Topology of Trace Deformed Yang-Mills Theory: a Lattice Study

Claudio Bonati, Marco Cardinali, Massimo D’Elia

University of Pisa, INFN sez. Pisa

21 June 2019

The 37th International Symposium on Lattice Field Theory
Wuhan 2019
Table of Contents

Motivations
   The Deformed Theory

$SU(3)$

$SU(4)$
   Preliminary Results $\rightarrow$ In progress

Conclusions
Motivations

**Yang-Mills** theory at **LOW-\(T\)**:
- Confined.
- Strongly coupled \(\Rightarrow\) **no** perturbative methods.
- Center symmetry is realized \(\Rightarrow\) \(\langle \text{Tr} P \rangle = 0\).

**Yang-Mills** theory at **HIGH-\(T\)**:
- Deconfined.
- Weakly coupled \(\Rightarrow\) perturbative/semiclassical methods.
- Center symmetry is spontaneously broken \(\Rightarrow\) \(\langle \text{Tr} P \rangle \neq 0\).

Is it possible to **use semiclassical methods** to study the low-\(T\) regime? Or, **most in general**:

How the **properties of the confined phase** and \(\langle \text{Tr} P \rangle = 0\) are related?
The Deformed Theory

Consider a deformed theory in which center symmetry is recovered even at high-\( T \).


\[
S^{\text{def}} = S_{W} + h \sum_{\vec{n}} |\text{Tr}P(\vec{n})|^{2}
\]

where \( S_{W} \) is the Wilson action.

- Gauge configurations with \( \langle \text{Tr}P \rangle \neq 0 \) are suppressed.
- The parameter \( h \) is chosen in order to restore center symmetry.
- The theory is on \( \mathcal{R}^{3} \times S^{1} + \text{PBC} \).
The Aim of This Work

1. **Start** deep in the *deconfined* phase.
2. **Switch** on the *deformation*.
3. **Study** the properties of the *re-confined phase*.

We want to investigate:

- How center symmetry is recovered.
- Compute observables in the re-confined phase and compare their values with the ones obtained in the usual confining phase. In particular *topological properties*.

Are the deformed theory and the usual one equivalent?
Summary of Topology

\[ \mathcal{L}_\theta = \mathcal{L}_{YM} - i\theta Q(x) \]

\[ F(\theta, T) = -\frac{1}{V_4} \ln \int [dA] \exp \left\{ -\int_0^1 dt \int d^3x \mathcal{L}_\theta \right\} \]

The free energy \( F(\theta, T) \) can be parametrized as follows:

\[ F(\theta, T) - F(0, T) = \frac{1}{2} \chi(T) \theta^2 \left( 1 + b_2(T) \theta^2 + b_4(T) \theta^4 + \cdots \right) \]

and it is easy to see that

\[ \chi = \frac{\langle Q^2 \rangle_{\theta=0}}{V_4} \]

\[ b_2 = -\frac{1}{12\langle Q^2 \rangle_{\theta=0}} \left[ \langle Q^4 \rangle_{\theta=0} - 3\langle Q^2 \rangle_{\theta=0}^2 \right] \]

\( b_2 \) is a very noisy observable. In order to measure it we used the imaginary \( \theta \) method.

Imaginary Theta Method

We add to the Lagrangian an imaginary $\theta$ term

$$S^{\text{def},i\theta} = S_W + h \sum_{\vec{n}} |\text{Tr} P(\vec{n})|^2 - \theta_L Q_L$$

where $Q_L$ is the clover discretisation of $Q$.

We perform simulations using different values of $\theta_L$ and we obtain $\chi$, $b_2$ and $Z$ with a combined fit of the first four cumulants

\[
\frac{\langle Q \rangle}{V_4} = \chi Z \theta_L \left( 1 - 2b_2 Z^2 \theta_L^2 + 3b_4 Z^4 \theta_L^4 + \cdots \right)
\]

\[
\frac{\langle Q^2 \rangle_c}{V_4} = \chi \left( 1 - 6b_2 Z^2 \theta_L^2 + 15b_4 Z^4 \theta_L^4 + \cdots \right)
\]

\[
\frac{\langle Q^3 \rangle_c}{V_4} = \chi \left( -12b_2 Z \theta_L + 60b_4 Z^3 \theta_L^3 + \cdots \right)
\]

\[
\frac{\langle Q^4 \rangle_c}{V_4} = \chi \left( -12b_2 + 180b_4 Z^2 \theta_L^2 + \cdots \right)
\]
Topology and Finite Temperature: MC Results

Fig. 3

SU(2) data
SU(3) data

Topological properties change drastically from the low-$T$ to the high-$T$ regime.

(sx) [Bonati, D’Elia, Panagopoulos, Vicari: PRL 110 (25) 2013]
(dx) [Allès, D’Elia, Di Giacomo: PLB 412 1997] See also:
[C. Gattringer, R. Hoffmann and S. Schaefer, PL B535, 358 (2002)]
[L. Del Debbio, H. Panagopoulos and E. Vicari, JHEP0409, 028 (2004)]
SU(3)
Restoration of Center Symmetry

Center Symmetry is recovered increasing $h$.

The local value of $\text{Tr}P$?

Adjoint Polyakov loop.

$P^{\text{adj}} = |\text{Tr}P|^2 - 1$.

A negative value implies that $\text{Tr}P$ is close to zero locally.
$r_0^4 \chi$ vs $h$ on $32^3 \times 8$ Lattice, $\beta = 6.4$

- $r_0$ is the Sommer parameter, used to fix the scale, and it is approximately 0.5 fm.
- We assumed that the deformation does not modify the lattice spacing.
- $r_0^4 \chi$ in YM $\rightarrow$ [C. Bonati et al: PRD 93, (2016) 025028].
$r_0^4 \chi$ on Different Lattices ($N_S = 32$)

- $\beta = 6.0$, $N_t = 6$
  - $L^{-1} = 365$ MeV
- $\beta = 6.2$, $N_t = 6$
  - $L^{-1} = 493$ MeV
- $\beta = 6.2$, $N_t = 8$
  - $L^{-1} = 372$ MeV
- $\beta = 6.4$, $N_t = 8$
  - $L^{-1} = 493$ MeV
$b_2$ on $N_t = 8 \quad N_s = 32$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a$. 

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a$. 

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a$. 

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$

- $b_2$ in YM
  - $\beta = 6.4$, $h = 1.10$
  - $L^{-1} = 493$ MeV

- Without im. $\theta$
  - $\beta = 6.2$, $h = 1.20$
  - $L^{-1} = 372$ MeV

- With im. $\theta$
  - $\beta = 6.4$, $h = 1.20$
  - $L^{-1} = 493$ MeV

- Dilute instanton gas

- $b_2$ is dimensionless ⇒ we do not need to fix $a.$
SU(4)
Two Deformations

\[ SU(4) \rightarrow \text{Center Symmetry has two breaking patterns:} \]

\[ \mathbb{Z}_4 \rightarrow \text{Id} \quad \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \]

The order parameter are

\[ \langle \text{Tr} P \rangle \quad \langle \text{Tr} P^2 \rangle \]

In order to recover the full center symmetry we must consider two deformations:

\[ S_{\text{def}} = S_W + h_1 \sum_{\vec{n}} \left| \text{Tr} \ P(\vec{n}) \right|^2 + h_2 \sum_{\vec{n}} \left| \text{Tr} P^2(\vec{n}) \right|^2 \]
Restoration of Center Symmetry

\[
\langle \text{TrP} \rangle_{(h,0)}, \langle \text{TrP} \rangle_{(0,h)}, \langle \text{TrP} \rangle_{(h,h)}
\]

\[6 \times 32^3 \quad \beta = 11.15 \quad T = 393 \text{ MeV}\]

\[6 \times 32^3 \quad \beta = 11.40 \quad T = 482 \text{ MeV}\]
Topological Susceptibility

$6 \times 32^3 \quad \beta=11.15 \quad T=393 \text{ MeV}$

$\chi_{\text{def}} / \chi_{T=0}$

$T = 0 \rightarrow [\text{C. Bonati et al: PRD 94, (2016) 085017}].$
Both $\langle \text{Tr}P \rangle$ and $\langle \text{Tr}P^2 \rangle$ must be zero to recover the correct $T = 0$ result.
$b_2$ Coefficient

$6 \times 32^3 \quad \beta=11.15 \quad T=393 \text{ MeV} \quad h=1.5$

Dilute Instanton Gas

$\cos(\theta/4)$ prediction

$h=1.5 \quad \beta=11.15 \quad T=393 \text{ MeV}$

- $b_2$ prediction
- $b_2$ for $T=0$
- $b_2$ for $h=1.5$
- $b_2$ for $\beta=11.15$
- $b_2$ for $T=393$ MeV

- $b_2$ for $h=1.5$ for $\cos(\theta/4)$

- $b_2$ for $T=0$ for $\cos(\theta/4)$
$b_2$ Coefficient

6\times 32^3 \quad \beta = 11.40 \quad T = 482 \text{ MeV} \quad h = 1.5

\begin{itemize}
  \item \cos(\theta/4) prediction
  \item (h,0)
  \item (0,h)
  \item (h,h)
\end{itemize}

Dilute Instanton Gas
Continuum Limit in $SU(3)$ (In progress)

![Graph showing the continuum limit in SU(3)]

- The plateau is more stable when the lattice spacing is finer.
Conclusions

- We study a deformed $SU(N)$ YM theory in which center symmetry is recovered even at high temperature.
- Once center symmetry is recovered the topological properties of the reconfined phase ($\chi$ and $b_2$) are in agreement with the values obtained at $T = 0$.
- For $SU(N)$ with $N > 3$ we need more than one deformation in order to avoid different breaking patterns of center symmetry.
- In order to obtain the $T = 0$ values of $\chi$ and $b_2$ in $SU(4)$ center symmetry must not be broken to any subgroup.
THANK YOU
BACK-UP SLIDES
Discretisation of The Topological Charge

- In our simulations we will use the discretisation of the topological charge with definite parity

\[ q_L(x) = -\frac{1}{2^9\pi^2} \sum_{\mu\nu\rho\sigma=\pm 1}^{\pm 4} \varepsilon_{\mu\nu\rho\sigma} \text{tr} \left[ \Pi_{\mu\nu} \Pi_{\rho\sigma} \right] \]

- In the continuum limit, \( q_L(x) \) must be corrected by a renormalization factor \( Z \) introduced by the lattice discretisation

\[ q_L(x) \rightarrow a^4 Z q(x) + O(a^6) \]

- We remove UV fluctuation using the Cooling procedure.
Dilute Instanton Gas Approximation (DIGA)

We can describe our system as a gas of weakly interacting objects called (anti-) instantons which carry a topological charge equal to (minus) one and a finite action.

The free energy of this system is given by

\[ F(\theta) \approx \chi (1 - \cos \theta) \rightarrow b_2 = -\frac{1}{12} \]
Lattice Spacing and the Deformation on $SU(3)$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$h$</th>
<th>$t_0/a^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.96</td>
<td>0.0</td>
<td>2.7854(62)</td>
</tr>
<tr>
<td>5.96</td>
<td>1.0</td>
<td>2.8087(69)</td>
</tr>
<tr>
<td>5.96</td>
<td>2.0</td>
<td>2.8063(74)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$h$</th>
<th>$t_0/a^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.17</td>
<td>0.0</td>
<td>5.489(14)</td>
</tr>
<tr>
<td>6.17</td>
<td>1.0</td>
<td>5.530(16)</td>
</tr>
<tr>
<td>6.17</td>
<td>2.0</td>
<td>5.498(16)</td>
</tr>
</tbody>
</table>

- To test the independence of the lattice spacing on $h$ we determined the scale $t_0$ defined by gradeient flow. See [M. Luscher: JHEP 1403, 092 (2014)].
- $\beta = 5.96 \rightarrow 24^4$ lattices.
  $\beta = 6.17 \rightarrow 32^4$ lattices.
- Data coincides with those at $h = 0$ up to less than 1%. 
Scatter Plots $SU(4) \beta = 11.15$
Scatter Plots $SU(4)$ $\beta = 11.40$

$6\times32^3$ $\beta=11.40$

ReTrP vs. ImTrP

$6\times32^3$ $\beta=11.40$

ReTrP$^2$ vs. ImTrP$^2$