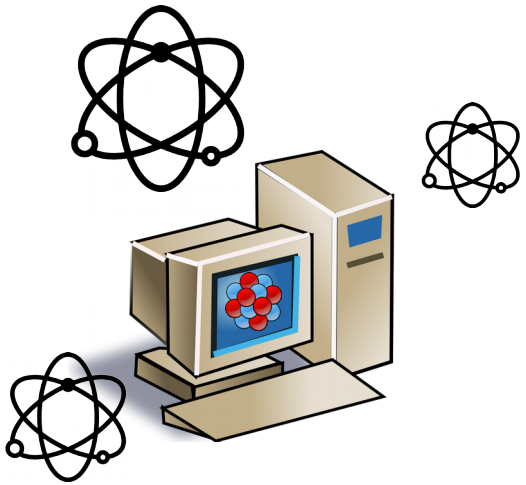
Tailoring Non-Abelian Gauge Theory for Digital Quantum Simulation



Jesse R. Stryker

Work done w/ David B. Kaplan (INT),
Indrakshi Raychowdhury (I.Ass. Kolkata)

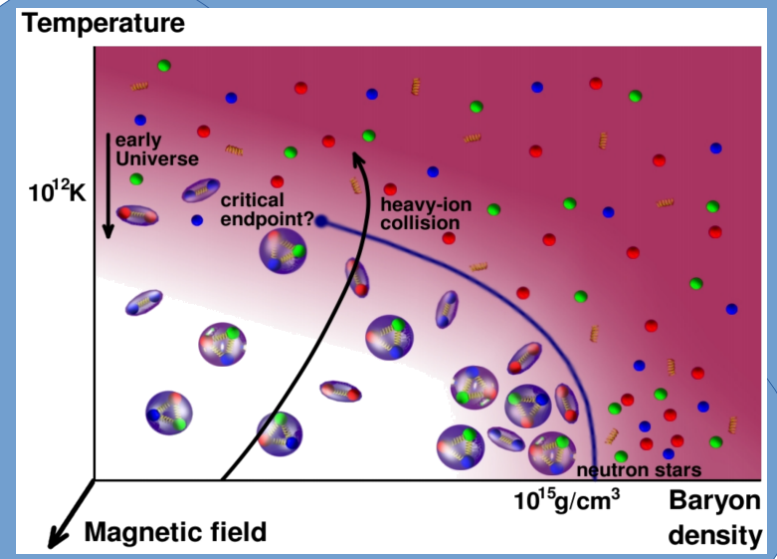
Outline

- **Big Picture**
- **Kogut-Susskind Lattice Gauge Theory**
- **Gauge-Invariant Variables: U(1)**
- **Non-Abelian Hilbert space**
  - **Loops and strings**
- **Summary**

Big Picture

Physics targets:

- **Simulation of QCD**
  - Hadronization
  - Detailed understanding of nuclear interactions
- **Complete phase diagram of QCD**
- **Nuclear equation of state**



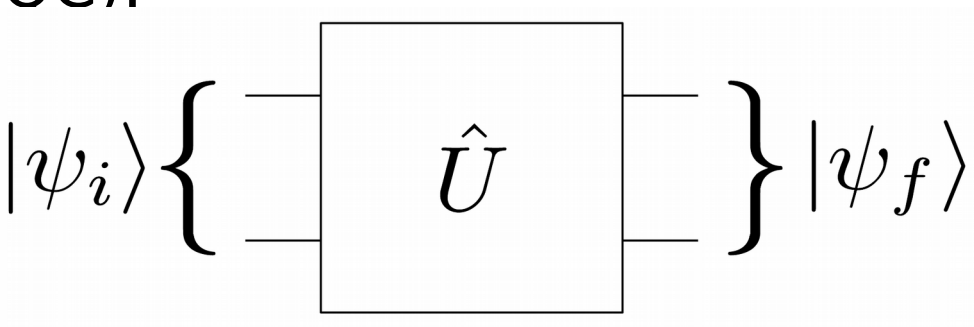
One branch of reaching these goals:

- Quantum simulate QCD degrees of freedom
- **Gauge theories like QCD present unique practical challenges for simulation**

Conjectured phase diagram credit: G. Endrödi J.Phys.Conf.Ser. 503 (2014) 012009

Digital Quantum Simulation

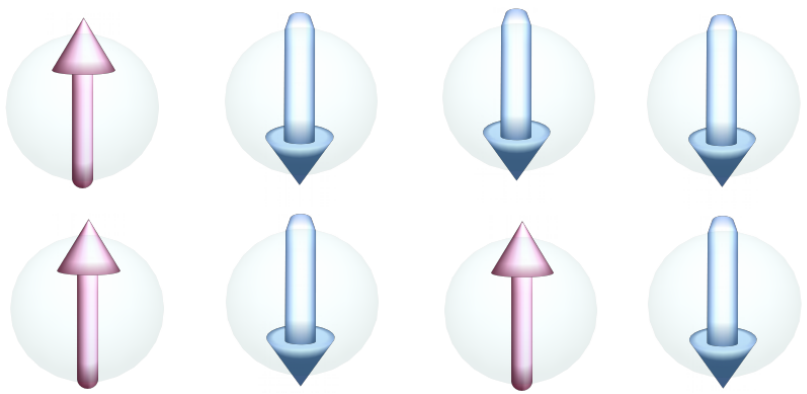
Digital quantum computers (QC):



- Unitary gates: $e^{-it\hat{H}}$ with your favorite Hamiltonian
- Want to simulate quantum field theory non-perturbatively → Lattice quantum field theory + quantum simulation

Near-term QC architectures will have very limited capabilities

- **How to most wisely spend those qubits?**



How to map the Hilbert space $\mathcal{H}$ and $\hat{H}$ on to the QC?

Hamiltonian Lattice Gauge Theory I

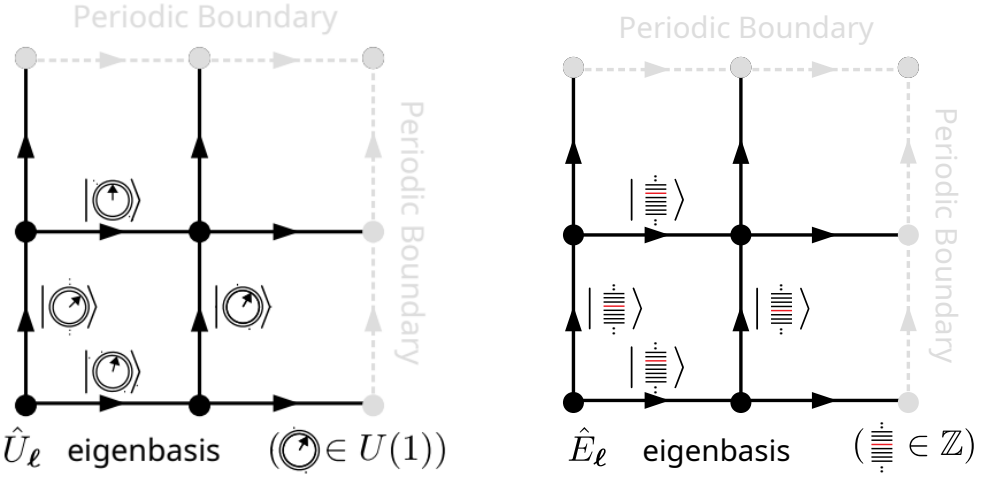
LGT Hilbert space, operator algebra:

- U(1)

$$\langle \phi | q \rangle = \frac{1}{\sqrt{2\pi}} e^{i\phi q}$$

group element or "coordinate" basis

representation or "momentum" basis



$$[E, U] = U$$

Canonical (same-link) commutation relation

$$U\,|q\rangle = |q+1\rangle$$

$$\hat{U}\,|\,\rangle = |\,\rangle$$

"*U* raises *E*"

$$\hat{H}_E = \frac{g^2}{2} \sum_{n,i} \hat{E}^2_{n,i} \qquad \hat{H}_B = -\sum_n \frac{1}{2g^2} \mathrm{Re}(\hat{U}_{n,\square})$$

Hamiltonian Lattice Gauge Theory II

LGT Hilbert space, operator algebra:

- Non-Abelian Lie group

Gauge transformations:

$$U_{x,i} \to \Omega_x U_{x,i} \Omega^{-1}_{x+e_i}$$

$$[E^a_{L/R}, E^b_{L/R}] = i f^{abc} E^c_{L/R}$$
$$[E^a_R, U] = U T^a$$
$$[E^a_L, U] = -T^a U$$

Left, right electric fields to generate left, right rotations.

$$\hat{H}_E = \frac{g^2}{2} \sum_{n,i} \hat{E}^a_{n,i} \hat{E}^a_{n,i} \quad \hat{H}_B = -\sum_{\square} \frac{1}{2g^2} \mathrm{tr}(\hat{U}_\square + \hat{U}^\dagger_\square)$$

$$\langle g | j, m, m' \rangle = \sqrt{\frac{d_j}{|G|}} D^{(j)}_{m,m'}(g)$$

group element

representation state

"*U* adds angular momentum"

$$U_{m,m'} |j, M, M'\rangle =$$
$$C_+ |j + 1/2, M + m, M' + m'\rangle$$
$$+ C_- |j - 1/2, M + m, M' + m'\rangle$$

Hamiltonian Lattice Gauge Theory III

LGT Hilbert space, operator algebra:
- Non-Abelian Lie group

$$C_\pm = \sqrt{\frac{\dim(J)}{\dim(J \pm 1/2)}} \langle J, M; \frac{1}{2}, m | J \pm \frac{1}{2}, M + m \rangle \langle J \pm \frac{1}{2}, M' + m' | J, M'; \frac{1}{2}, m' \rangle$$

- Link operators couple states via Clebsch-Gordon (CG) coefficients
  - **Plaquettes** implement this x4
- A QC would internally compute these in each time step

$$\hat{H}_E = \frac{g^2}{2} \sum_{n,i} \hat{E}^a_{n,i} \hat{E}^a_{n,i} \quad \hat{H}_B = -\sum_{\square} \frac{1}{2g^2} \mathrm{tr}(\hat{U}_\square + \hat{U}^\dagger_\square)$$

"*U* adds angular momentum"

$$U^{(1/2)}_{m,m'} |J, M, M'\rangle = C_+ |J + 1/2, M + m, M' + m'\rangle + C_- |J - 1/2, M + m, M' + m'\rangle$$

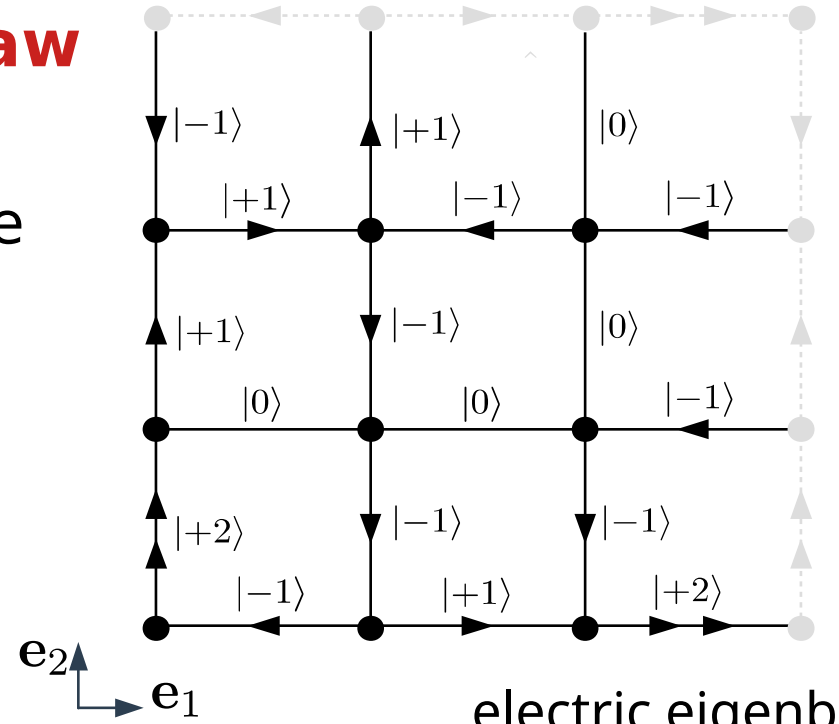
Hamiltonian Lattice Gauge Theory IV

Plus constraints: Gauss law

$$\hat{\mathcal{G}}_n \left|\Psi\right\rangle = 0$$

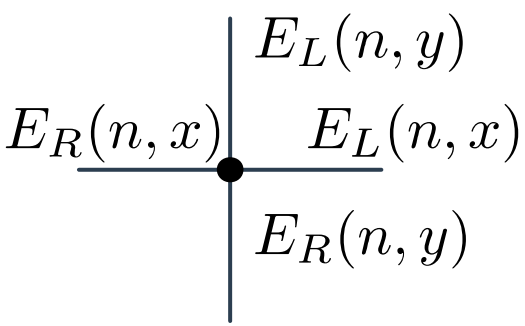
Gauss law ↔ gauge invariance

U(1) $$\hat{\mathcal{G}}_n = \underbrace{\nabla \cdot \mathbf{E}}_{\text{lattice discrete "divergence"}} - \rho$$



electric eigenbasis

SU(N) $$\hat{\mathcal{G}}^a_n = \nabla \cdot \mathbf{E}^a - \rho^a$$

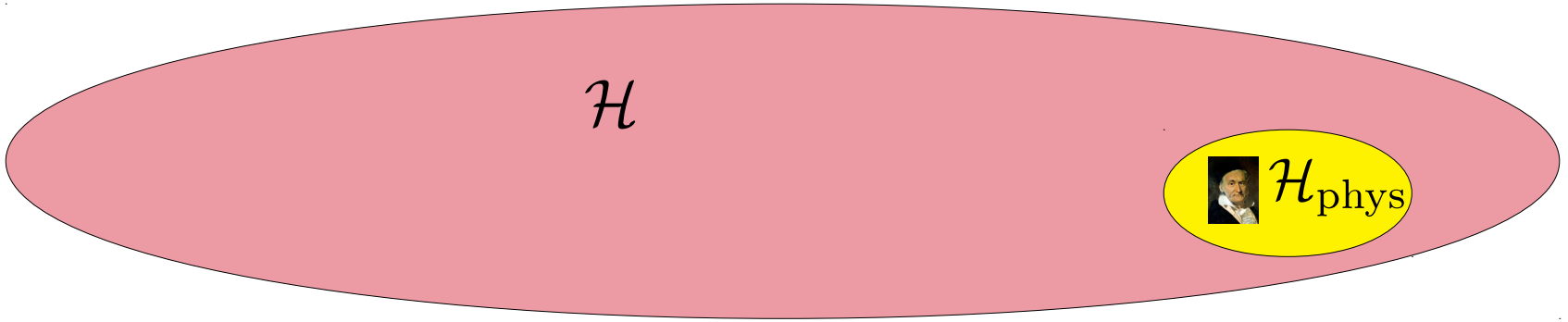


$$\sum_{i=1}^{d} E^a_L(n,i) + E^a_R(n,i)$$

(also: $E^a_L E^a_L = E^a_R E^a_R$)

Bugs in This Framework

- **Qubits <u>wasted</u> on unphysical states**



- **Quantum noise <u>will</u> create components along unphysical directions**
- **In practice, gauge invariance could suffer systematic errors from <u>approximated evolution</u>**

Gauge-Invariant Variables I

Q: Build gauge invariance in from start?

- **Start simple:**
  - Pure U(1) in d=2+1
  - Consider strong-coupling vacuum (SCV): $|\Omega\rangle \equiv \otimes_{\boldsymbol{\ell}} |0\rangle_{\boldsymbol{\ell}}$

Ask: **What states does *H* actually visit**? (States connected to SCV?)

$$\begin{aligned} \hat{H} &= \hat{H}_E + \hat{H}_B \ , \\ \hat{H}_B &\propto \sum_n \left(2 - \hat{U}_{n,\square} - \hat{U}^\dagger_{n,\square}\right) \\ \hat{H}_E &= \text{function of the } \hat{E}_{n,i}. \text{ (diagonal)} \end{aligned}$$

See also work by Y. Meurice, J. Unmuth-Yockey et al.

For more: See JRS & D.B. Kaplan PoS LATTICE2018 (2018) 227 1806.08797

Gauge-Invariant Variables II

What does H do? (in electric basis)

- $\hat{H}_E \supset \hat{E}_{n,i}$: **rescale basis states**
- $\hat{H}_B \supset \hat{P}_{\boldsymbol{p}}$: **excite electric flux loops**
- That's it.

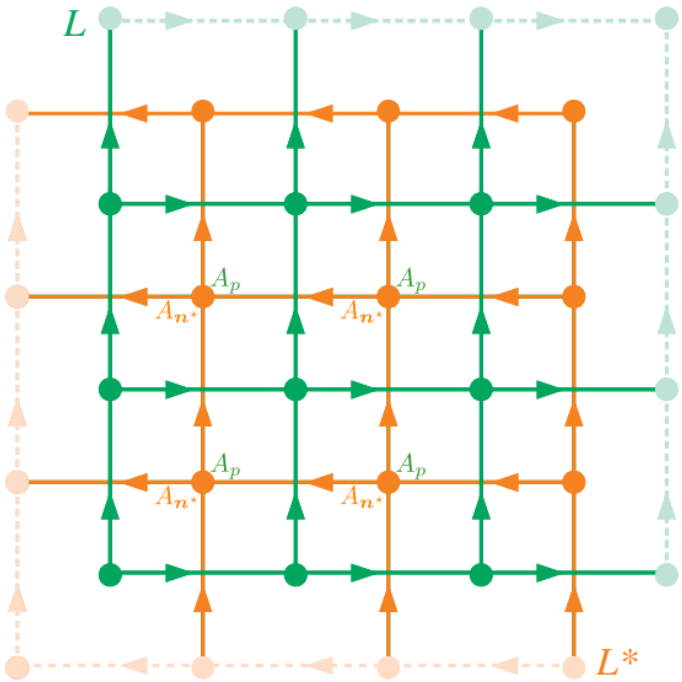
$$\hat{U}_{\square} \left| \begin{array}{c} |0\rangle \\ |0\rangle \quad\quad |0\rangle \\ |0\rangle \end{array} \right\rangle = \left| \begin{array}{c} |-1\rangle \\ |-1\rangle \quad\quad |+1\rangle \\ |+1\rangle \end{array} \right\rangle$$

$\Rightarrow$ **Basis for $\mathcal{H}_{\text{phys}}$ is generated by acting with plaquettes on SCV**

For more: See JRS & D.B. Kaplan
PoS LATTICE2018 (2018) 227

U(1) Punchline

- **d=2+1 E&M is dual to a scalar field theory (known) – dual uses less variables**
  - Naturally emerges from gauge invariant building blocks
  - Resulting *H* describes same physics more concisely



Limitations

- $\oint d^2x\ B = 0$ **is not automatic**
  - **Enforcing for 2+1 (w/ periodic BC) impractical for large volumes**
- **Not obvious how generalize to matter, non-Abelian gauge groups**

Want to preserve: Local Hilbert spaces, Hamiltonian built from local operators, local constraints

Non-Abelian LGT Hilbert space I

Previous slides: Challenges existing already with Abelian

New complications for non-Abelian? (here SU(2))

- Truncation on basis states

$$|j, m, n\rangle\,, \quad 0 \leq j \leq J \qquad\qquad d_j = (2j+1)^2$$

$$\Rightarrow \mathrm{dim}(\mathcal{H}) = (8/3)(J+1/2)(J+3/4)(J+1) \neq 2^n$$

Not ideal for qubits

- Gauge invariance

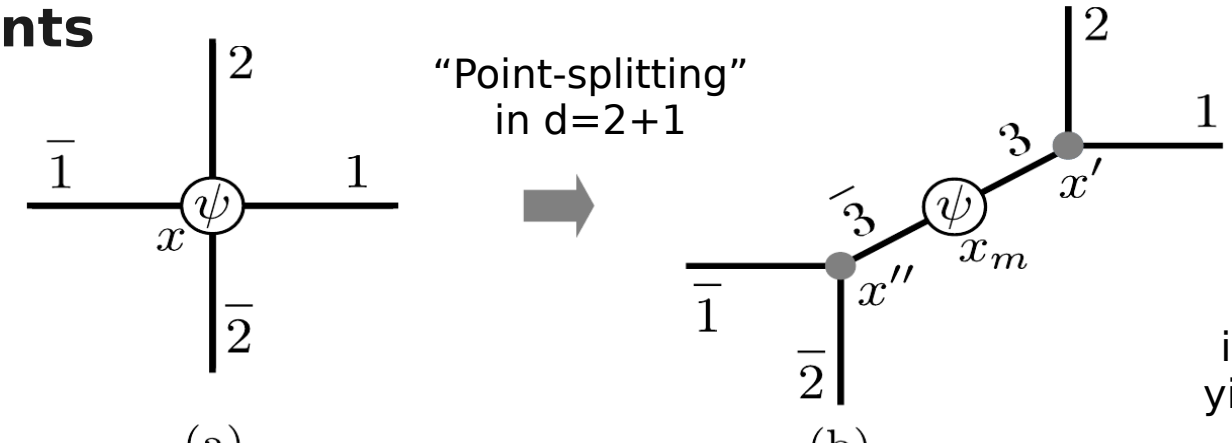
$$[\mathcal{G}^a, \mathcal{G}^b] \neq 0$$

→I. e., constraints not simultaneously diagonalizable
→Simply probing physicality involves change of basis

Non-Abelian LGT Hilbert space II

- Situation does not bode well, before even *thinking about* state preparation, time evolution, or measurements
- Enter: Schwinger boson formulation of LGT *
  - Angular momentum basis ↔ Many bosonic harmonic oscillators ("prepotential" formulation)
  - Newer loop reformulation ** **solves all non-Abelian constraints**

JRS & I. Raychowdhury, 1812.07554

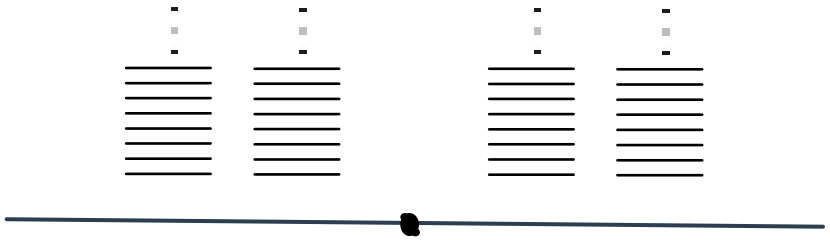


Virtual links introduced to yield **trivalent** lattice

* See papers by e.g. Mathur, Anishetty, Raychowdhury, Sreeraj
** Raychowdhury, I. Eur. Phys. J. C (2019) 79: 235

SU(2)-invariant variables

Kogut-Susskind $\quad |j, m, m'\rangle\,, \quad 0 \leq m, m' \leq j$

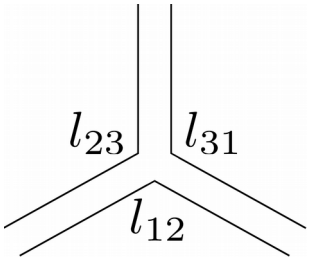


$|n_{L,1}, n_{L,2}, n_{R,1}, n_{R,2}\rangle$

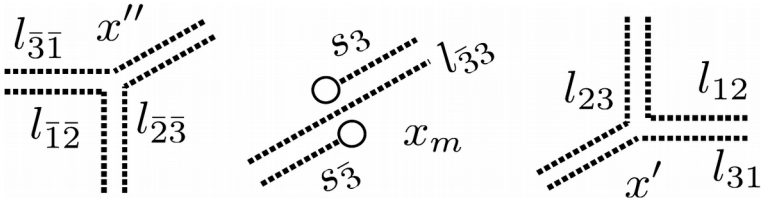
Schwinger bosons / "prepotentials"

Loop-String

- Gauge-invariant quantum numbers follow from Schwinger boson formulation
- Fundamental quarks
- Matter always locally reduced to a 1D problem



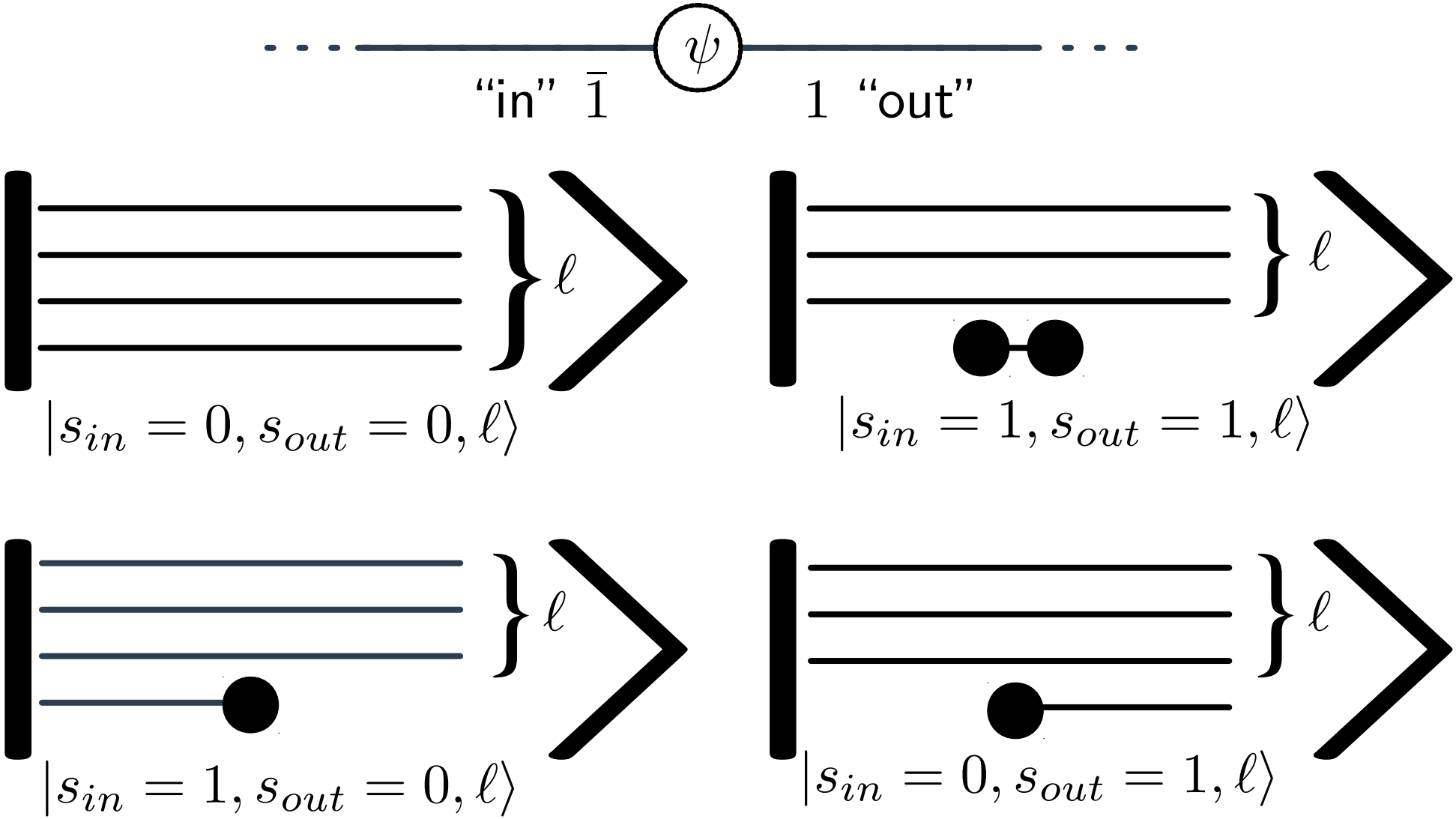
Matterless vertices



Matterful vertex

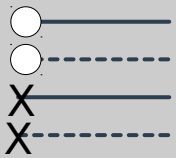
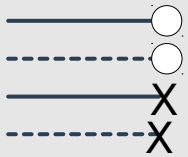
Example operators: $\quad L^{++}_{1\bar{1}} = b^\dagger \epsilon a^\dagger \qquad S^{++}_{1} = \psi^\dagger \epsilon a^\dagger$

On-(matter)site Hilbert space



All SU(2)-invariant bilinears act on these states like simple ladder operators.

Loop-String Repackaging

Operators	Behavior	#	Cartoons	Example
$n_L,\ n_R,\ n_\psi$	Measure E or quark number	3		$\psi^\dagger\psi\,\lvert\ell, s_i, s_o\rangle = (s_i + s_o)\,\lvert\ell, s_i, s_o\rangle$
"Loop" $\mathcal{L}^{\pm\pm}$	Add/destroy/reroute gauge flux	4		$\mathcal{L}^{+-}\,\lvert\ell, 0, 1\rangle \propto \lvert\ell, 1, 0\rangle$
"String" $S_{out}^{\pm\pm}$	Add/destroy/reroute a string end	4		 $\mathcal{S}_{out}^{++}\,\lvert\ell, s_i, 0\rangle \propto \lvert\ell + s_i, s_i, 1\rangle$
"String" $S_{in}^{\pm\pm}$	Add/destroy/reroute a string end	4		 $\mathcal{S}_{in}^{+-}\,\lvert\ell, 0, 0\rangle = 0$

Loop-String Repackaging

- Schwinger bosons = theory described using many SHOs
  - All operators in terms of a's and $a^\dagger$'s
  - Replaces Kogut-Susskind irrep states with HO states
  - Clebsch-Gordons implicit in SHO operator normalizations
    - Issue: *Individual* modes aren't SU(2)-invariant
    - Issue: Need Casimir constraint

Loop(-String) Formulation = theory of more general 'oscillators'

- All operators either count or have a simple ladder action (next slide)
- Replaces Schwinger modes with loop and string quantum numbers
  - All d.o.f.s strictly SU(2)-invariant
- More efficient use of Hilbert space
- Still need Casimir constraint (solved)

Advantages of loop-string quantum numbers

- Remnant constraints are Abelian ($E_L^a E_L^a = E_R^a E_R^a$ no longer automatic); **they commute**
- So techniques from U(1) can be ported over for wave function validation
  - Validation of SU(2) wave functions not previously described
  - **Gauss oracles* designed for U(1) immediately port over**
- Much greater similarity with U(1) in "how *H* acts on states" and simpler to understand.
  - Borrow quantum algorithms?

The framework we propose is a very promising starting point for digital quantum simulation

* JRS PRA 99, 042301 (2019)

Summary and Future Directions

- **Consequences of LGT basis choice are far-reaching in DQS**
  - **Very long, interesting road to QCD**
- **SU(2) + loop-string formulation is promising for DQS**
  - **Structurally closer to U(1)**
  - **Easy to enforce gauge invariance**
  - **Clearer path to SU(3)?**

[Anishetty & Sreeraj 1903.07956]

In preparation:

- **Time evolution**

See also talk by R. Brower

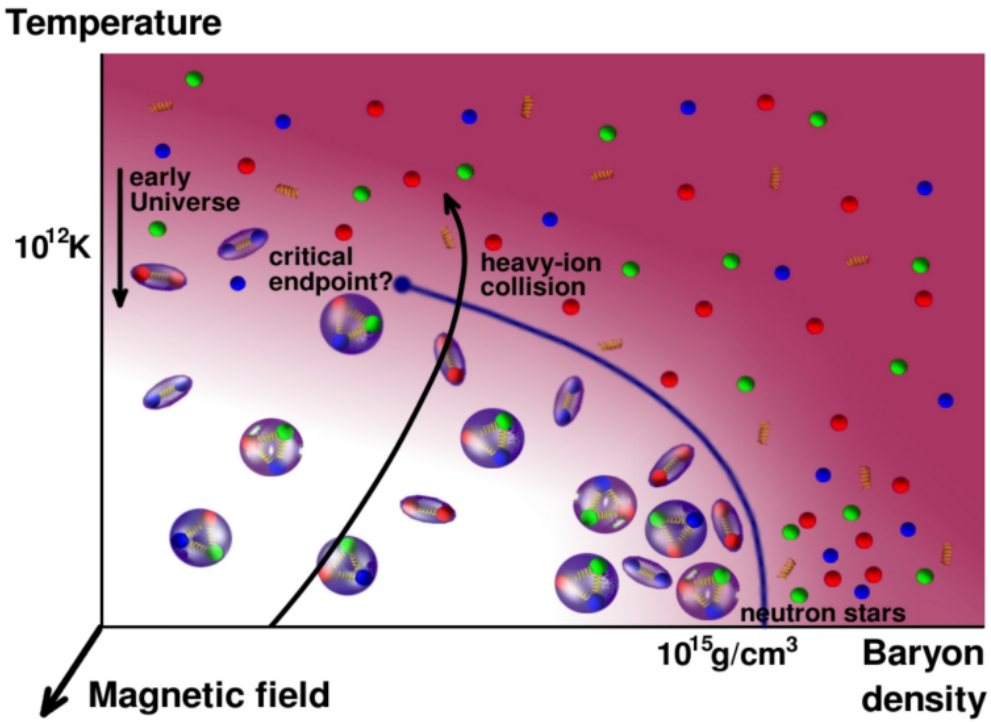
FIN

Thank you for your attention!

Questions?

Other helpful colleagues at INT: N. Klco, A. Roggero, M. Savage
This work was supported by DOE Grant No. DE-FG02-00ER41132, and by the National Science Foundation Graduate Research Fellowship under Grant No.1256082.

Phase diagram graphic credit



QCD phase diagram: Overview of recent lattice results - Scientific Figure on ResearchGate. Available from: https://www.researchgate.net/figure/Conjectured-QCD-phase-diagram_fig1_261701898 [accessed 23 Jan, 2019]

Oracles for U(1) Gauge Invariance

Basic idea