Lattice QCD calculation of the two-photon contributions to $K_L \rightarrow \mu^+ \mu^-$ and $\pi^0 \rightarrow e^+ e^-$ decays

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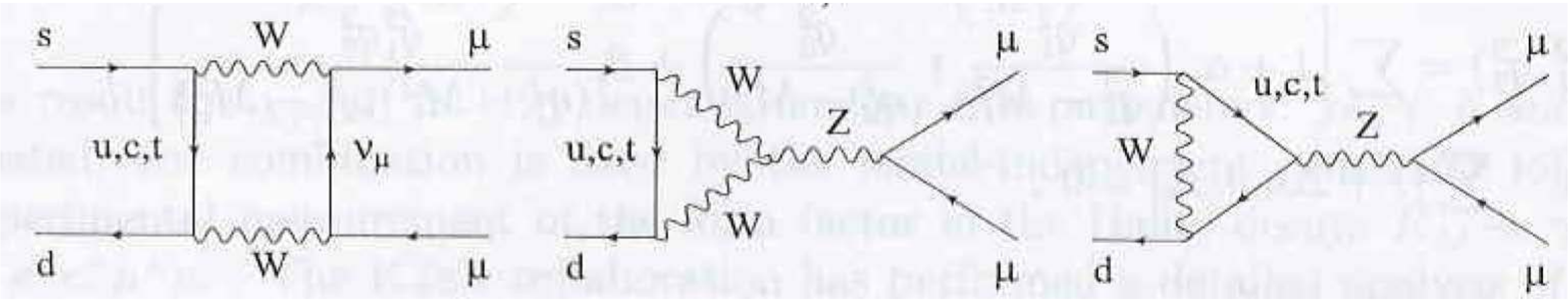
Outline

- Physics of $K_L \to \mu^+ \mu^-$
  - Interesting 2nd order weak decay
  - Large $(\alpha_{EM})^2 G_F$ background – target for lattice QCD?
- An easier example: $\pi^0 \to e^+ e^-$
  - Method presented here
  - Calculation described next by Yidi Zhao
- Prospects for $K_L \to \mu^+ \mu^-$

Physics of $K_L \rightarrow \mu^+ \mu^-$

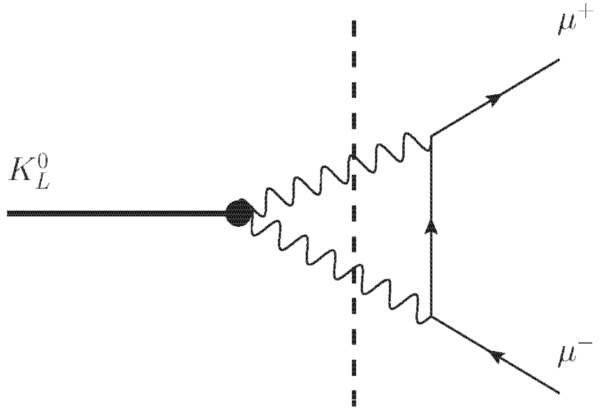
- A second order weak, ``strangeness changing neutral current''
- A potential test of the standard model at high energies and one loop



- Similar in importance to the $K^+ \rightarrow \pi^+ \nu \overline{\nu}$ decay now studied by NA62 at CERN

$K_{S/L} \rightarrow \mu^+ \mu^-$

- $K_L \rightarrow \mu^+ \mu^-$ and $K_S \rightarrow \mu^+ \mu^-$ rates are similar but K_S branching ratio is too small to be observed:
  - BR($K_S \rightarrow \mu^+ \mu^-$) < 9 x 10^{-9}
  - BR($K_L \rightarrow \mu^+ \mu^-$) = (6.84 ± 0.11) x 10^{-9}



- Two photon decays are known
  - BR($K_S \rightarrow \gamma\gamma$) < (2.63 ± 0.17) x 10^{-6}
  - BR($K_L \rightarrow \gamma\gamma$) = (5.47 ± 0.04) x 10^{-4}
- Define:

$$\frac{\Gamma(K_L \rightarrow \mu^+\mu^-)}{\Gamma(K_L \rightarrow \gamma\gamma)} = 2\beta_\mu \left(\frac{\alpha}{\pi}\frac{m_\mu}{M_K}\right)^2 \left(|F_{\rm imag}|^2 + |F_{\rm real}|^2\right)$$

- The optical theorem determines $F_{\rm imag}$

$$F_{\rm imag} = \frac{\pi}{2\beta_\mu} \ln\left(\frac{1-\beta_\mu}{1+\beta_\mu}\right) = -5.209 \pm 0.03$$

Top quark physics

- Optical theorem and measured decay rates determine:

$$|F_{\text{real}}| = |(F_{\text{real}})_{E\&M} + (F_{\text{real}})_{\text{Weak}}| = 1.167 \pm 0.094$$

- Short-distance second-order weak contribution is:

$$(F_{\text{real}})_{\text{Weak}} = 4.965 \times 10^3 \cdot \left[\text{Re}(\lambda_t) Y(x_t) + \text{Re}(\lambda_c) Y_{\text{RN}}\right]$$

$$= -1.82 \pm 0.04 \qquad (*)$$

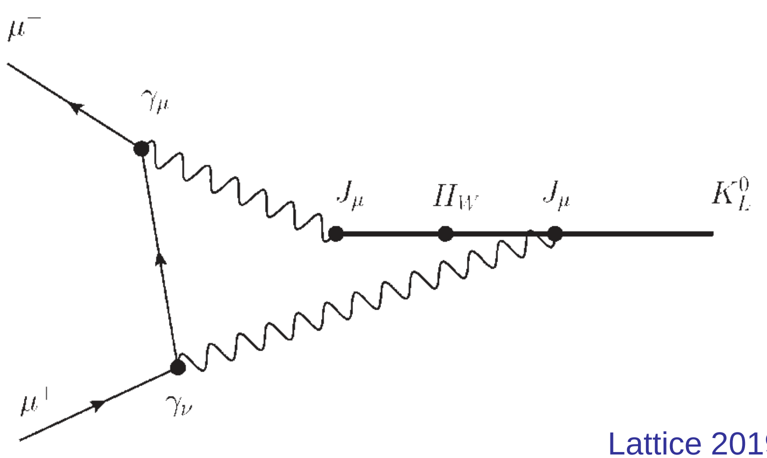
- A 10% lattice calculation of $(F_{\text{real}})_{E\&M}$ would allow a test of $(*)$ to 13%.
- Useful review article, including ChPT efforts to calculate $(F_{\text{real}})_{E\&M}$:

Cirigliano, *et al.* , Rev. Mod. Phys., **84**, 2012, p 399

Lattice Calculation of $K_L \rightarrow \mu^+ \mu^-$

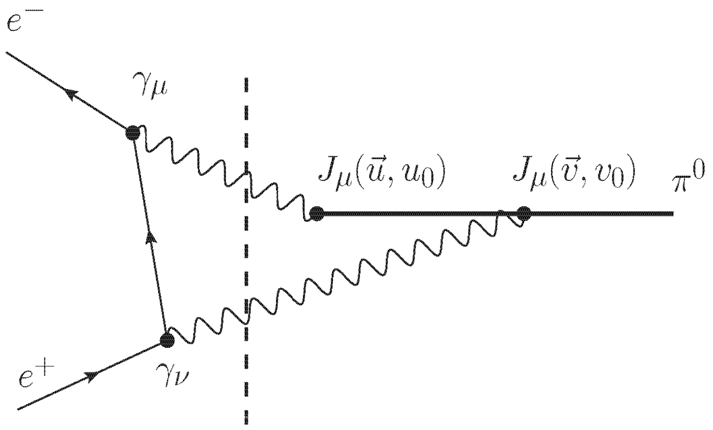
- Similar to calculation of ΔM_K
  - 2 vertices, one time order
  - Exponentially growing terms, one time order.

- But much more difficult
  - 5 vertices, 60 time orders
  - many states |n> with $E_n < M_K$



Consider simpler $\pi^0 \rightarrow e^+ e^-$

- Four vertices, 12 time orders, $E_{\gamma\gamma} < M_{\pi^0}$



- Try something different:
  - Evaluate in Minkowski space
  - Wick rotate integral over time arguments:

    $< 0|\,\mathrm{T}\{J_\mu(\vec{u},u_0), J_\nu(\vec{v},v_0)\}|\pi^0>$

Minkowski-space strategy

- Begin in Minkowski space with infinite space-time volume:

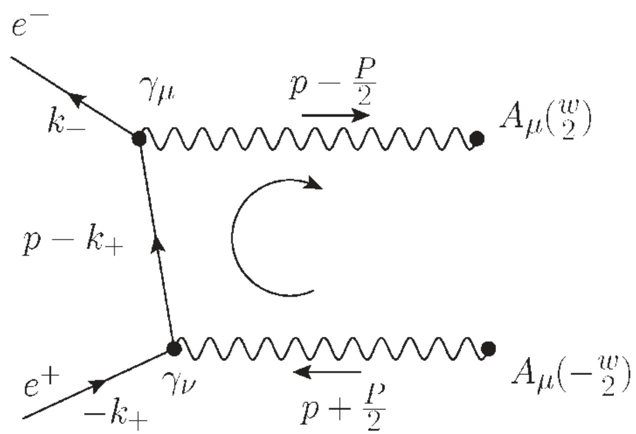
$$\mathcal{A}_{\pi^0\to e^+e^-} = \int d^4u \int d^4v \; L(u,v)_{\mu\nu} \langle 0 | T\left\{ J_\mu(u) J_\nu(v) \right\} | \pi^0(\vec{P}=0) \rangle$$

- Define: $W=(u+v)/2$, $w=u-v$.
- Integrate over W to impose four-momentum conservation.

$$\mathcal{A}_{\pi^0\to e^+e^-} \to \int d^4w \; \widetilde{L}(k_-,k_+,w)_{\mu\nu} \langle 0 | T\left\{ J_\mu(\frac{w}{2}) J_\nu(-\frac{w}{2}) \right\} | \pi^0(\vec{P}=0) \rangle$$

- Express $L(k_-,k_+,w)_{\mu\nu}$ in momentum space.

Minkowski-space strategy – con't



- Evaluate in QED perturbation theory:

$$\widetilde{L}(k_-, k_+, w)_{\mu\nu} = \int dp_0 \int d^3p\, \overline{u}(\vec{k}_-)\gamma_\mu \frac{\not{p} - \not{k}_+ + m_e}{(p-k_+)^2 + m_e^2 - i\epsilon}\gamma_\nu v(\vec{k}_+) \frac{1}{(p-\frac{P}{2})^2 - i\epsilon}\frac{1}{(p+\frac{P}{2})^2 - i\epsilon} e^{-ip\cdot w}$$

- Known analytic function of p_0 and w_0.

Minkowski-space strategy – con't

$< 0|\, T\{J_\mu(\vec{w}/2, w_0/2),\, J_\nu(-\vec{w}/2, -w_0/2)\}|\pi^0>$ also depends on w_0 in a known way:

$$\langle 0|\, T\left\{ J_\mu\left(\frac{\vec{w}}{2}, \frac{w_0}{2}\right) J_\nu\left(-\frac{\vec{w}}{2}, -\frac{w_0}{2}\right)\right\} |\pi^0(\vec{P}=0)\rangle$$

$$= \sum_n \langle 0|J_\mu(0)|n\rangle\langle n|J_\nu(,0)|\pi^0(\vec{P}=0)\rangle e^{i\vec{p}_n\cdot\vec{w} - iw_0(E_n - \frac{m_\pi}{2})}\,\theta(w_0)$$

$$+\left\{(w,\mu,\nu)\to(-w,\nu,\mu)\right\}$$

Minkowski-space strategy – con't

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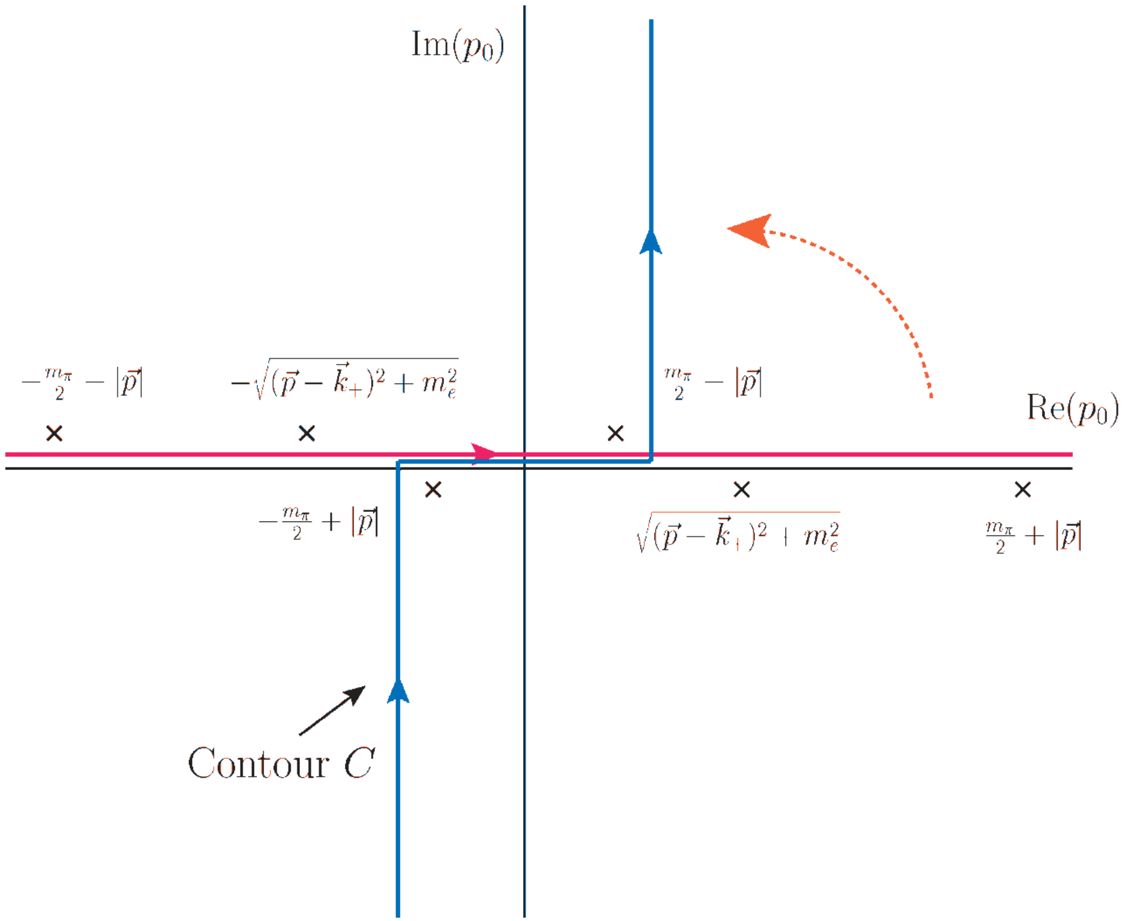
$$+\left\{(w,\mu,\nu)\to(-w,\nu,\mu)\right\}$$

Minkowski-space strategy – con't

- Wick rotate both the p_0 and w_0 integrals::



$$\sim e^{i p_0 w_0}$$

$$\downarrow$$

$$e^{w_0 p_0} \sim e^{|w_0| \frac{m_\pi}{2}}$$

Minkowski-space lattice calculation

- Wick rotated integral converges:

$$\mathcal{A}_{\pi^0\to e^+e^-} = \int d^4w\, \widetilde{L}_C(k_-,k_+,w)_{\mu\nu}\langle 0|T\left\{J_\mu(\frac{w}{2})J_\nu(-\frac{w}{2})\right\}|\pi^0(\vec{0})\rangle_E$$

$$\langle 0|T\left\{J_\mu(\frac{w}{2})J_\nu(-\frac{w}{2})\right\}|\pi^0(\vec{P}=0)\rangle_E \sim e^{-|w_0|(E_n-\frac{m_\pi}{2})} \quad E_n \geq 2m_\pi$$

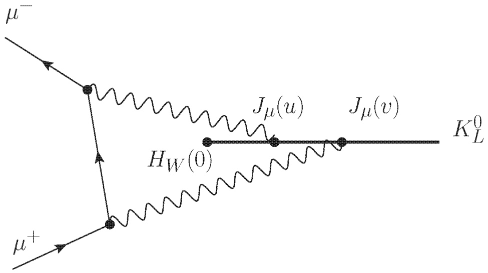
$$\widetilde{L}_C(k_-,k_+,w)_{\mu\nu} \sim e^{|w_0|\frac{m_\pi}{2}}$$

1. Problem of $E_{\gamma\gamma} < m_{\pi^0}$ has been solved by Minkowski-space perturbation theory.
2. Decay amplitude is complex with correct real and imaginary parts.
3. Hadronic amplitude can be evaluated in finite volume with exponentially suppressed errors.

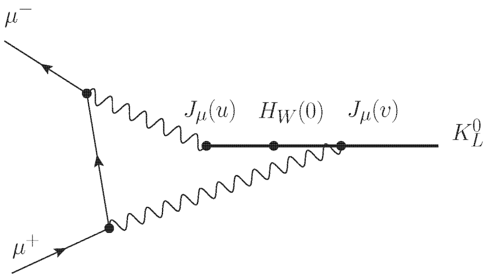
Apply to $K_L \rightarrow \mu^+ \mu^-$

Examine three time orders for $\langle 0 | T\left\{ J_\mu(u) J_\nu(v) H_W(0) \right\} | \pi^0(\vec{0}) \rangle$

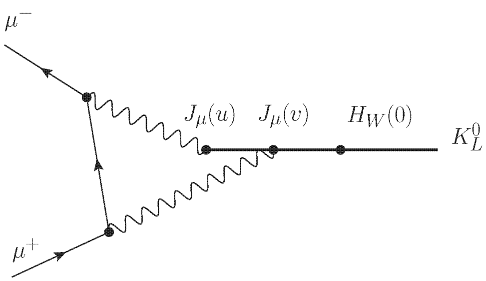
A. $0 > u_0 , v_0$



B. $u_0 > 0 > v_0$

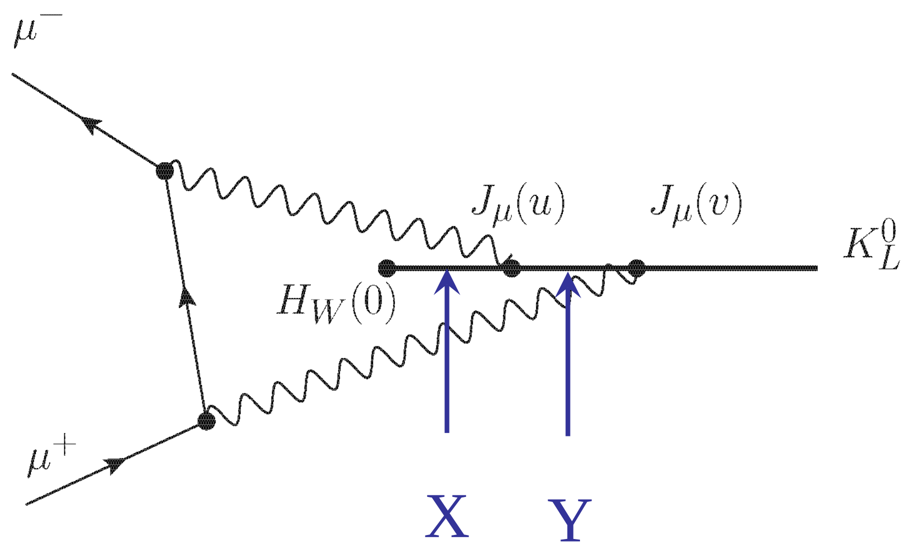


C. $u_0 , v_0 > 0$



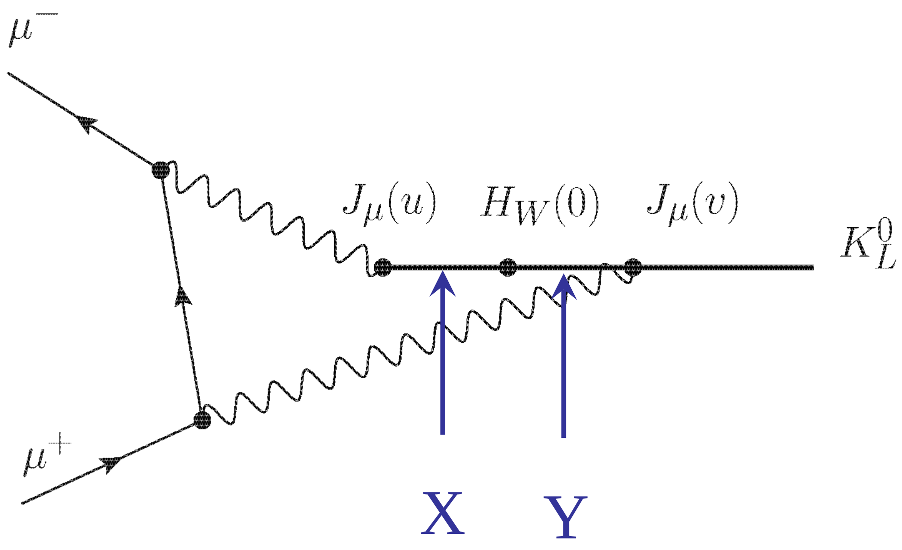
Ordering A

- $0 > u_0$, v_0
- $E_X \geq M_K$
- $E_Y \geq M_K + M_\pi$
- u_0 integral: converges
- v_0 integral: converges
- A is easily evaluated.



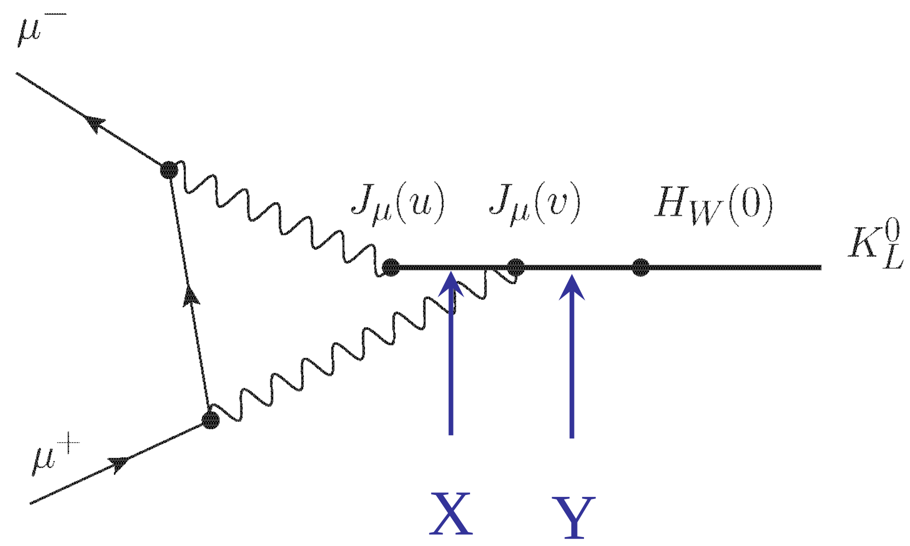
Ordering B

- u_0 ,> 0 > v_0
- $E_Y \geq M_K + M_\pi$
- $E_X \geq 2M_\pi$
- v_0 integral: converges
- u_0 integral: remove states with $E_X < M_K$.
- X = π , η not allowed
- Remove X = 2π
- For $K_L \rightarrow \gamma\gamma$, $|\vec{p}_X| > 0$ removes problem.



Ordering C

- u_0 , $v_0 > 0$
- $E_Y \geq 0$
- $E_X \geq 2M_\pi$
- u_0,v_0 integrals: remove states with E_X & $E_Y < M_K$.
- Remove X = 2π
- Remove Y = |0>, π , 2π, η
- For $K_L \rightarrow \gamma\gamma$, $|\vec{p}_X| > 0$ removes problem with X



Conclusion

- QED portion of some combined QCD + QED amplitudes can be evaluated in Minkowski space.
- Position space formulation can lead to exponentially suppressed finite-volume errors ($\pi^0 \rightarrow e^+ e^-$ case).
- Compute complex $\pi^0 \rightarrow e^+ e^-$ amplitude decay using this method (Stay for the next talk of Yidi Zhao and see Xu Feng's $\pi^0 \rightarrow \gamma\gamma$ poster.)
- A 10% calculation of E&M contribution to $K_L \rightarrow \mu^+ \mu^-$ would provide a new test of the Standard Model
  - Much simplified by the proposed method.
  - Still difficult with many graphs and subtractions.
  - Need theory of finite-volume effects (arXiv:1504.01).
  - $K_L \rightarrow \gamma\gamma$ somewhat easier.