The 600-cell is regarded as the 4-dimensional analog of the icosahedron, since it has five tetrahedra meeting at every edge, just as the icosahedron has five triangles meeting at every vertex. Its boundary is composed of 600 tetrahedral cells with 20 meeting at each vertex. Together they form 1200 triangular faces, 720 edges, and 120 vertices. We first tessellate the 600 cell and then project its points to the $S^3$ where we use Qhull to construct our Delaunay triangulation of $S^3$.

A tetrahedral cell on the $S^3$ projected down to 3D.

The matrix elements of our discrete Laplacian on $S^3$ as dictated by Discrete Exterior Calculus are the following:

$$
\frac{1}{2} \sum_{ij} V_{ij} (\phi_i - \phi_j)^2 + \frac{1}{2} \sum_i m \sqrt{\mu_i} \phi_i^2 = \frac{1}{2} \phi_i^T M \phi_i \quad M_{ij} = \sum_{k=1}^n V_{ij} + m \sqrt{\mu_i} \quad \text{when } i \neq j \quad \sqrt{\mu_i} = a \sigma_i(l, j) = l_{ij}
$$

$$
V_{ij} = |\sigma_i \land \sigma_j|^2 = \frac{l_{ij} S_{ij}}{d}, \quad S_{ij} \text{ is the volume of an } d \text{-dimensional polytope } \sigma_i \text{ normal to the link } l_{ij} \text{ of the simplex in the Delaunay graph.}
$$

A visual demonstration of this in 2D is the following.

Our simplicial complex on the $S^3$ is composed of a number of $d$-dimensional simplices $|\sigma_j|$, glued to each other via $d$-2-dimensional simplices $\delta \sigma_j$. This allows us to view simplices in terms of the following hierarchy $\sigma_d \rightarrow \sigma_{d-1} \rightarrow \cdots \rightarrow \sigma_0$. We restrict the structure of our simplicial complex so that it satisfies the requirements to be a Delaunay graph. Mathematically, our hierarchy we impose on the simplices of our a Delaunay graph can be specified by the boundary operator

$$
\partial \sigma_n (l_0 l_1 \ldots l_n) = \sum_{k=0}^n (-1)^k \sigma_{n-k} (l_0 \bar{l}_1 \ldots \bar{l}_k \ldots l_n) \quad \text{where } \bar{u} \text{ means to exclude this site}
$$

The Voronoi dual lattice is constructed by inserting the circumcenters of the hierarchical structures previously mentioned and joining their nearest neighbors circumcenters to form a polytope. In 3D a visual sample of the above formula that went into the computing the spectrum of the Laplace operator on $S^3$ is

$$
|\sigma_n \land \sigma_n^*| = \frac{n!(d-n)!}{d!} |\sigma_n||\sigma_n^*|
$$

Hybrid cells, constructed from simplices and their orthogonal duals give a proper tessellation of the discrete $d$-dimensional manifold with the special case $|\sigma_d(i)| = 1$ and $|\sigma_d(i)^*| = \sqrt{d}$. In 3D a visual sample of the above formalism that went into the computing the spectrum of the Laplace operator on $S^3$ is a Delaunay triangulation with all the circumcircles and their centers (in red). The key feature of a Delaunay graph is that the circumcircle of one of its simplex contains no other vertices belonging to another simplex besides the vertices of a shared edge.