Atiyah-Patodi-Singer (APS) index theorem on a lattice

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Atiyah-Patodi-Singer (APS) index theorem and bulk-edge correspondence

APS index theorem is important to understand bulk-edge correspondence in topological insulator.

[Witten 2015]

3+1-dim bulk — 2+1-dim edge

$$Z_{\text{edge}} \propto \exp\left(-i\pi\eta\left(iD^{3\text{D}}/2\right)\right)$$ T-anomalous

$$Z_{\text{bulk}} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon^{\mu\nu\rho\sigma}\text{tr}F_{\mu\nu}F_{\rho\sigma}\right)$$ T-anomalous

$$Z_{\text{edge}}Z_{\text{bulk}} \propto (-1)^{\mathcal{J}}$$ T-symmetry is protected

APS index

$$\boxed{\mathcal{J} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon^{\mu\nu\rho\sigma}\text{tr}_c F_{\mu\nu}F_{\rho\sigma} - \frac{\eta\left(iD^{3\text{D}}\right)}{2}}$$

Eta-invariant

$$\mathcal{J} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}_c F_{\mu\nu}F_{\rho\sigma} - \boxed{\frac{\eta\left(iD^{\mathrm{3D}}\right)}{2}}$$

APS eta-invariant

Eta-invariant is defined by summing up all signs of eigenvalues.

$$\eta(H) = \sum_i \mathrm{sgn}\lambda_i = \mathrm{Tr}\frac{H}{\sqrt{H^2}} \qquad H\phi_i = \lambda_i\phi_i$$

Eta-invariant in APS index

$$\frac{\eta\left(iD^{\mathrm{3D}}\right)}{2} = \frac{\mathrm{CS}}{2} + \mathrm{integer} \qquad \mathrm{CS} = \frac{1}{4\pi}\int d^3x \mathrm{tr}_c\left[\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\right)\right]$$

Eta-invariant = Chern-Simons term and integer(non-local effect)

APS index theorem using domain-wall fermion

Topological insulator and domain-wall fermion [Kaplan 1992] share the same properties.

Massive fermion in the bulk + edge localized mode

The “new” APS index is formulated using domain-wall fermion in continuum theory.

[Fukaya-Onogi-Yamaguchi 2017]

$$\mathcal{J} = -\frac{1}{2}\eta(H_{\rm DW}) \qquad H_{\rm DW} = \gamma_5(D_{\rm 4D} - M\epsilon(x_4))$$

⇓ **Fujikawa method**

$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon^{\mu\nu\rho\sigma}{\rm tr}_c F_{\mu\nu}F_{\rho\sigma} - \frac{\eta\left(iD^{\rm 3D}\right)}{2}$$

Atiyah-Singer (AS) index theorem on lattice

[Hasenfratz 1998]

[Neuberguer 1998]

AS index theorem on lattice

$$Ind\left(D_{ov}\right) = \mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right) \qquad D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_{\mathrm{W}}}{\sqrt{H_{\mathrm{W}}^2}}\right)$$

$$= -\frac{1}{2}\mathrm{Tr}\frac{H_{\mathrm{W}}}{\sqrt{H_{\mathrm{W}}^2}} = -\frac{1}{2}\eta(\gamma_5(D_{\mathrm{W}} - M))$$

AS index can be written by eta-invariant of massive Wilson-Dirac operator.

Chiral symmetry is NOT important.

Index theorems on continuum and on lattice

	continuum	lattice
AS	$-\frac{1}{2}\eta\left(\gamma_5(D^{4\mathrm{D}}-M)\right)$	$-\frac{1}{2}\eta\left(\gamma_5(D_{\mathrm{W}}-M)\right)$
APS	$-\frac{1}{2}\eta\left(\gamma_5(D^{4\mathrm{D}}-M\epsilon(x_4))\right)$	Not Known

Index theorems on continuum and on lattice

	continuum	lattice
AS	$-\frac{1}{2}\eta\left(\gamma_5(D^{4\mathrm{D}}-M)\right)$	$-\frac{1}{2}\eta\left(\gamma_5(D_{\mathrm{W}}-M)\right)$
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Index theorems on continuum and on lattice

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AS	$-\frac{1}{2}\eta\left(\gamma_5(D^{4\mathrm{D}}-M)\right)$	$-\frac{1}{2}\eta\left(\gamma_5(D_{\mathrm{W}}-M)\right)$
APS	$-\frac{1}{2}\eta\left(\gamma_5(D^{4\mathrm{D}}-M\epsilon(x_4))\right)$	$-\frac{1}{2}\eta\left(\gamma_5(D_{\mathrm{W}}-M\epsilon(x_4))\right)$

Eta-invariant is always an integer on 4-dim lattice

Eta-invariant

$$\eta(H) = \sum_i \mathrm{sgn}\lambda_i = \mathrm{Tr}\frac{H}{\sqrt{H^2}} \qquad H\phi_i = \lambda_i\phi_i$$

$-\frac{1}{2}\eta(H_{\mathrm{DW}})$ is always an integer.

Its variation is always zero. $H_{\mathrm{DW}} = \gamma_5\left(D_{\mathrm{W}} - M\epsilon\left(x_4 - \frac{a}{2}\right)\right)$

$$\delta\left(\eta(H_{\mathrm{DW}})\right) = \mathrm{Tr}\left[\delta H_{\mathrm{DW}}(H_{\mathrm{DW}}^2)^{-1/2} - \frac{1}{2}H_{\mathrm{DW}}(H_{\mathrm{DW}}^2)^{-3/2}(\delta H_{\mathrm{DW}}H_{\mathrm{DW}} + H_{\mathrm{DW}}\delta H_{\mathrm{DW}})\right]$$

$$= 0$$

Our goal

Our goal is to show

$$\mathcal{J} = -\frac{1}{2}\eta(H_{\rm DW})$$

$$H_{\rm DW} = \gamma_5 \left(D_{W} - M\epsilon \left(x_4 - \frac{a}{2} \right) \right)$$

$$= \frac{1}{32\pi^2} \int_{x_4>0} d^4x \epsilon^{\mu\nu\rho\sigma} {\rm tr}_c F_{\mu\nu} F_{\rho\sigma} - \frac{\eta\left(iD^{\rm 3D}\right)}{2} + \mathcal{O}(a)$$

in continuum limit

Contents

How to evaluate eta-invariant

$\eta(H_{\mathrm{DW}})$

How to evaluate eta-invariant

Insert complete set

$$\eta(H_{\mathrm{DW}}) = \mathrm{Tr}\frac{H_{\mathrm{DW}}}{\sqrt{H_{\mathrm{DW}}^2}} = a^4 \sum_{x,n} \Phi_n^\dagger(x) \frac{H_{\mathrm{DW}}}{\sqrt{H_{\mathrm{DW}}^2}} \Phi_n(x)$$

Step1: find a good complete set

Step2: perturbative calculation

How to evaluate eta-invariant

Insert complete set

$$\eta(H_{\rm DW}) = {\rm Tr}\frac{H_{\rm DW}}{\sqrt{H_{\rm DW}^2}} = a^4 \sum_{x,n} \boxed{\Phi_n^\dagger(x)} \frac{H_{\rm DW}}{\sqrt{H_{\rm DW}^2}} \boxed{\Phi_n(x)}$$

Step1: find the complete set

x_1, x_2, x_3 -direction have no kink structure

Plane wave

x_4 -direction has kink structure

Plane wave and Edge localized mode

$$\Rightarrow \quad \Phi_n(x) = \phi_{\rm plane}^{\rm 3D}(\vec{x}) \otimes \phi_\pm(x_4)$$

How to evaluate eta-invariant

$$\eta(H_{\mathrm{DW}}) = \mathrm{Tr}\frac{H_{\mathrm{DW}}}{\sqrt{H_{\mathrm{DW}}^2}} = a^4 \sum_{x,n} \Phi_n^\dagger(x) \frac{H_{\mathrm{DW}}}{\sqrt{H_{\mathrm{DW}}^2}} \Phi_n(x)$$

<u>Step2: perturbative calculation</u>

$$H_{\mathrm{DW}}^2 = H_{\mathrm{DW0}}^2(\text{free part}) + (\text{including } A_\mu)$$

Expand the denominator in gauge field

We choose $\Phi_n(x)$ as eigenfunction of the free part.

$$a^2 H_{\mathrm{DW0}}^2 \Phi_n(x) = \lambda^2 \Phi_n(x)$$

Set up

Domain-wall Dirac operator

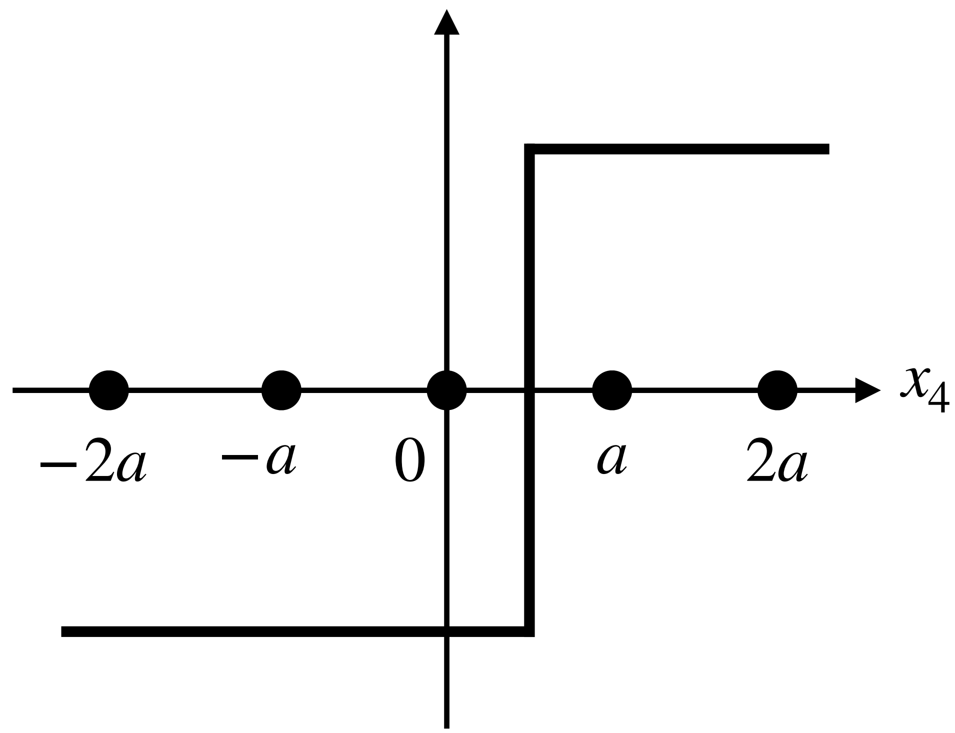
$$H_{\mathrm{DW0}} = \gamma_5 \left(\gamma_\mu \partial_\mu + R - M \epsilon \left(x_4 - \frac{a}{2} \right) \right)$$

Kink on Link

Wilson term

$$R = -\frac{ar}{2} \nabla^*_\mu \nabla_\mu$$

We set Wilson parameter = 1



Domain-wall mass is asymmetric due to Wilson term.

Contents

Free domain-wall eigenfunction set

We need to solve eigenvalue problem of the free domain-wall Dirac operator .

$$a^2 H_{\mathrm{DW0}}^2 \Phi_n(x) = \lambda^2 \Phi_n(x)$$

Assuming the plane wave form in the x_1, x_2, x_3-directions.

$$\Phi_n(x) = \phi_{\mathrm{plane}}^{\mathrm{3D}}(\vec{x}) \otimes \phi_{\pm}(x_4)$$

Free domain-wall eigenfunction set

We need to solve eigenvalue problem of the free domain-wall Dirac operator .

$$a^2 H^2_{\mathrm{DW0}} \Phi_n(x) = \lambda^2 \Phi_n(x)$$

Assuming the plane wave form in the x_1, x_2, x_3-directions.

$$\Phi_n(x) = \phi^{\mathrm{3D}}_{\mathrm{plane}}(\vec{x}) \otimes \phi_\pm(x_4)$$

Denoting $\sin(p_i a) = s_i$ $\cos(p_i a) = c_i$

$$a^2 H^2_{\mathrm{DW0}} \phi_\pm(x_4) = \lambda^2 \phi_\pm(x_4)$$

$$a^2 H^2_{\mathrm{DW0}}$$

$$= s_i^2 + M_\pm^2 - a^2 \left[1 + M_\pm\right] \nabla_4^* \nabla_4 + 2Ma \left[P_+ \delta_{x_4,0} \hat{S}_+ - P_- \delta_{x_4,a} \hat{S}_-\right]$$

$$M_\pm = \sum_i (1 - c_i) \mp Ma \qquad P_\pm = \frac{1}{2}(1 \pm \gamma_4) \qquad \hat{S_\pm} f(x) = f(x \pm a)$$

Complete set

Solve this difference equation

$$\left[s_i^2 + M_\epsilon\right]\phi_\pm(x_4) - [1 + M_\epsilon]\left[\phi_\pm(x_4 + a) + \phi_\pm(x_4 - a) - 2\phi_\pm(x_4)\right]$$
$$+ 2Ma\left[P_+\delta_{x_4,0}\phi_\pm(x_+a) - P_-\delta_{x_4,a}\phi_\pm(x_4 - a)\right] = \lambda^2\phi_\pm(x_4)$$

Complete set

Solve this difference equation

$$\left[s_i^2 + M_\epsilon\right] \phi_\pm(x_4) - [1 + M_\epsilon] [\phi_\pm(x_4 + a) + \phi_\pm(x_4 - a) - 2\phi_\pm(x_4)]$$
$$+ 2Ma \left[P_+ \delta_{x_4,0} \phi_\pm(x_+ a) - P_- \delta_{x_4,a} \phi_\pm(x_4 - a)\right] = \lambda^2 \phi_\pm(x_4)$$

(1) Edge localized mode $\qquad \gamma_4 u_\pm = \pm u_\pm$

$$\phi_-^{\mathrm{edge}}(x_4) = \begin{cases} A u_- e^{-\kappa_+ (x_4 - a)} & (x_4 \geqq a), \\ A u_- e^{\kappa_- (x_4 - a)} & (x_4 \leqq 0), \end{cases}$$

$$A = \sqrt{\frac{M_+ M_- (2 + M_+)(2 + M_-)}{a (M_+ - M_-)(M_+ + M_- + 2)}}$$

Normalizable only if

$$|1 + M_+| < 1$$

$e^{-\kappa_+ a} = 1 + M_+ \qquad e^{\kappa_- a} = 1 + M_-$ ($0 < Ma < 2$ in the continuum limit)

Complete set

Solve this difference equation

$$\left[s_i^2 + M_\epsilon\right] \phi_\pm(x_4) - \left[1 + M_\epsilon\right] \left[\phi_\pm(x_4 + a) + \phi_\pm(x_4 - a) - 2\phi_\pm(x_4)\right]$$
$$+ 2Ma \left[P_+ \delta_{x_4,0} \phi_\pm(x_+ a) - P_- \delta_{x_4,a} \phi_\pm(x_4 - a)\right] = \lambda^2 \phi_\pm(x_4)$$

(2) Bulk mode 1 : Plane wave on both side of the kink

$$\phi^{\text{bulk}}_{1,\pm}(x_4) = \begin{cases} u_\pm T_1^\pm e^{i\omega_+ x_4} & (x_4 \geqq a) \\ u_\pm \left[e^{-i\omega_- x_4} + R_1^\pm e^{-i\omega_- x_4}\right] & (x_4 \leqq 0) \end{cases}$$

$$\phi^{\text{bulk}}_{2,\pm}(x_4) = \begin{cases} u_\pm \left[e^{-i\omega_+ x_4} + R_2^\pm e^{i\omega_+ x_4}\right] & (x_4 \geqq a) \\ u_\pm T_2^\pm e^{-i\omega_- x_4} & (x_4 \leqq 0) \end{cases}$$

orthogonal

$$R_1^\pm = -\frac{(1+M_\pm) - (1+M_\mp)e^{-i(\omega_+ - \omega_-)a}}{(1+M_\pm) - (1+M_\mp)e^{-i(\omega_+ + \omega_-)a}} \qquad T_1^\pm = \frac{(1+M_-)e^{-i\omega_+ a} 2i \sin(\omega_- a)}{(1+M_\pm) - (1+M_\mp)e^{-i(\omega_+ + \omega_-)a}}$$

$$R_1^\pm = -\frac{(1+M_\mp) - (1+M_\mp)e^{-i(\omega_+ - \omega_-)a}}{(1+M_\mp) - (1+M_\mp)e^{-i(\omega_+ + \omega_-)a}} \qquad T_2^\pm = \frac{-(1+M_+)e^{-i\omega_- a} 2i \sin(\omega_+ a)}{(1+M_\mp) - (1+M_\pm)e^{-i(\omega_+ + \omega_-)a}}$$

Complete set

Solve this difference equation

$$\left[s_i^2 + M_\epsilon\right]\phi_\pm(x_4) - \left[1 + M_\epsilon\right]\left[\phi_\pm(x_4+a) + \phi_\pm(x_4-a) - 2\phi_\pm(x_4)\right] + 2Ma\left[P_+\delta_{x_4,0}\phi_\pm(x_+a) - P_-\delta_{x_4,a}\phi_\pm(x_4-a)\right] = \lambda^2\phi_\pm(x_4)$$

(3) Bulk mode 2 : Plane wave on one side of the kink and localized mode on the other side

$$\phi_\pm^{\mathrm{bulk}}(x_4) = \begin{cases} \frac{u_\pm}{\sqrt{2\pi}|A_\pm|}\left[A_\pm e^{i\omega x_4} - A_\pm^* e^{-i\omega x_4}\right] & (x_4 \geqq a) \\ \frac{u_\pm}{\sqrt{2\pi}|A_\pm|} B_\pm e^{k x_4} & (x_4 \leqq 0) \end{cases}$$

$$A_\pm = \frac{1+M_\mp}{1+M_\pm}e^{\kappa a} - e^{-i\omega a} \qquad B_+ = e^{i\omega a} - e^{-i\omega a}$$

$$B_- = \frac{1+M_+}{1+M_-}\left[e^{i\omega a} - e^{-i\omega a}\right]$$

Shamir-type domain-wall

Introduce an additional mass M_2

[Shamir 1993]

[Furman-Shamir 1995]

$$H_{\mathrm{DW0}} = \gamma_5 \left[\gamma_\mu \partial_\mu + R - M_1 \epsilon\left(x_4 - \frac{a}{2}\right) + M_2\right]$$

$$a^2 (H_{\mathrm{DW0}})^2$$

$$= s_i^2 + M_\pm^2 - a^2 \left[1 + M_\pm\right] \nabla_4^* \nabla_4 + 2 M_1 a \left[P_+ \delta_{x_4,0} \hat{S}_+ - P_- \delta_{x_4,a} \hat{S}_-\right]$$

$M_1 + M_2 = \infty$ while $M_1 - M_2 = M$ is fixed

$$M_+ = \sum (1 - c_i) - M_1 a + M_2 a$$

$$M_- = \sum (1 - c_i) + M_1 a + M_2 a$$

$\Rightarrow$

$$M_+ = \sum (1 - c_i) - M a$$

$$M_- = \infty$$

The index is independent of M_2.

[Fukaya-Onogi-Yamaguchi 2017]

Shamir-type domain-wall

Complete set is reduced to

(1) Edge localized mode

$$\gamma_4 u_\pm = \pm u_\pm$$

$$\phi_-^{\mathrm{edge}}(x_4) = \sqrt{-\frac{1}{a}M_+(2+M_+)}u_- e^{-\kappa_+(x_4-a)}$$

(2) Bulk mode

$$\phi_+^{\mathrm{bulk}}(x_4) = \frac{1}{\sqrt{2\pi}}u_+\left[e^{i\omega x_4} - e^{-i\omega x_4}\right]$$

$$\phi_-^{\mathrm{bulk}}(x_4) = \frac{1}{\sqrt{2\pi}}u_-\left[C_\omega e^{i\omega x_4} - C_\omega^* e^{-i\omega x_4}\right]$$

$$C_\omega = -\frac{(1+M_+) - e^{-i\omega a}}{|(1+M_+) - e^{-i\omega a}|}$$

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Calculation of eta-invariant

$$\mathcal{J} = -\frac{1}{2}\eta(H_{\mathrm{DW}}) = -\frac{1}{2}\mathrm{Tr}H_{\mathrm{DW}}\frac{1}{\sqrt{H_{\mathrm{DW}}^2}}$$

Bulk and edge decomposition

$$= -\frac{1}{2}a^4 \sum_{\vec{x},x_4 \geq a} \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p \int_0^{\frac{\pi}{a}} d\omega \sum_{\eta=\pm} \mathrm{tr}_c \Phi_\eta^{\mathrm{bulk}\dagger}(x) \left[\frac{H_{\mathrm{DW}}}{\sqrt{H_{\mathrm{DW}}^2}}\right] \Phi_\eta^{\mathrm{bulk}}(x)$$

bulk part

$$\Phi_\eta^{\mathrm{bulk}}(x) = \phi_\eta^{\mathrm{bulk}}(x_4) \otimes \phi_p^{\mathrm{3D}}(\vec{x})$$

$$-\frac{1}{2}a^4 \sum_{\vec{x},x_4 \geq a} \int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p \,\mathrm{tr}_c \Phi_-^{\mathrm{edge}\dagger}(x) \left[\frac{H_{\mathrm{DW}}}{\sqrt{H_{\mathrm{DW}}^2}}\right] \Phi_-^{\mathrm{edge}}(x)$$

edge part

$$\Phi_-^{\mathrm{edge}}(x) = \phi_-^{\mathrm{edge}}(x_4) \otimes \phi_p^{\mathrm{3D}}(\vec{x})$$

Calculation of eta-invariant (1) : bulk part

$$-\frac{1}{2}a^4\sum_{\vec{x},x_4\geq a}\int_{\frac{\pi}{a}}^{\frac{\pi}{a}}d^3p\int_0^{\frac{\pi}{a}}d\omega\sum_{\eta=\pm}\mathrm{tr}_c\Phi_\eta^{\mathrm{bulk}\dagger}(x)\left[\frac{H_{\mathrm{DW}}}{\sqrt{H_{\mathrm{DW}}^2}}\right]\Phi_\eta^{\mathrm{bulk}}(x)$$

$$H_{\mathrm{DW}}^2=H_{\mathrm{DW0}}^2+\Delta H_{\mathrm{DW}}^2 \qquad \Delta H_{\mathrm{DW}}^2\ni[\gamma^\mu,\gamma^\nu][D_\mu,D_\nu]\ ,\ [\gamma^\mu D_\mu,R]$$

$$[H_{\mathrm{DW}}^2]^{-\frac{1}{2}}=[H_{\mathrm{DW0}}^2+\Delta H_{\mathrm{DW}}^2]^{-\frac{1}{2}}$$

$$=\frac{1}{(H_{\mathrm{DW0}})^{\frac{1}{2}}}-\frac{1}{2}\Delta H_{\mathrm{DW}}^2\frac{1}{(H_{\mathrm{DW0}}^2)^{\frac{3}{2}}}+\frac{3}{8}(\Delta H_{\mathrm{DW}}^2)^2\frac{1}{(H_{\mathrm{DW0}}^2)^{\frac{5}{2}}}+\cdots$$

= 0 due to the spinor structure

Calculation of eta-invariant (1) : bulk part

$$-\frac{3}{16}a^4\sum_{\vec{x},x_4\geq a}\int_{\frac{\pi}{a}}^{\frac{\pi}{a}}d^3p\int_0^{\frac{\pi}{a}}d\omega\sum_{\eta=\pm}\mathrm{tr}_c\Phi_\eta^{\mathrm{bulk}\dagger}(x)\left[H_{\mathrm{DW}}(\Delta H_{\mathrm{DW}})^2\frac{1}{(H_{\mathrm{DW0}})^{\frac{5}{2}}}\right]\Phi_\eta^{\mathrm{bulk}}(x)$$

$$=\frac{1}{32\pi^2}a^4\sum_{\vec{x},x_4>a}I(M)\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}_cF_{\mu\nu}F_{\rho\sigma}+\mathcal{O}\left(e^{-Mx_4}\right)$$

It is the same expression as [Suzuki 1999] which derives AS index on lattice by using perturbative expansion.

$$I(Ma)\to\theta(Ma)-4\theta(Ma-2)+6\theta(Ma-4)-4\theta(Ma-6)+\theta(Ma-8)$$

We choose $0<Ma<2$ in continuum limit.

$$\frac{1}{32\pi^2}a^4\sum_{\vec{x},x_4\geq a}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}_cF_{\mu\nu}F_{\rho\sigma}+\mathcal{O}(a)$$

[Kikukawa-Yamada 1998]
[Luscher 1998]
[Fujikawa 1998]
[Adams 1998]

Calculation of eta-invariant (2) : edge part

Domain-wall Dirac operator in $A_4 = 0$ gauge.

$$H_{\rm DW} = \gamma_5 \left[-aP_-\nabla_4 + aP_+\nabla_4^* + \gamma_i D_i(U_i(x)) + M_+(U_i(x)) \right]$$

Expanding in $||U^\dagger \partial_{x_4} U||/M$ (adiabatic expansion)

∇_4 : forward derivative

∇_4^* : backward derivative

$\phi_-^{\rm edge}(x_4)$ satisfies

$$-aP_-\nabla_4 \phi_-^{\rm edge}(x_4) = -M_+ \phi_-^{\rm edge}(x_4)$$

Then

$$H_{\rm DW}\phi_-^{\rm edge}(x_4) = \gamma_5\gamma_i D_i \phi_-^{\rm edge}(x_4)$$

$$= \begin{pmatrix} -i\sigma_i D_i & \\ & i\sigma_i D_i \end{pmatrix} \phi_-^{\rm edge}(x_4)$$

$$\longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & i\sigma_i D_i \end{pmatrix} \phi_-^{\rm edge}(x_4)$$

Calculation of eta-invariant (2) : edge part

$$-\frac{1}{2}a^4\sum_{\vec{x},x_4\geq a}\int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p\,\mathrm{tr}_c\,\Phi_-^{\mathrm{edge}\dagger}(x)\left[\frac{H_{\mathrm{DW}}}{\sqrt{H_{\mathrm{DW}}^2}}\right]\Phi_-^{\mathrm{edge}}(x)$$

$$= -\frac{1}{2}a^4\sum_{\vec{x},x\geq a}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p\,\mathrm{tr}_c\,\phi_p^{\mathrm{3D}\dagger}(\vec{x})\left[\frac{iD^{\mathrm{3D}}}{\sqrt{(iD^{\mathrm{3D}})^2}}\right]\phi_p^{\mathrm{3D}}(\vec{x})\underline{\phi_-^{\mathrm{edge}}(x_4)\phi_-^{\mathrm{edge}\dagger}(x_4)}$$

$$-M_+(2+M_+)e^{-2\kappa_+(x_4-a)}$$

$$D^{\mathrm{3D}}(x) = \sigma_i D_i(x)$$

In continuum and low-energy limit.

$$\longrightarrow \quad \delta_{x_4,a}$$

$$= -\frac{1}{2}\mathrm{Tr}_{\mathrm{3D}}\frac{iD^{\mathrm{3D}}}{\sqrt{(iD^{\mathrm{3D}})^2}}|_{x_4=a} + \mathcal{O}(a) = -\frac{1}{2}\eta\left(iD^{\mathrm{3D}}\right)|_{x_4=a} + \mathcal{O}(a)$$

Calculation of eta-invariant : Result

$$\mathcal{J} = -\frac{1}{2}\eta(H_{\rm DW}) = -\frac{1}{2}{\rm Tr}H_{\rm DW}\frac{1}{\sqrt{H_{\rm DW}^2}}$$

$$= -\frac{1}{2}a^4 \sum_{\vec{x},x_4\geq a}\int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p \int_0^{\frac{\pi}{a}} d\omega \sum_{\eta=\pm} {\rm tr}_c \Phi_\eta^{{\rm bulk}\dagger}(x)\left[\frac{H_{\rm DW}}{\sqrt{H_{\rm DW}^2}}\right]\Phi_\eta^{\rm bulk}(x)$$

$$\Rightarrow \frac{1}{32\pi^2}a^4\sum_{\vec{x},x_4\geq a}\epsilon^{\mu\nu\rho\sigma}{\rm tr}_c F_{\mu\nu}F_{\rho\sigma} + \mathcal{O}(a)$$

$$-\frac{1}{2}a^4 \sum_{\vec{x},x_4\geq a}\int_{\frac{\pi}{a}}^{\frac{\pi}{a}} d^3p\,{\rm tr}_c \Phi_-^{{\rm edge}\dagger}(x)\left[\frac{H_{\rm DW}}{\sqrt{H_{\rm DW}^2}}\right]\Phi_-^{\rm edge}(x)$$

$$\Rightarrow -\frac{1}{2}\eta\left(iD^{\rm 3D}\right) + \mathcal{O}(a)$$

Contents

Summary

- The eta invariant of massive Dirac operator gives a unified view of index theorems.

	continuum	lattice
AS	$-\frac{1}{2}\eta\left(\gamma_5(D^{4\mathrm{D}}-M)\right)$	$-\frac{1}{2}\eta\left(\gamma_5(D_{\mathrm{W}}-M)\right)$
APS	$-\frac{1}{2}\eta\left(\gamma_5(D^{4\mathrm{D}}-M\epsilon(x_4))\right)$	$-\frac{1}{2}\eta\left(\gamma_5(D_{\mathrm{W}}-M\epsilon(x_4))\right)$

- We confirmed that the eta-invariant of Wilson-Dirac operator with domain-wall mass converges to the APS index in the continuum limit.

$$\mathcal{J} = -\frac{1}{2}\eta(H_{\mathrm{DW}}) = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}_c F_{\mu\nu}F_{\rho\sigma} - \frac{\eta\left(iD^{3\mathrm{D}}\right)}{2} + \mathcal{O}(a)$$

Outlook

- Our computation was approximately done in $x_4 \geq 0$ semi-infinite spacetime.

  $\longrightarrow$ x_4 should be compactified with anti-domain-wall.

- Generalization to any even dimensional lattices.

- Application to odd dimensions (Matsuki's talk Thursday)

- Discussion of locality and smoothness(admissibility condition).