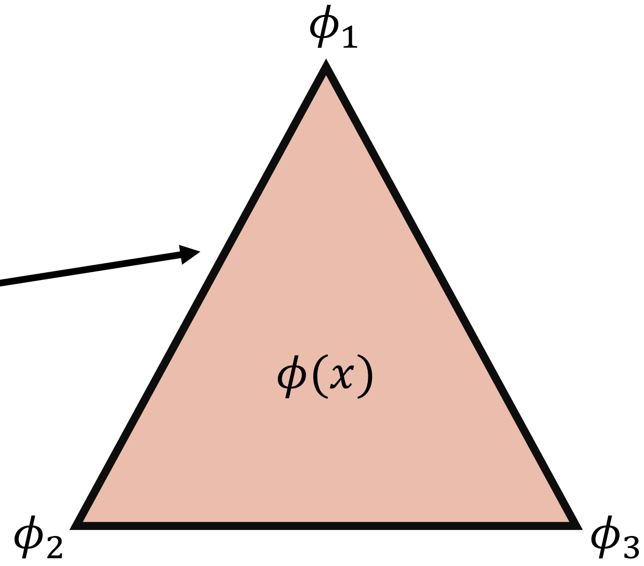
- Partition space into simplices
- Simplicial Complexes

Geometry, $g^{\mu\nu} \to g_\sigma^{\mu\nu}$



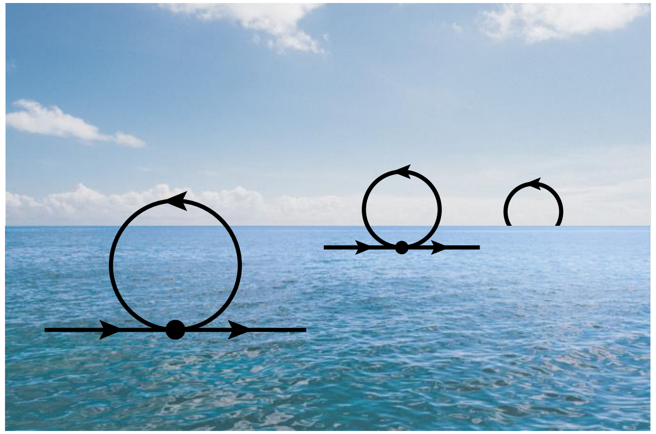
- Define metric by assigning lengths
- Regge Calculus

Hilbert Space



- Expand fields in finite basis
- Finite Elements
- Discrete Exterior Calc.

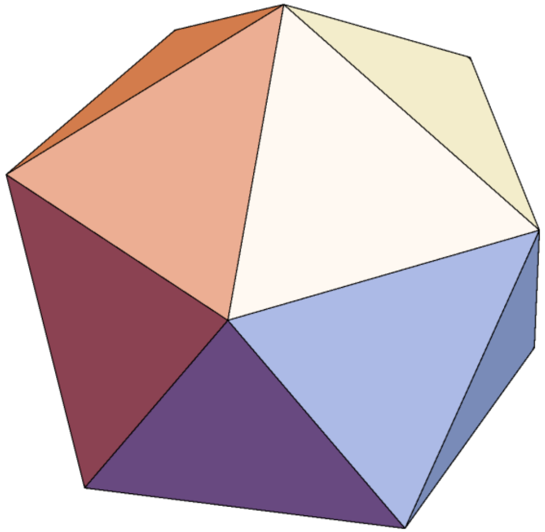
Quantum Effects and Renormalization



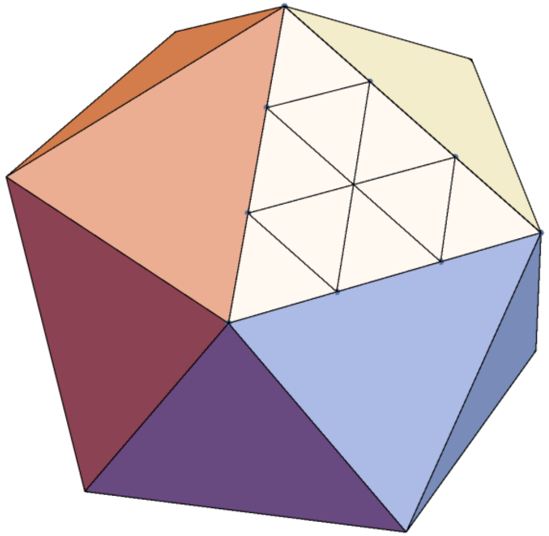
- Quantum loops sensitive to curvature
- “Quantum Finite Elements”

Simplicial Lattice for S^2

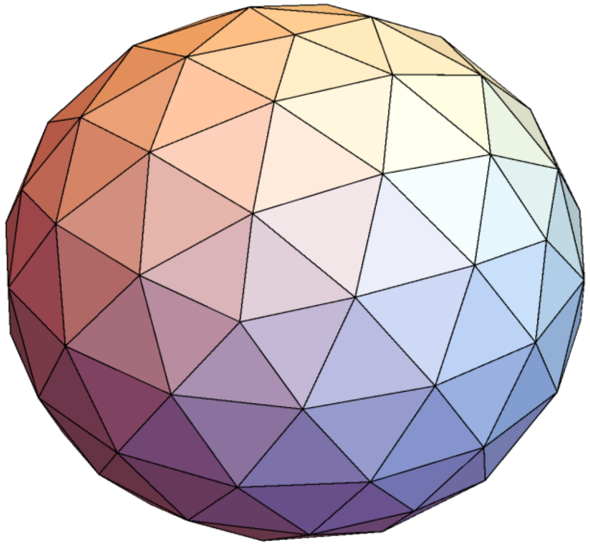
- Construction of refined simplicial lattice (L= $3 \sim 1/a$)



Icosahedron
a.k.a L=1
"Plato's best sphere"



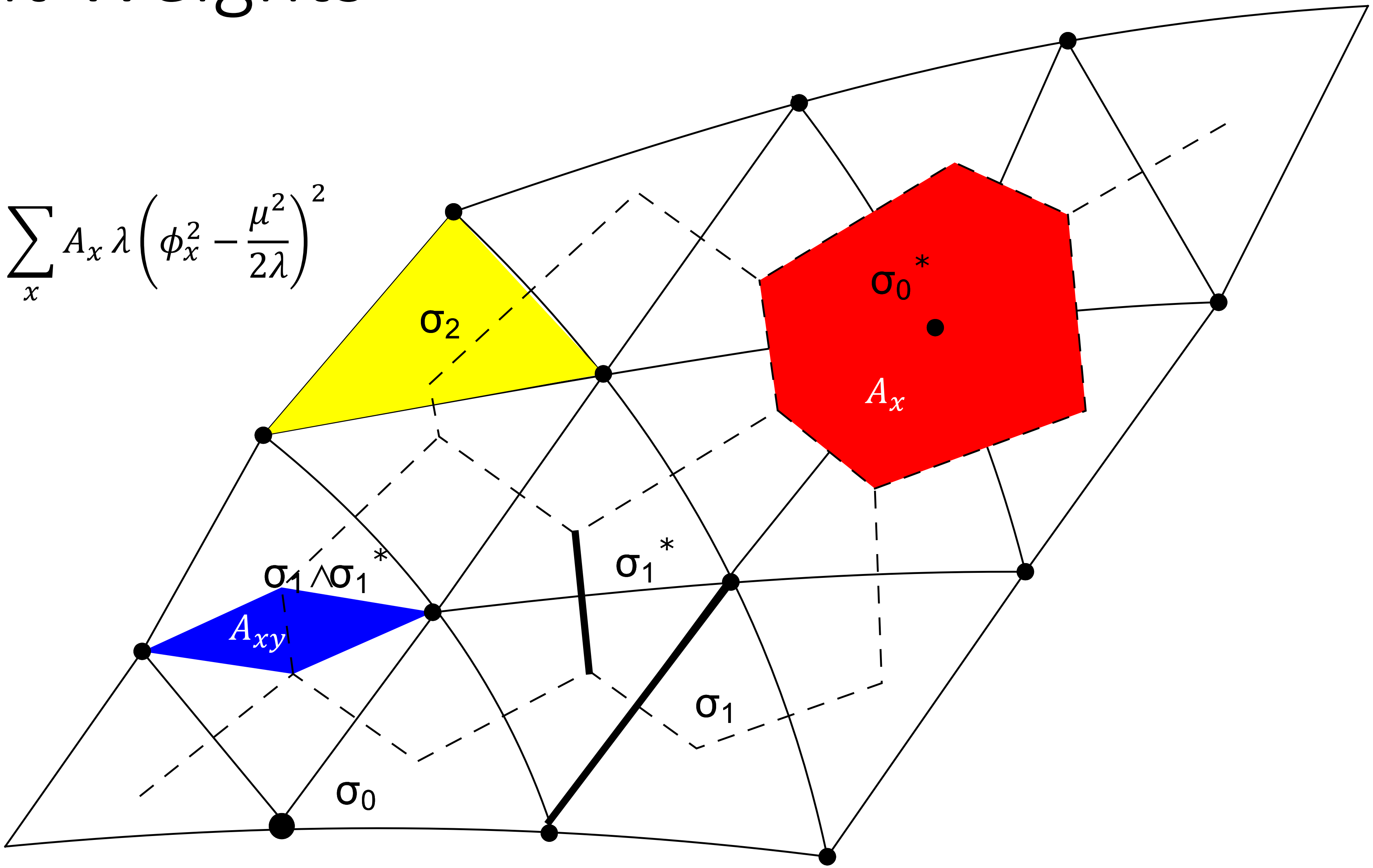
"Refine"
Divide each face into smaller
Equilateral triangles (L-1) times



Project
All points now lie on sphere
Distances given by secant distances
In embedding space

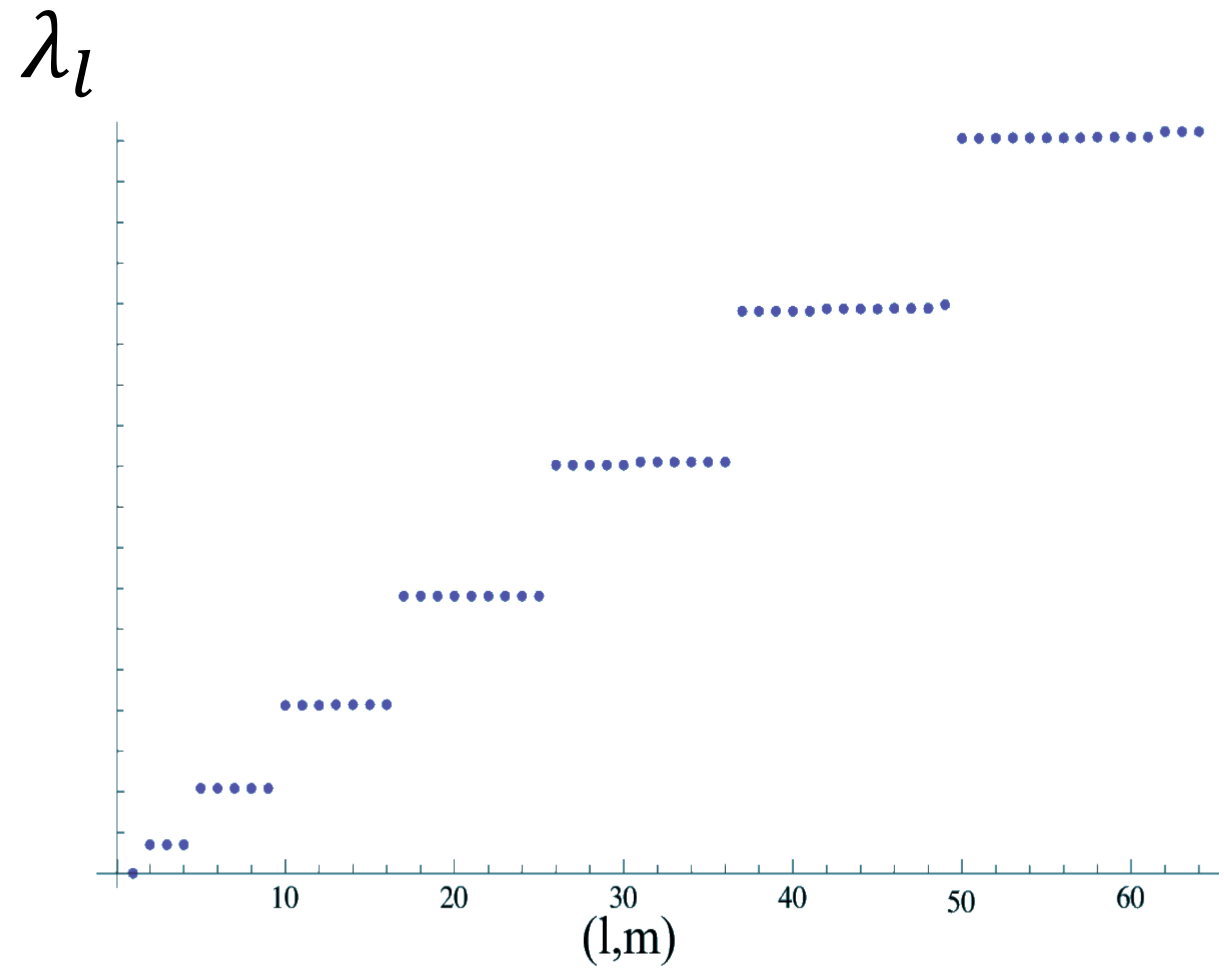
Finite Element Weights

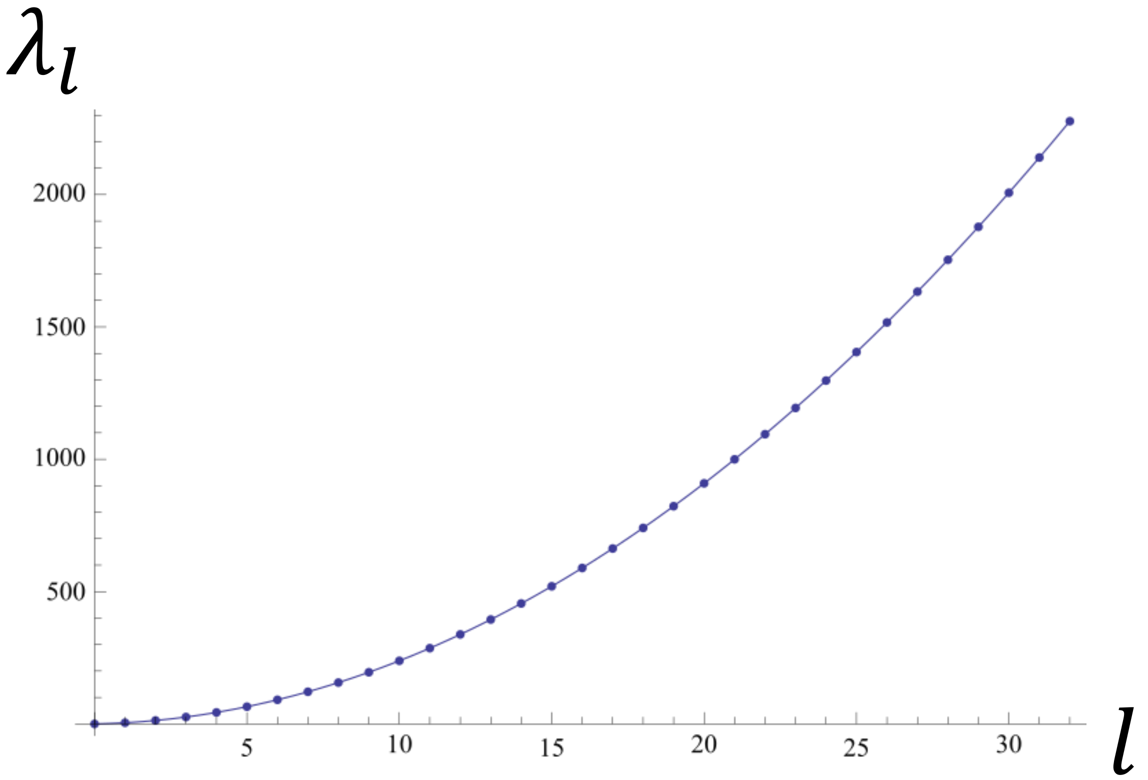
$$S_{FEM} = \sum_{\langle x,y \rangle} A_{xy} \frac{(\phi_x - \phi_y)^2}{l_{xy}^2} + \sum_x A_x \lambda \left(\phi_x^2 - \frac{\mu^2}{2\lambda} \right)^2$$



FEM Laplacian Spectrum on S^2

$$S_{FEM}^{kinetic} = \frac{1}{2}\sum_{\langle i,j\rangle} A_{ij}\frac{(\phi_i - \phi_j)^2}{l_{ij}^2}$$



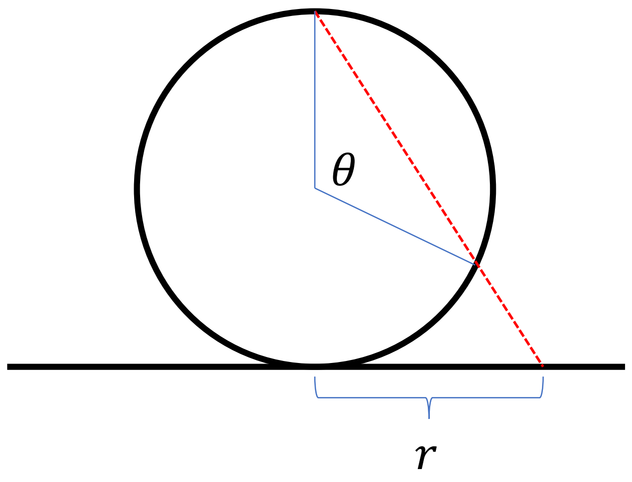


$$\lambda(l)_{fit} = l + 1.00012l^2 - 0.000013l^3 - 0.000005l^4$$

ii.) Scalar ϕ^4 Theory on S^2

- Warm up to ϕ^4 Theory on $R \times S^2$ (Radial Quant.)
- Stereographic Projection maps R^2 to S^2
  - Equivalent up to Weyl factor

$$ds^2_{R^2} = dr^2 + \mathrm{r}^2 \mathrm{d}\phi^2 = \Omega^2_{\mathrm{W}}(\theta,\phi)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

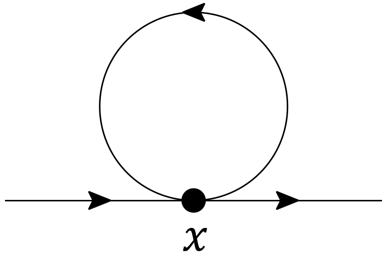


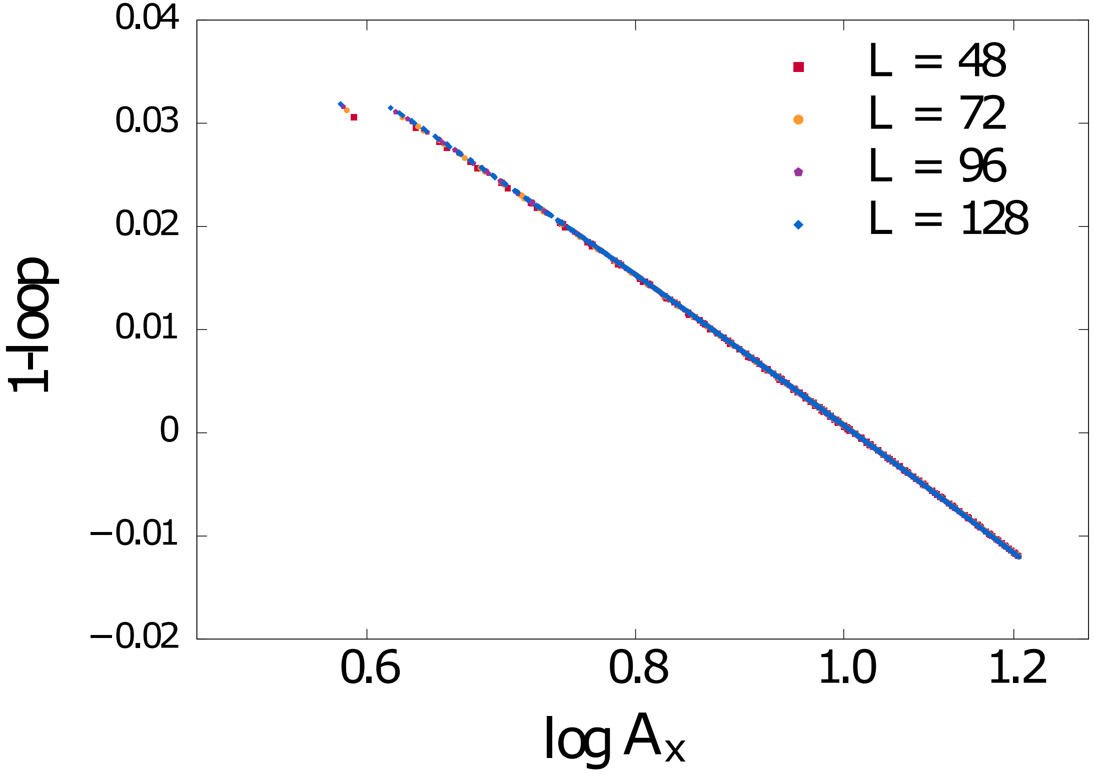
- Study Wilson Fischer fixed point
  - CFT is minimal c=1/2 model
- Classical FEM action:

$$S_{FEM} = \sum_{\langle x,y\rangle} A_{xy} \frac{(\phi_x - \phi_y)^2}{l^2_{xy}} + \sum_x A_x\, \lambda \left(\phi_x^2 - \frac{\mu^2}{2\lambda}\right)^2$$

ii.) Scalar ϕ^4 Theory on S^2

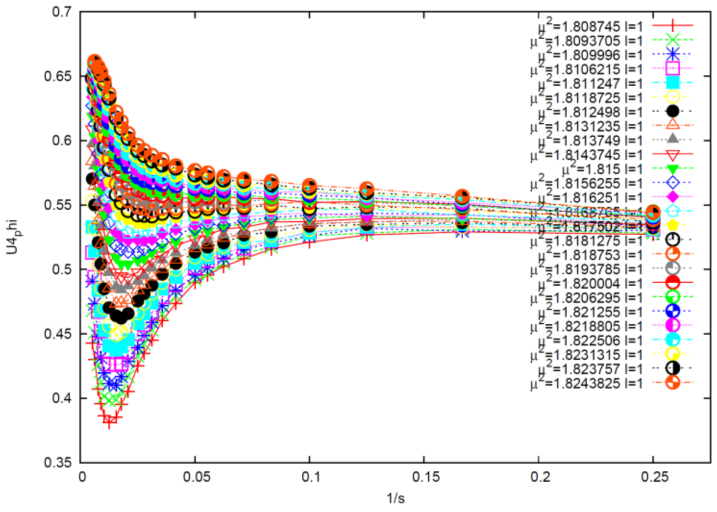
Quantum loop sensitive to lattice spacing in UV and contributes to breaking SO(3) sym in IR

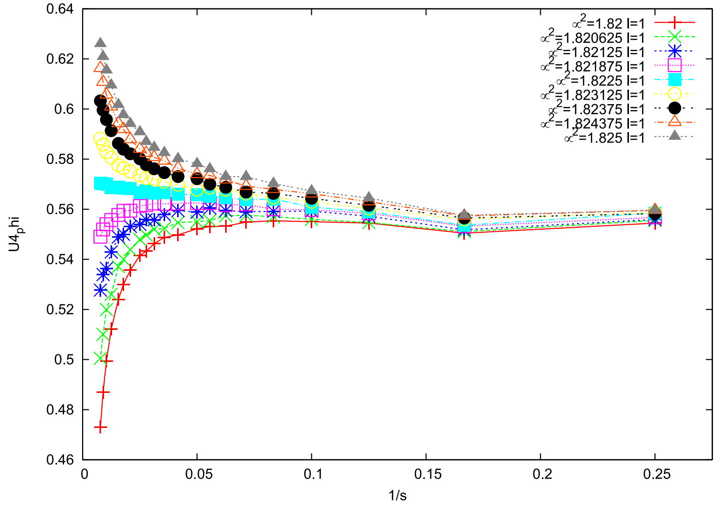




Scan for critical surface
With **binder cumulants**

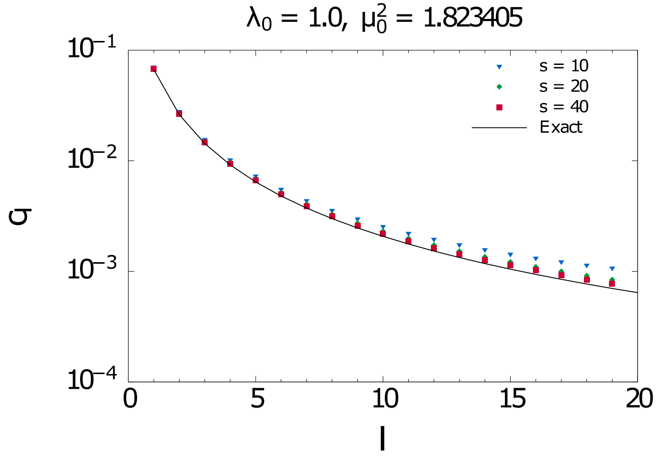
$$U_4(\mu, \lambda, s) = \frac{3}{2}\left(1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}\right)$$

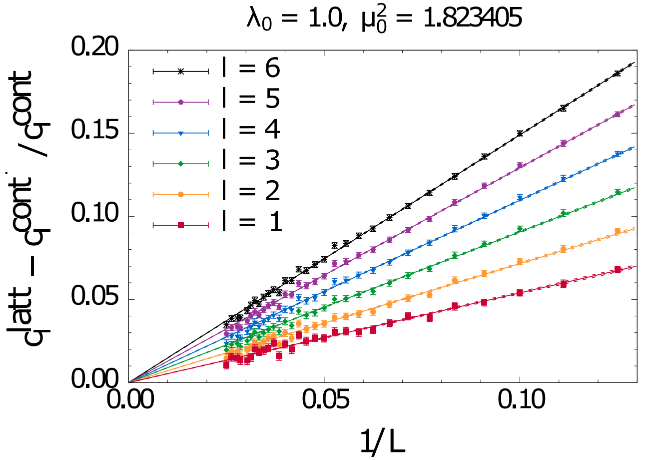




ii.) Scalar ϕ^4 Theory on S^2

Two Point Function $$c_l = \sum_{x,x'} A_x A_{x'} \langle \phi_x \phi_{x'} \rangle Y_x^{lm} Y_{x'}^{lm}$$

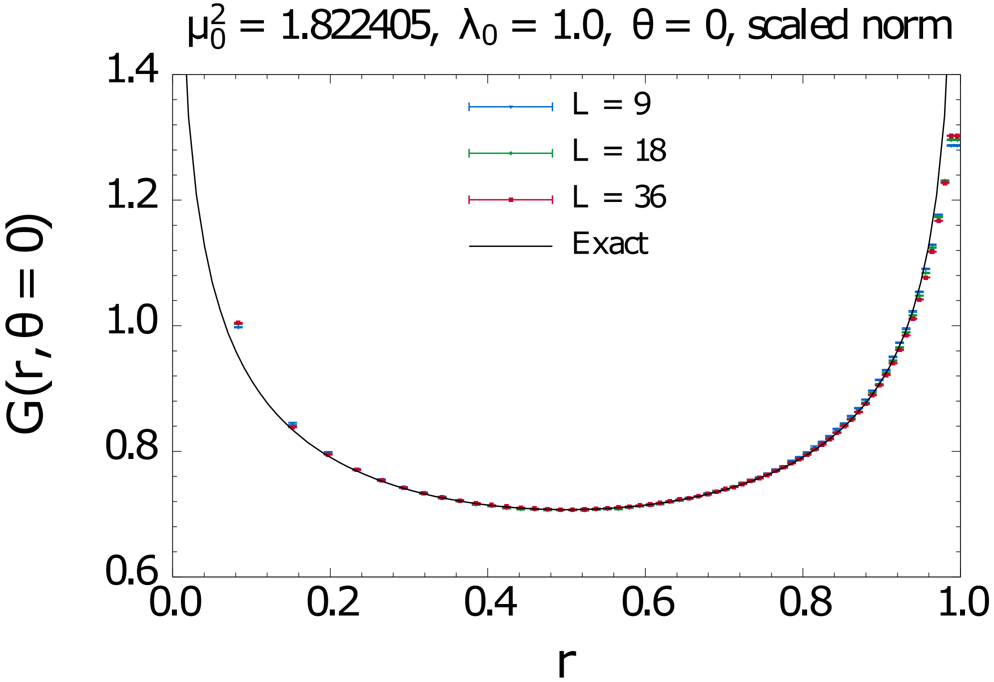




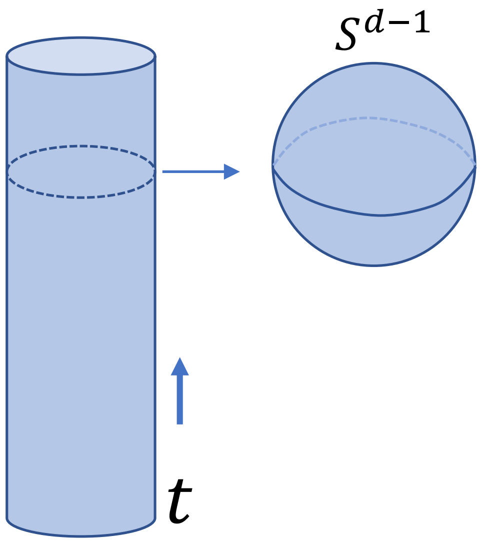
Four Point Function $$g(u,v) = \frac{\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle}{\langle \phi_1 \phi_2 \rangle \langle \phi_3 \phi_4 \rangle}$$

$$u = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 r_{24}^2}, v = \frac{r_{14}^2 r_{23}^2}{r_{13}^2 r_{24}^2}$$

$$u = z\bar{z}$$
$$v = (1-z)(1-\bar{z})$$
$$z = re^{i\theta}$$



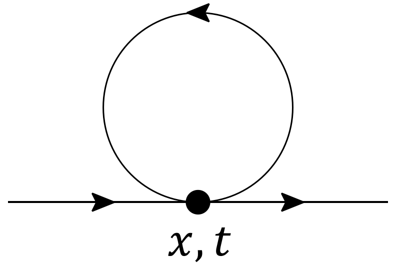
iii.) Scalar ϕ^4 Theory on $\mathrm{R} \times S^2$

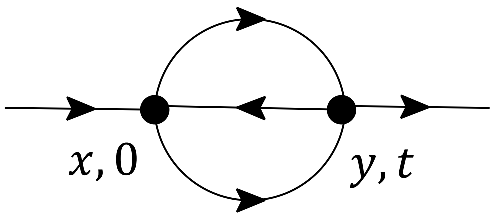


- Classical FEM Action

$$S_{FEM} = \sum_{\langle x,y \rangle ,t} A_{xy} \frac{(\phi_{x,t} - \phi_{y,t})^2}{l_{xy}^2} + \sum_{x,t} A_x \, \lambda \left(\phi_{x,t}^2 - \frac{\mu^2}{2\lambda} \right)^2 + \sum_{x,t} A_x (\phi_{x,t+1} - \phi_{x,t})^2$$

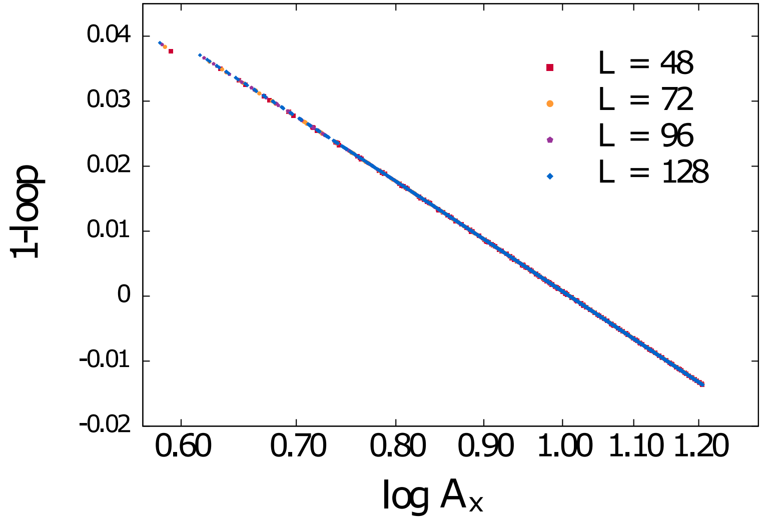
- In 3D, two UV divergent diagrams
  - 1-loop diagram treated as on S^2
  - 2-loop diagram nonlocal
    - Interpret in derivative expansion
    - UV Divergence only in the 0-derivative piece
    - k-derivative pieces should become SO(3) invariant in CL

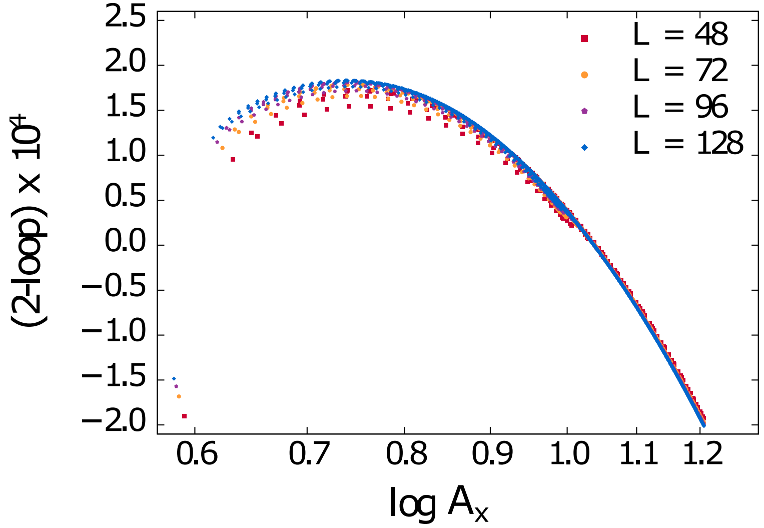


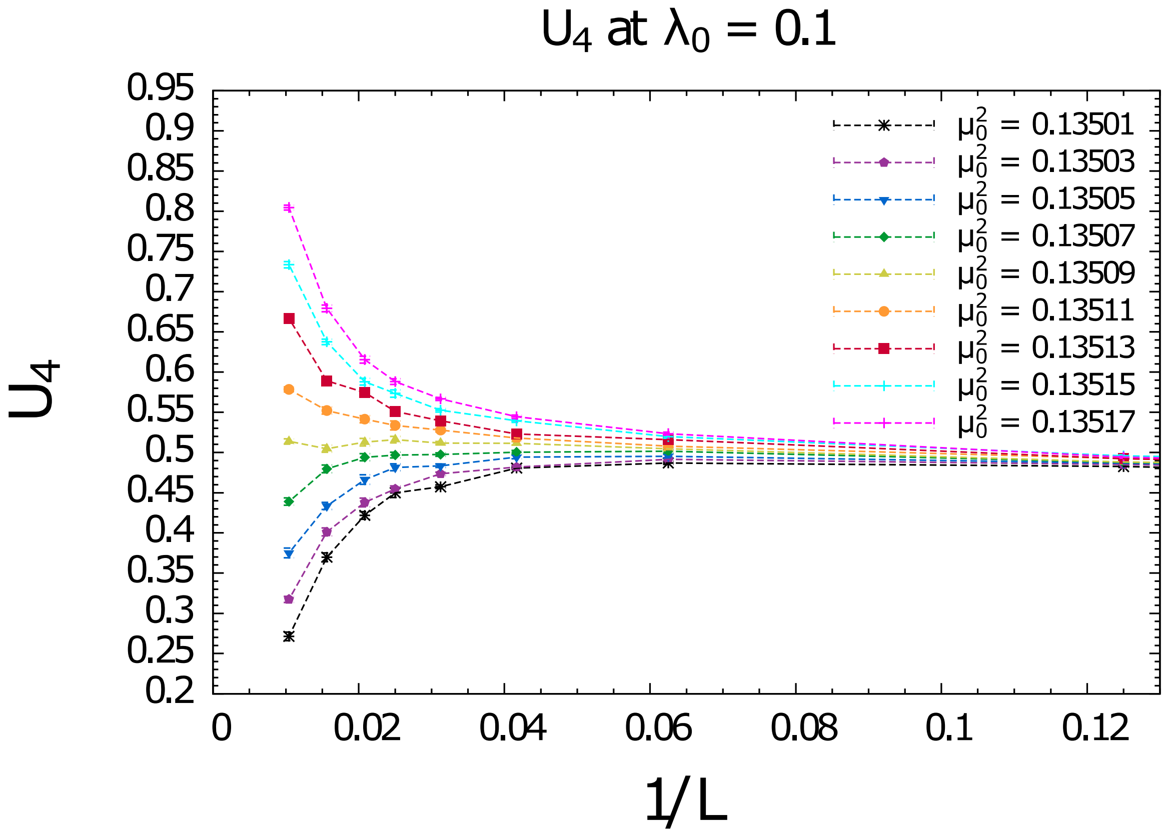


$$\Gamma_{2-loop}^{0-deriv}{}_{x} = \sum_{y,t} A_y \left[G_{x,0;y,t} \right]^3$$

iii.) Scalar ϕ^4 Theory on $\mathrm{R} \times S^2$



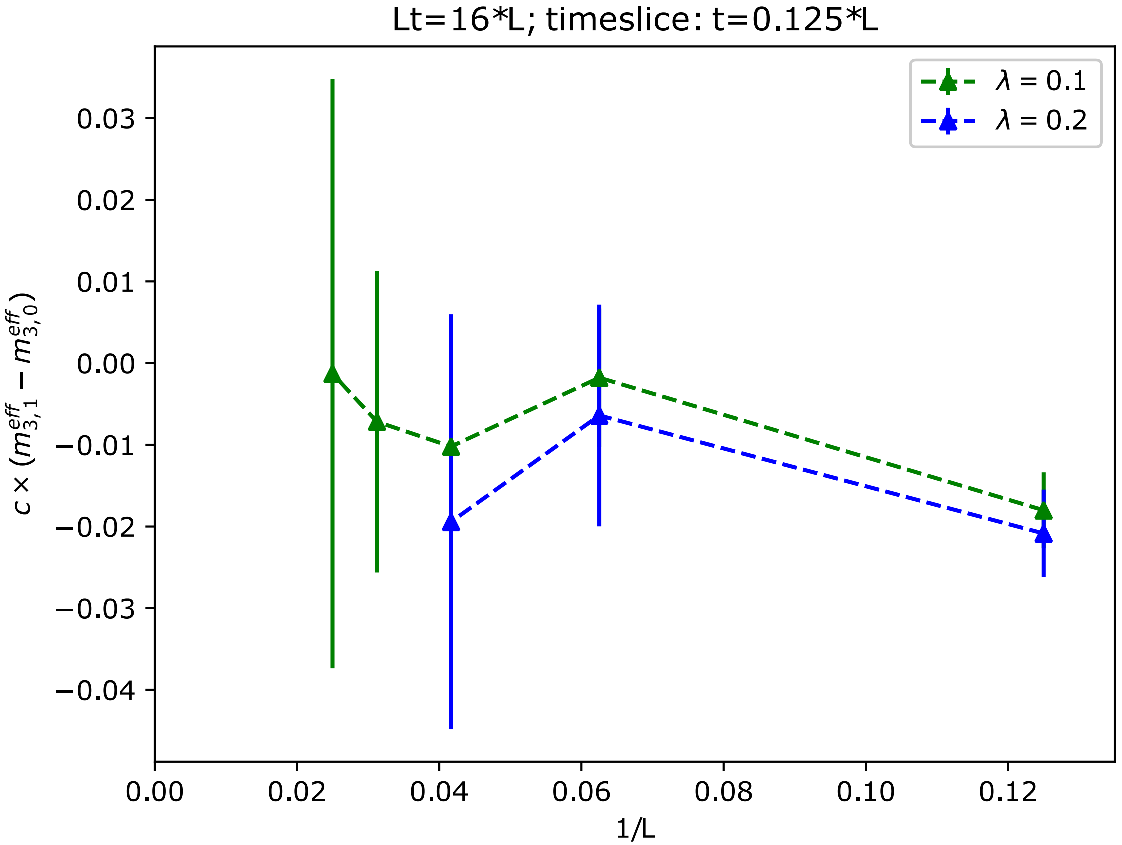


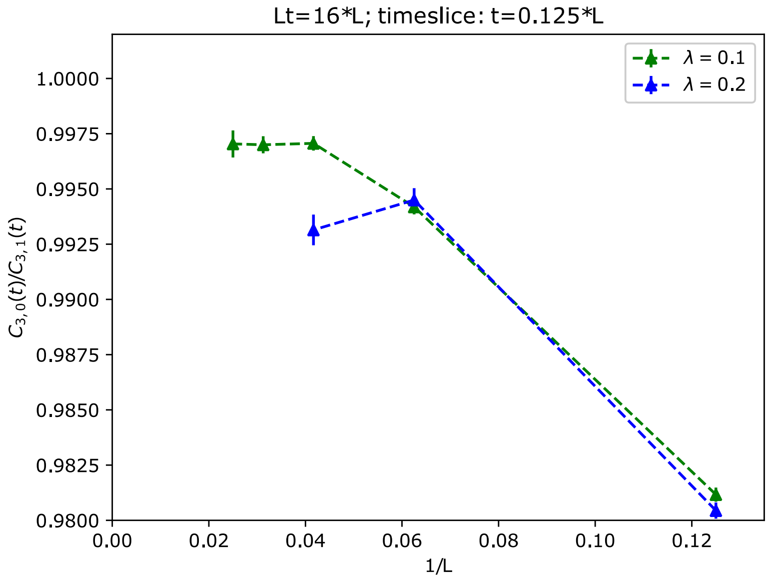


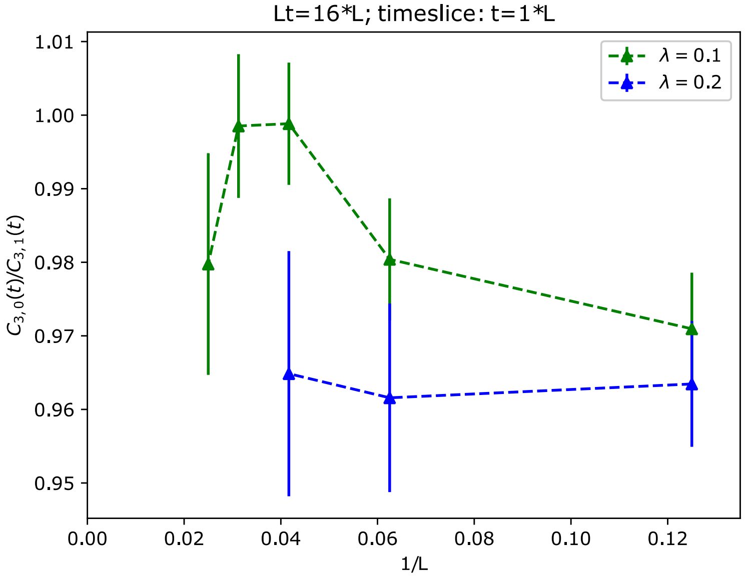
Criticality at $(\lambda, \mu^2) = (0.1, 0.13509)$

iii.) Scalar ϕ^4 Theory on $\mathrm{R} \times S^2$

Testing rotation symmetry







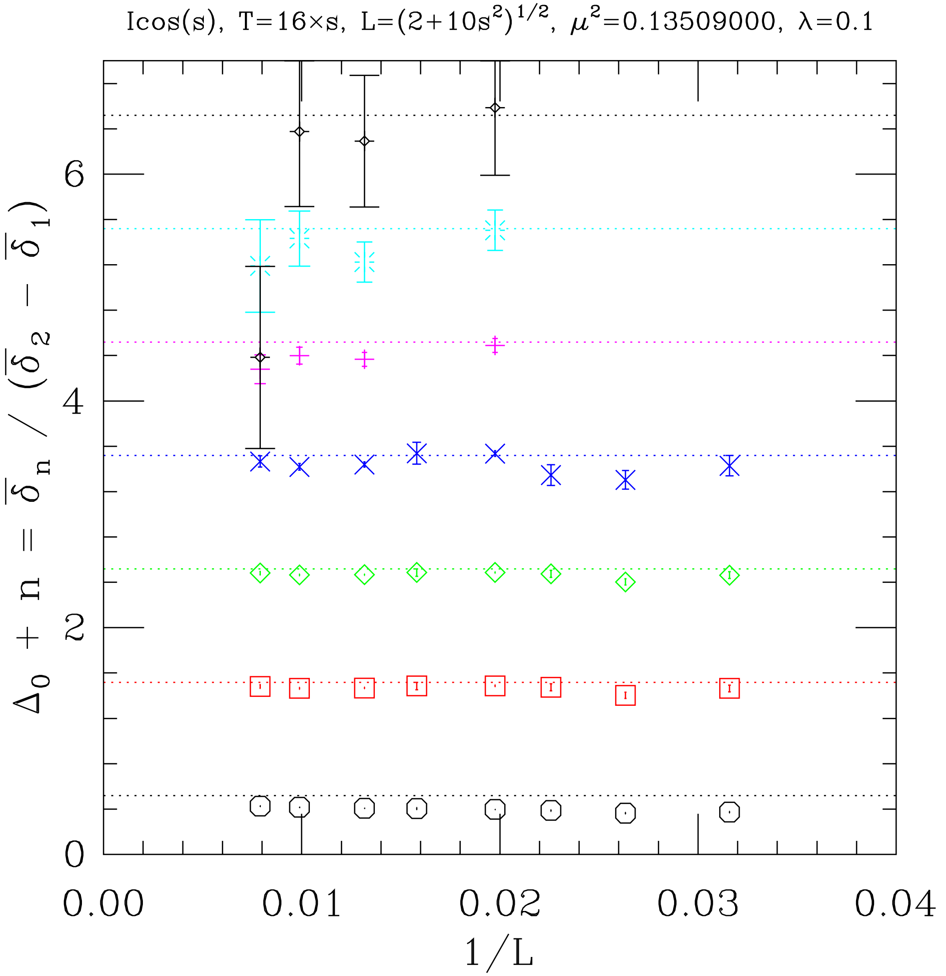
iii.) Scalar ϕ^4 Theory on $\mathrm{R} \times S^2$

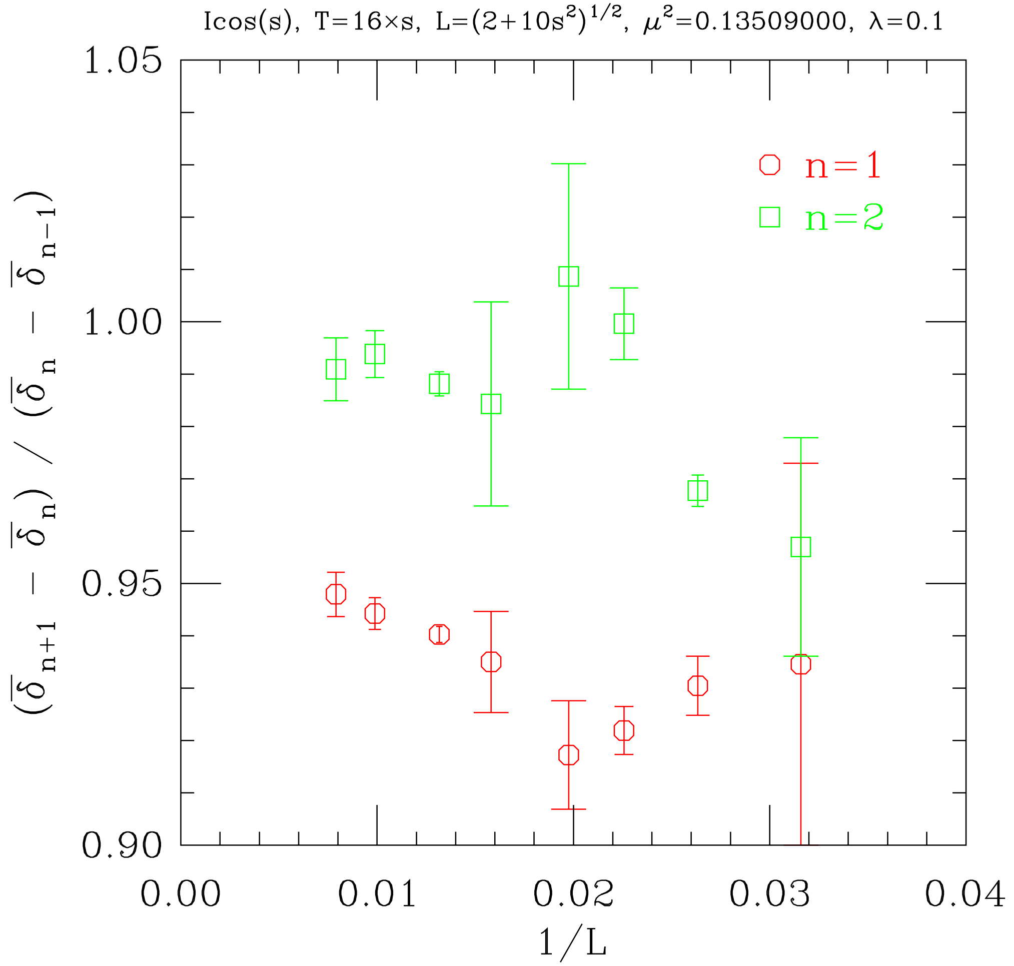
Testing translation + dilation symmetry

$$[D, K_\mu] = -iK_\mu$$
$$[D, P_\mu] = iP_\mu \quad \rightarrow$$
$$[K_\mu, P_\mu] = 2iD$$

Algebra of ladder operators

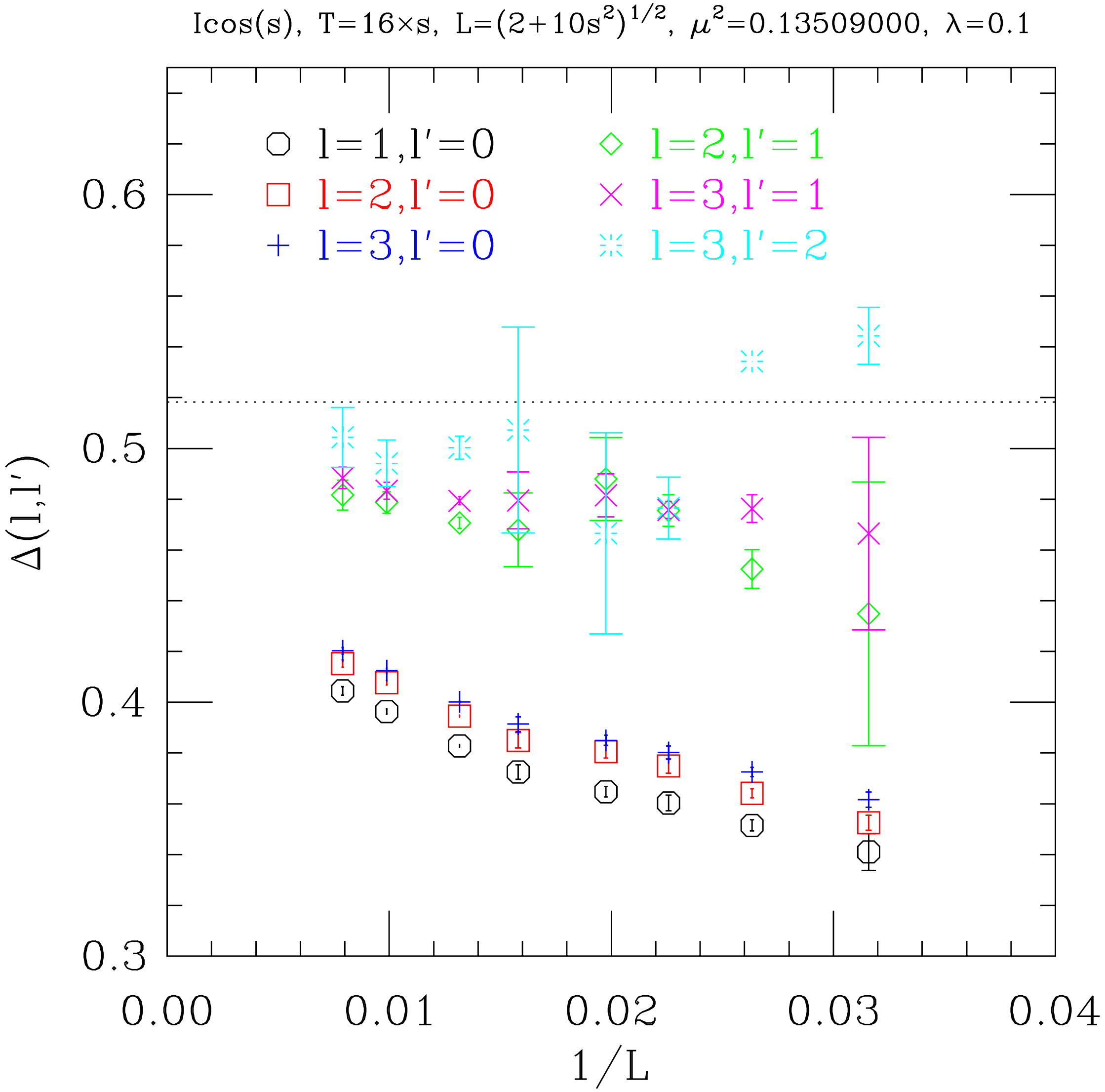
Integer spaced descendents





iii.) Scalar ϕ^4 Theory on $\mathrm{R} \times S^2$

Z2 Odd Primary Scaling Dimension



iv.) On going and future

- Operator product expansion of 4pt function

$$\frac{\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle}{\langle \phi_1 \phi_2 \rangle \langle \phi_3 \phi_4 \rangle} = g_s(u,v) = \sum_O \lambda_O^2 g_{\Delta_O,l}(u,v)$$

- Can truncate to first few terms at small r

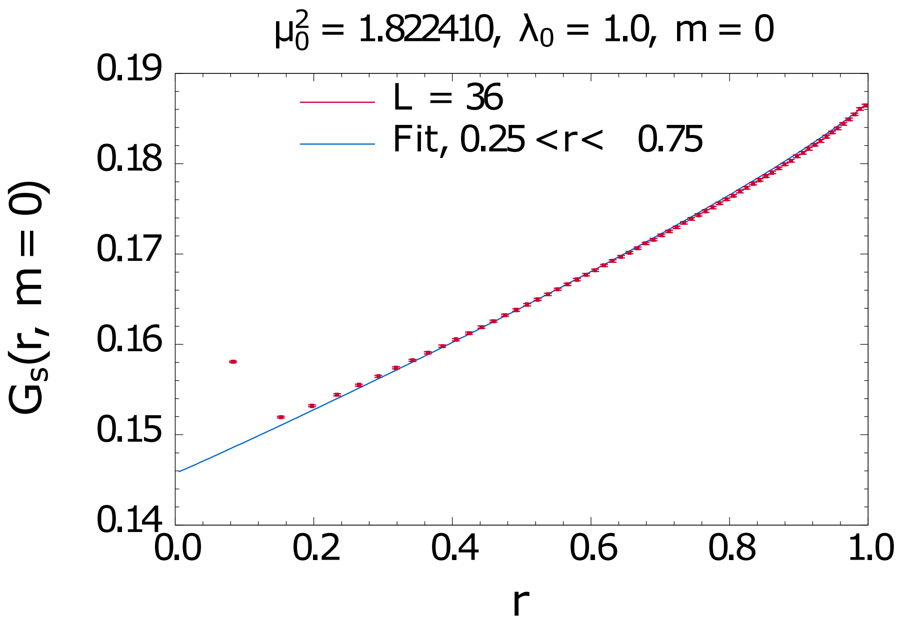
$$u = z\bar{z}$$
$$v = (1-z)(1-\bar{z})$$
$$z = re^{i\theta}$$

$$g_s \propto 1 + \lambda_\epsilon^2 g_{\epsilon,0}(r,\theta) + \lambda_T^2 g_{T,2}(r,\theta) + \cdots$$

- λ_T related to central change, c
- In 2D, conformal blocks have closed form expressions
- We fit to these expressions to extract $\Delta_\epsilon, \lambda_\epsilon^2, c$

- In 3D, conformal blocks don't have simple closed form but can be computed numerically
  - Often done in 3D conformal bootstrap

- Working on measurements of 4pt function on $R \times S^2$!



μ^2	s	$r_{\min} \le r \le r_{\max}$	norm	Δ_ϵ	λ_ϵ^2	c
1.82241	9	$0.25 \le r \le 0.75$	0.2900	1.075	0.2536	0.4668
1.82241	9	$0.30 \le r \le 0.70$	0.2901	1.075	0.2533	0.4704
1.82241	9	$0.35 \le r \le 0.65$	0.2902	1.077	0.2533	0.4738
1.82241	9	$0.40 \le r \le 0.60$	0.2902	1.016	0.2427	0.4747
1.82241	18	$0.25 \le r \le 0.75$	0.2051	1.068	0.2563	0.4866
1.82241	18	$0.30 \le r \le 0.70$	0.2051	1.056	0.2544	0.4878
1.82241	18	$0.35 \le r \le 0.65$	0.2051	1.050	0.2535	0.4904
1.82241	18	$0.40 \le r \le 0.60$	0.2051	1.046	0.2526	0.4884
1.82241	36	$0.25 \le r \le 0.75$	0.1457	1.031	0.2528	0.4926
1.82241	36	$0.30 \le r \le 0.70$	0.1458	1.026	0.2519	0.4932
1.82241	36	$0.35 \le r \le 0.65$	0.1458	1.018	0.2508	0.4931
1.82241	36	$0.40 \le r \le 0.60$	0.1458	1.007	0.2486	0.4933

iv.) On going and future

- Connection to the Large Charge Expansion

- For CFT with global symmetry in fixed charge sector, charge density introduces a scale
- Well below this scale, dynamics dominated by goldstone modes and an approximately scale inv Lagrangian
- E.x.

$$\mathcal{L} = \frac{k_{3/2}}{27}\left(\partial_\mu \chi \partial^\mu \chi\right)^{3/2} + \frac{k_{1/2}R}{3}\left(\partial_\mu \chi \partial^\mu \chi\right)^{1/2} + \cdots$$

- Which predicts a ground state energy as a function of charge

$$E_\Sigma(Q) = \sqrt{\frac{Q^3}{V}}\left(c_{3/2} + c_{1/2}\left(\frac{RV}{2Q}\right)\right) + q_\Sigma + O\left(\frac{1}{Q}\right)$$

q_Σ can be computed exactly for simple manifolds

- On $R \times S^{d-1}$ the ground state "energy" is the conformal scaling dimension

$$\Delta(Q) = \sqrt{\frac{Q^3}{4\pi}}\left(c_{3/2} + c_{1/2}\left(\frac{4\pi}{Q}\right)\right) - 0.94 + O\left(\frac{1}{Q}\right)$$

- Requires finite density simulation

Large charge EFT:
1610.04495
1707.00710
Numerical studies:
1707.00711
1902.09542

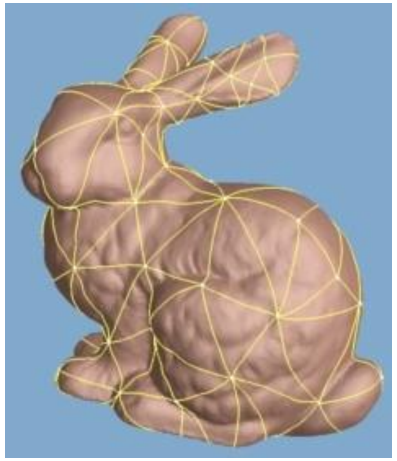
Conclusions

- Current capabilities
  - Classical FEM/DEC formalism describe classical field theories on arbitrary Riemannian manifold
    - Including Dirac-Wilson (not covered here), **1610.08587**
  - Perturbative corrections allow one to reach continuum limit for superrenormalizable QFT
  - We use highly efficient cluster algorithms to study scalar field theories
  - 2D scalar phi4 theory studied in great detail on 2-sphere, **1803.08512**
  - 3D scalar phi4 theory on $R \times S^2$ appears to reach continuum limit and recover full conformal symmetry
    - Detailed report on 2- and 4-point functions to follow
- Ongoing work
  - 3D Ising 4-point function on $R \times S^2$, and connection to conformal block expansion
  - Parallel code to speed up computations in 3D
  - O(N) models and connection to large charge
  - Tooling up for $R \times S^3$, see poster by **D. Berkowitz "Laplace Operator on Discretized 3-Sphere"**
- **Many** future directions
  - Nonperturbative scheme for renormalizable QFT
  - Gauge theories
  - Hyperbolic spaces and connection to holography
  - Massive (non conformal) theories
    - Luscher method on $R \times S^{d-1}$
  - …

Thank you!

Backup Slides

Topology and Simplicial Complexes

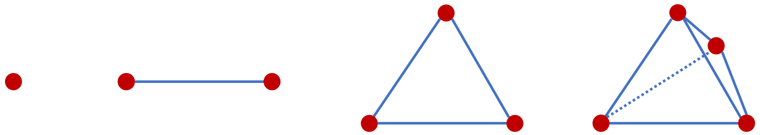


- Replace target manifold with a sequence of increasingly dense simplicial partitions of "refinement" s

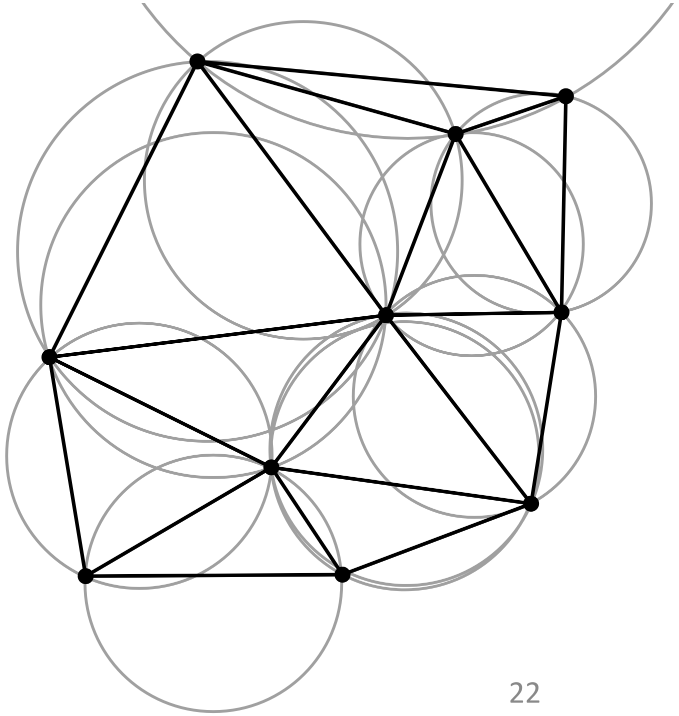
$$M \rightarrow \{M_s\}^{s \in 1,2,\ldots}$$

- At the moment, no metric. Purely topological.
  - How the simplices are glued together determines the topology of the space
  - Practically speaking, at each refinement we have a list of points and a neighbor table (amenable to intrinsic geometry)
- Simplicial complex provides an organized foundation on which to build geometrical structures (metric, vierbein, spin connection, etc)

$$\sigma_1 \rightarrow \sigma_2 \rightarrow \ldots \rightarrow \sigma_d$$



- Given a set of vertices, simplicial complex can always be constructed via the Delaney / Voronoi construction
  - Establishes links between vertices by maximizing smallest angle in simplices
  - Relies on knowing something about Geometry first, so slightly out of order

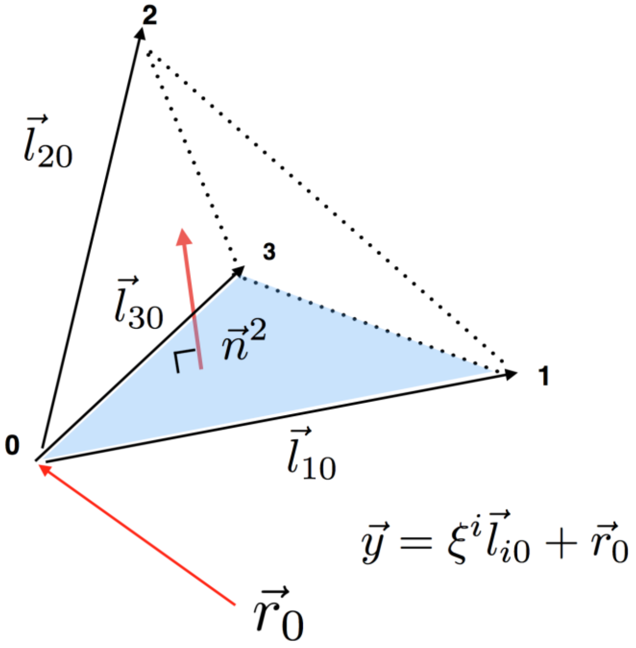


Geometry and Regge Calculus

- Define metric distance along edges by assigning lengths
  $$|\sigma_1(i,j)| \equiv l_{ij} = some\ \#$$
- Continue metric to interior of each simplex to be flat (not the only choice)
  - Then, geometry of each simplex known entirely in terms of edge lengths
- Very clean coordinate choice: Barycentric Coordinates
  - For a point $\vec{y}$ in a d-simplex

$$\vec{y} = \sum_{i=0}^{d} \xi^i \, \vec{r_i} \qquad 0 \le \xi^i \le 1 \qquad \sum_{i=0}^{d} \xi^i = 1 \qquad \rightarrow \qquad \vec{y} = \vec{r_0} + \sum_{i=1}^{d} \xi^i \, \overrightarrow{l_{i0}}$$

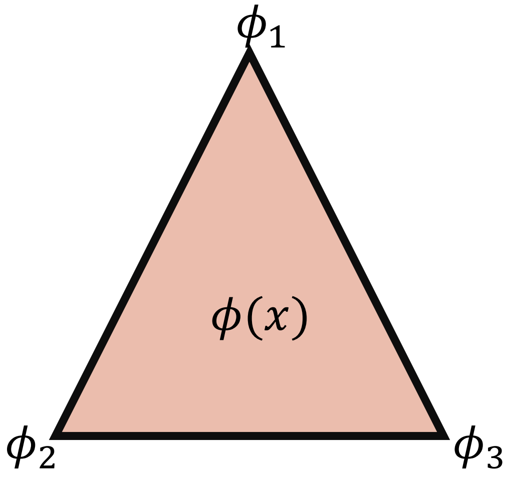
$$ds^2 = d\vec{y} \cdot d\vec{y} = g_{ij}\xi^i\xi^j \qquad g_{ij} = \overrightarrow{l_{i0}} \cdot \overrightarrow{l_{j0}} = \frac{1}{2}\left(l_{i0}^2 + l_{j0}^2 - l_{ij}^2\right)$$



- Constant flat metric everywhere inside simplex
- Can construct, e.g., Einstein Hilbert term and find EH action given entirely in terms of deficit angles
  - "GR without Coordinates", T. Regge, 1960

$$S_s = \frac{1}{2} \sum_{\sigma_d \in M_s} \int_{\sigma_d} d^d\,\vec{y}\,\left[\vec{\nabla}\phi(y) \cdot \vec{\nabla}\phi(y) + m^2\phi(y)^2\right] = \frac{1}{2} \sum_{\sigma_d \in M_s} \int_{\sigma_d} d^d\,\xi\,\sqrt{|g_{ij}|}\left[g^{ij}\,\partial_i\phi(\xi)\partial_j\phi(\xi) + m^2\phi(\xi)^2\right]$$

Hilbert Space and Finite Elements



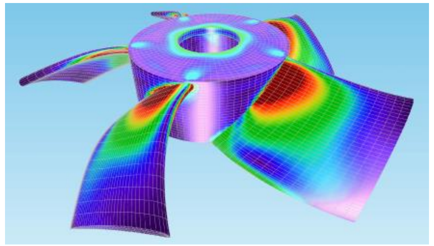
- To regulate the QFT, we truncate the Hilbert space by expanding in a finite field basis on each simplex called a **finite element basis**.

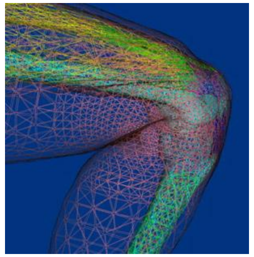
$$\phi_\sigma(\xi) = \sum_{i=0}^{d} E^i(\xi)\phi_i \qquad \sum_{i=0}^{d} E^i(\xi) = 1 \qquad E^i(\vec{r_j}) = \delta^i_j$$

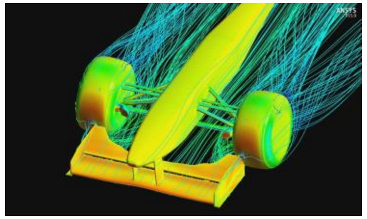
- Common tool for solving classical PDEs in engineering, E&M (cf. Jackson), fluid dynamics, …
- We use simplest case, **linear finite elements** , $E^i(\xi) = \xi^i$
- Gradients are constants everywhere in the simplex, $\partial_i \phi_\sigma(\xi) = \phi_i - \phi_0$
- Plugging expansion into action, arrive at discrete action in terms of lattice degrees of freedom located at vertices

$$S_\sigma = \frac{1}{2}\sum_{i,j=1}^{d} |\sigma_d|\, g^{ij}\, (\phi_i - \phi_0)(\phi_j - \phi_0) = \frac{1}{2}\sum_{\langle i,j\rangle} V_{ij} \frac{(\phi_i - \phi_j)^2}{l_{ij}^2}$$

"vertex form" "edge form"

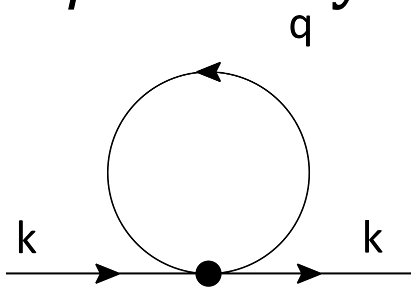






How does lattice describe the continuum (quantum)

- Nonperturbative proofs are hard
- One can prove renormalizability and continuum limit in perturbation theory
  - "Power counting theorem" for lattice perturbation theory (T. Reisz 1988-1989)
    - $\deg(I) < 0 \rightarrow$ Integral is finite and given by naïve continuum limit as $a \rightarrow 0$
- Consider ϕ^4 theory in d=2,3 dimensions. At one loop:



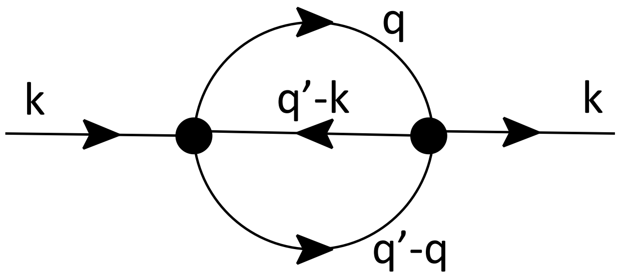
$$= I_1(k, m; a) = \frac{\lambda}{2} \int_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d} \frac{1}{\tilde{q}^2 + m^2}$$

$\deg(I_1) = d - 2$
divergent in 2 and 3 dimensions

- Divergent constant can be absorbed into counterterm δm^2
  - Diagram only has support on k=0 due to *translation invariance*
  - **This will change on a curved lattice without translation invariance!**

How does lattice describe the continuum (quantum), ctd

- At two loops:



$$= I_2(k,m;a) = \frac{\lambda^2}{3}\int_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d}\frac{d^d q'}{(2\pi)^d}\frac{1}{(\tilde{q}^2+m^2)}\frac{1}{((\widetilde{q'-q})^2+m^2)}\frac{1}{((\widetilde{q'-k})^2+m^2)}$$

$\deg(I_2) = 2d-6$ → Divergent in d=3

- Can check that divergence is independent of k and renormalized perturbation theory can be made Lorentz Invariance

$$I_2(k,m;a) = I_2(0,m;a) + D_2(k,m;a)$$

$$D_2(k,m;a) = \frac{\lambda^2}{3}\int_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d}\frac{d^d q'}{(2\pi)^d}\frac{1}{(\tilde{q}^2+m^2)}\frac{1}{((\widetilde{q'-q})^2+m^2)}\left[\frac{\widetilde{q'^2}-(\widetilde{q'-k})^2}{((\widetilde{q'-k})^2+m^2)(\widetilde{q'^2}+m^2)}\right]$$

$\deg(D_2) = 2d-7$ → Well defined continuum limit in d=3

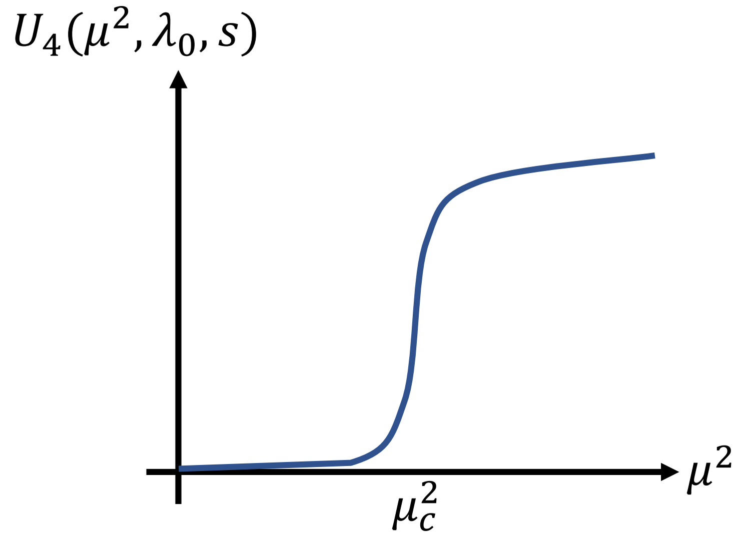
Renormalizable QFTs are also renormalizable in lattice regularization
The renormalized LFT becomes Lorentz invariant in the continuum limit

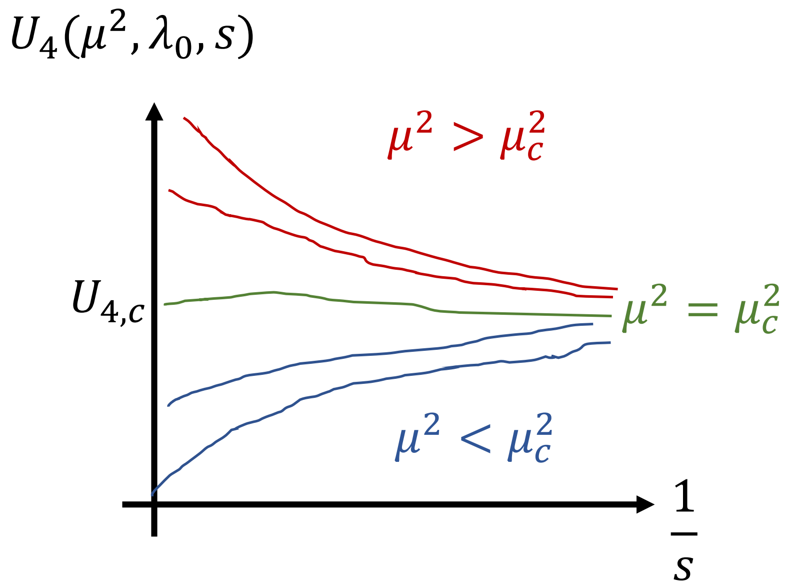
Binder Cumulant (Binder, K. 1981. Z. Physik B 43 119)

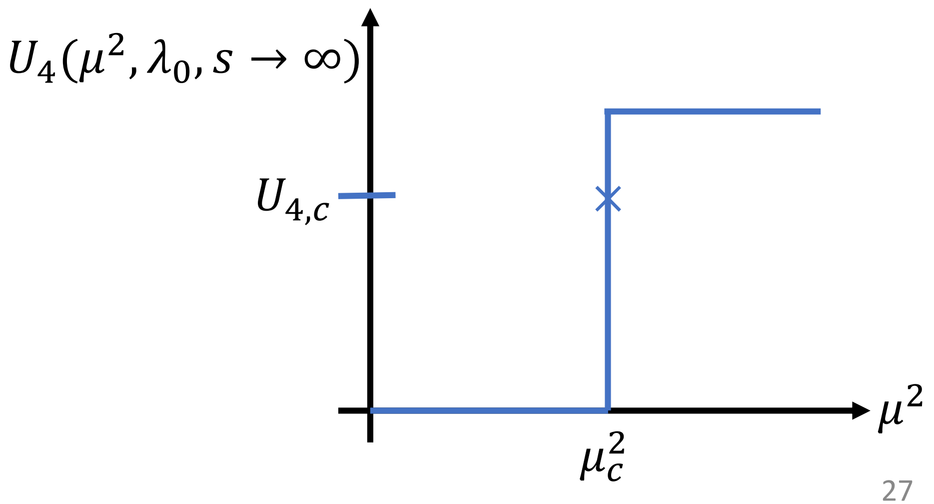
$$U_4(\mu, \lambda, s) = \frac{3}{2}\left(1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}\right)$$

$$M = \sum_x w_x \phi_x$$

- Ordered phase, $U_4 = 1$
- Disordered phase, $U_4 = 0$



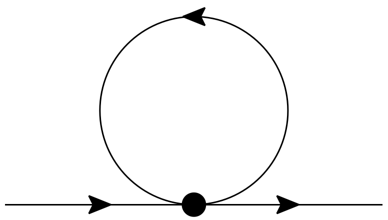


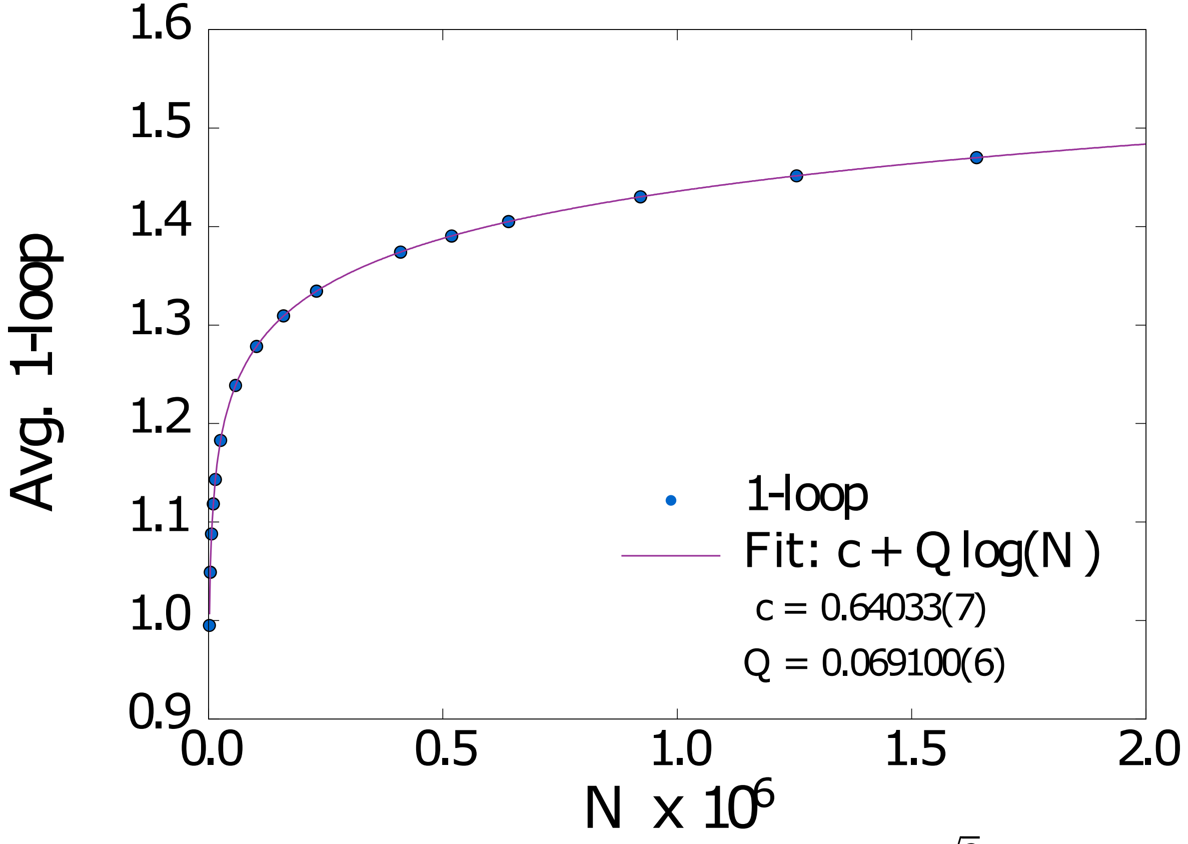


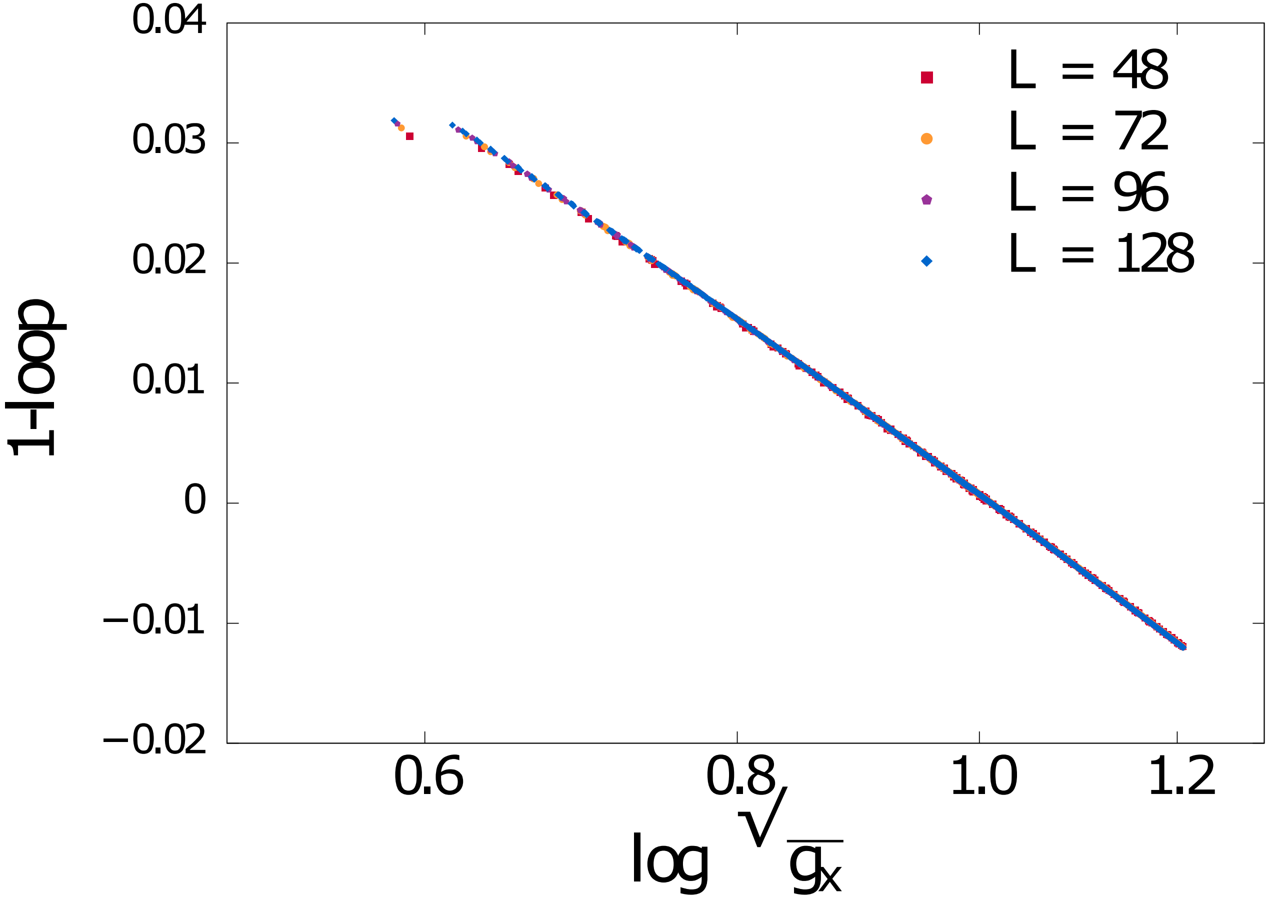
Quantum Corrections on a Curved Lattice

- General proof of renormalizability on curved lattice is hard
  - No translation symmetry, no Fourier techniques
  - No closed form for the propagator at finite lattice spacing
- Nonetheless, we propose a scheme which follows the spirit of the perturbative renormalization scheme of Reitz
- The scheme assumes the following
  1. Only divergent diagrams are sensitive to the lattice spacing in the deep UV, so only divergent diagrams remain position dependent as $a \to 0$
  2. The divergence is “universal” (the same at all positions)
- If (1) and (2) are true, then one only needs to add a **finite** position dependent counterterm to the FEM Laplacian to cancel the position dependence in the finite pieces of the UV divergent diagrams
- Then the divergence is removed as in usual lattice theory: either by explicit subtraction by a universal counterterm in perturbation theory, or nonperturbatively by tuning the universal bare mass to reach the critical surface
- We refer to this scheme as “quantum finite elements”

Quantum Corrections for ϕ^4 theory in d=2



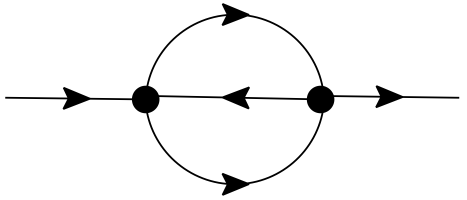


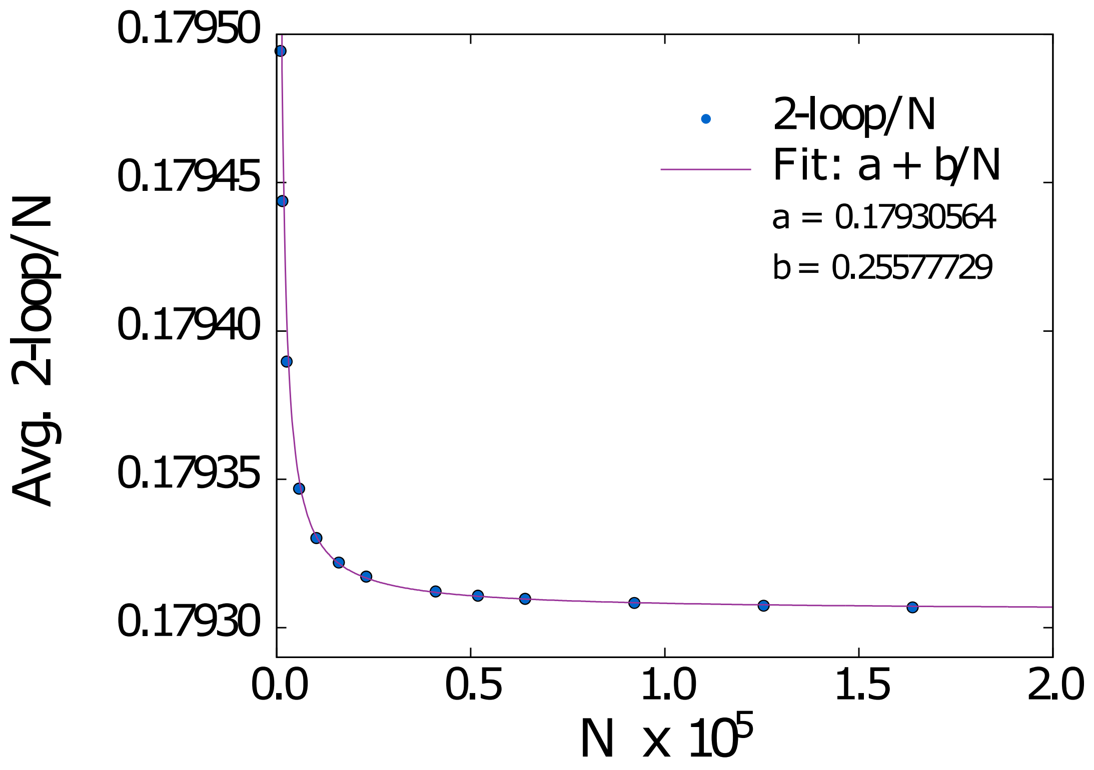


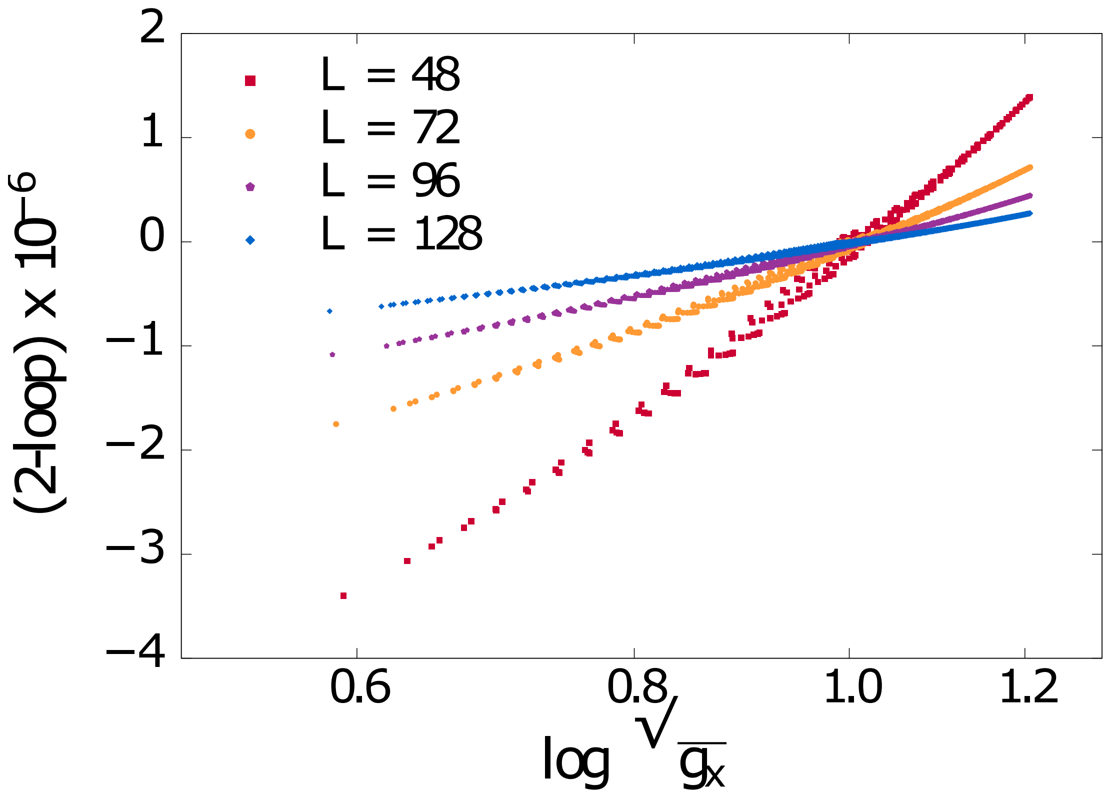
$$\frac{\sqrt{3}}{8\pi} = 0.0689$$

Quantum Corrections for ϕ^4 theory in d=2

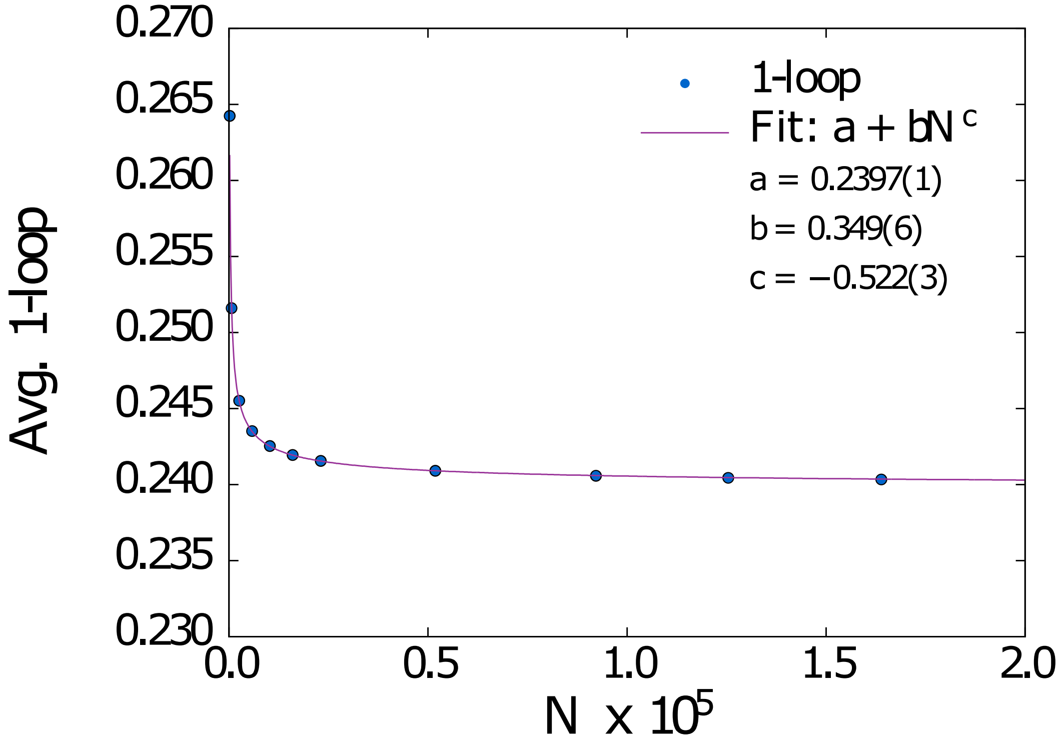
- Look at first convergent diagram, two loops

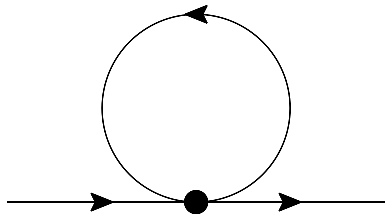


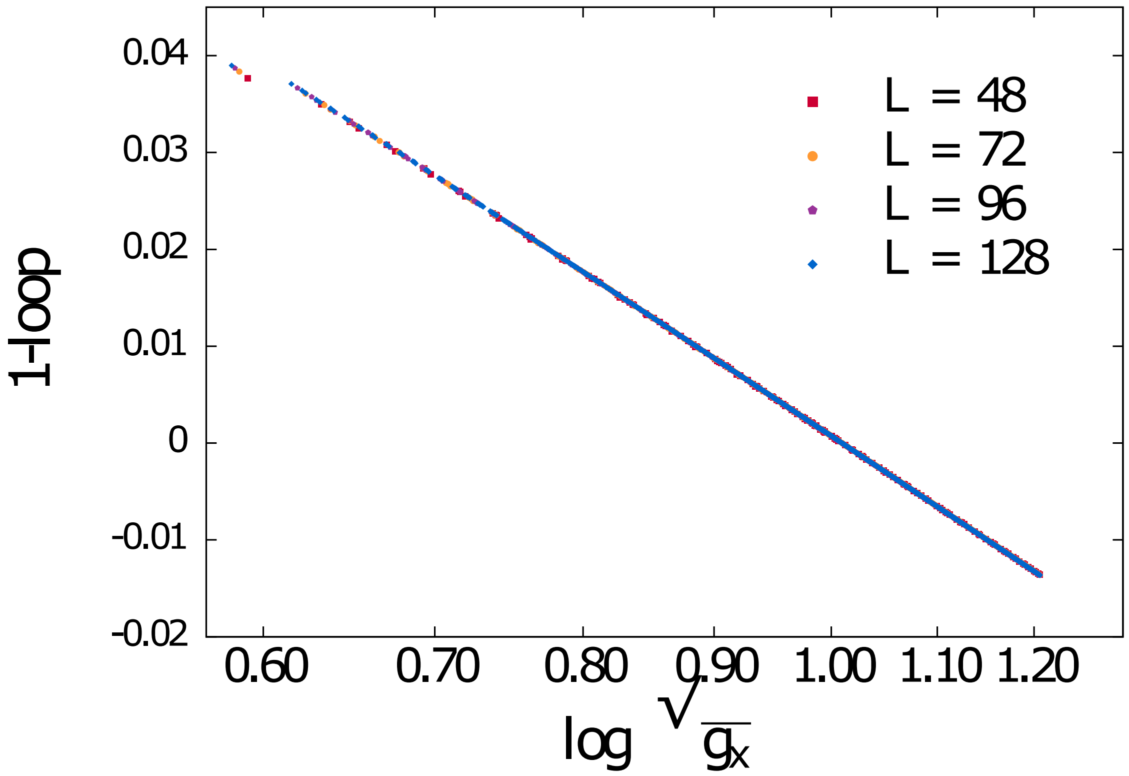




Quantum Corrections for ϕ^4 theory on $R \times S^2$







Quantum Corrections for ϕ^4 theory on $R \times S^2$