Spectral Projectors Method for Staggered Fermions

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Topological charge discretizations

Many different discretizations of the topological charge exist. They can be divided in two broad classes:

Gluonic

Discretizations expressed via the gauge links + smoothing method to dump UV fluctuations (e.g. cooling). For example:

$$Q_L = -\frac{1}{2^9\pi^2}\sum_x \sum_{\mu\nu\rho\sigma=\pm 1}^{\pm 4} \epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\{\Pi_{\mu\nu}(x)\Pi_{\rho\sigma}(x)\}$$

Fermionic

Discretizations expressed using the Dirac lattice operator D and based on the index theorem, which states, in the continuum theory:

$$Q = \mathrm{Index}\{\not{D}\} = n_+ - n_-$$

The spectral projectors method

Spectral projectors (Lüsher, 2004; Giusti & Lüscher, 2008) can be used to obtain a fermionic definition of the topological charge which leads to a non-singular, universal definition of the topological susceptibility χ and of all higher-order cumulants of the topological charge distribution.

Introducing the orthogonal projectors $\mathbb{P}_M = \sum_{|\lambda|<M} \mathbb{P}_\lambda$ on eigenspaces of the Wilson operator D_W with $|\lambda| \leq M$, the topological susceptibility χ can be written as:

$$\chi = \frac{Z_S^2}{Z_P^2} \frac{\langle Q_L^2 \rangle}{V};$$

where

$$Q_L = \mathrm{Tr}\{\gamma_5 \mathbb{P}_M\}, \qquad \frac{Z_S^2}{Z_P^2} = \frac{\langle \mathrm{Tr}\{\mathbb{P}_M\}\rangle}{\langle \mathrm{Tr}\{\gamma_5 \mathbb{P}_M \gamma_5 \mathbb{P}_M\}\rangle}.$$

The spectral projectors method for Wilson fermions has been tested both in the quenched theory (Lüscher & Palombi, 2010; Cichy et al., 2015) and in full QCD (Alexandrou et al., 2017).

The goal of the present work is to extend this method to staggered fermions, which is another commonly adopted discretization, and test it in on the pure Yang-Mills theory with $N_c = 3$.

To do so, we need to start from the "lattice index theorem" for staggered fermions (Adams, 2010):

$$\text{Index}\{D_{\text{st}}\} = 2^{d/2} Q_L$$

The $2^{d/2}$ factor is crucial, it cancels out the mode over-counting due to unphysical taste degeneration.

Topological susceptibility using staggered spectral projectors

Since $\text{Index}\{D_{\text{st}}\} \equiv \text{Tr}\{\Gamma_5 e^{tD_{\text{st}}^2}\}\ \forall t > 0$, we can evaluate the trace over the eigenspaces of D_{st} with $|\lambda| \leq M$. One gets:

$$Q_L = 2^{-d/2}\,\text{Tr}\{\Gamma_5 \mathbb{P}_M\} \tag{1}$$

Conversely, for the ratio of the renormalization constants Z_S/Z_P no extra factor appears, since the over-counting cancels out between numerator and denominator:

$$\frac{Z_S^2}{Z_P^2} = \frac{\langle \text{Tr}\{\mathbb{P}_M\}\rangle}{\langle \text{Tr}\{\Gamma_5 \mathbb{P}_M \Gamma_5 \mathbb{P}_M\}\rangle}. \tag{2}$$

Collecting (1) and (2) one obtains the expression for χ:

$$\chi = 2^{-d}\frac{\langle \text{Tr}\{\mathbb{P}_M\}\rangle}{\langle \text{Tr}\{\Gamma_5 \mathbb{P}_M \Gamma_5 \mathbb{P}_M\}\rangle}\frac{\langle \text{Tr}\{\Gamma_5 \mathbb{P}_M\}^2\rangle}{V}.$$

Numerical strategies

In order to measure χ on the lattice, one can express $\mathbb{P}_M$ using normalized orthogonal eigenvectors of iD_{st}:

$$\mathbb{P}_M = \sum_{|\lambda_i| \leq M_0} u_i u_i^\dagger, \quad iD_{\mathrm{st}} u_i = \lambda_i u_i, \quad \lambda_i \in \mathbb{R}.$$

Now, the traces can be rewritten using only such eigenvectors:

$$\mathrm{Tr}\{\mathbb{P}_M\} = \nu(M_0) \quad \text{(number of eigenvalues } |\lambda| \leq M_0\text{)},$$

$$\mathrm{Tr}\{\Gamma_5 \mathbb{P}_M\} = \sum_{|\lambda_i| \leq M_0} u_i^\dagger \Gamma_5 u_i,$$

$$\mathrm{Tr}\{\Gamma_5 \mathbb{P}_M \Gamma_5 \mathbb{P}_M\} = \sum_{|\lambda_i|, |\lambda_j| \leq M_0} u_i^\dagger \Gamma_5 u_i \; u_j^\dagger \Gamma_5 u_j.$$

Pure-gauge ensembles were generated using the standard plaquette Wilson action: $S = -\dfrac{\beta}{N_c} \sum_{x, \mu > \nu} \Re \, \mathrm{Tr} \, \Pi_{\mu\nu}(x)$.

Some technical remarks on the cut-off mass M

The cut-off mass M is irrelevant in the continuum limit since only zero-modes contribute to topology. As $a \to 0$, it must be kept constant to guarantee that $O(a)$ corrections are canceled out:

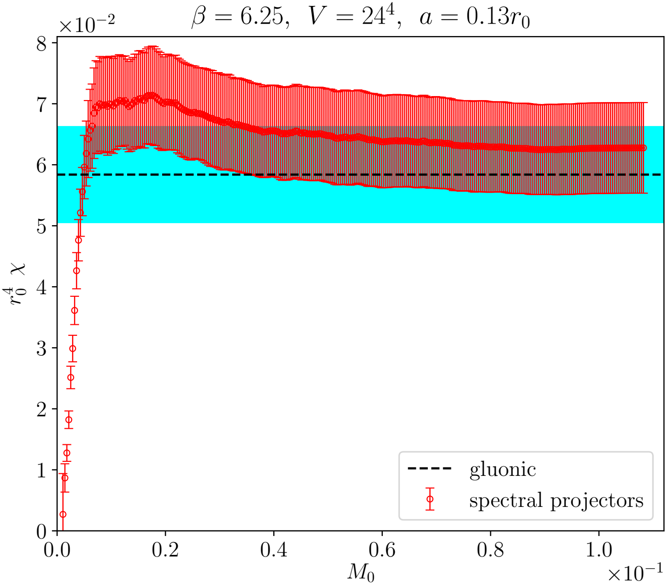
$$\chi(a, M) = \chi + c(M)a^2 + O(a^4).$$

Using chiral perturbation theory (Giusti & Lüscher, 2008) it is possible to show that, when $V \to \infty$ and $m_{\text{quark}} \to 0$:

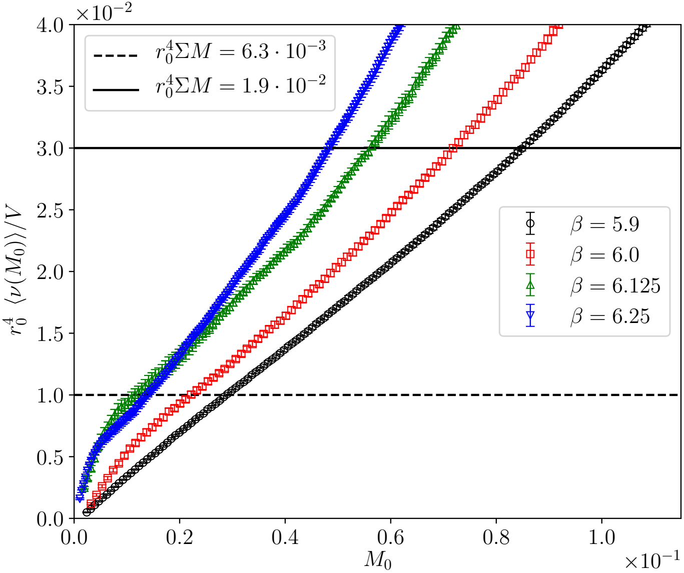
$$\frac{\langle \nu(M) \rangle}{V} = \frac{\pi}{2}\Sigma M, \quad \Sigma = \langle \bar{\psi}\psi \rangle \,. \tag{3}$$

Since the right hand of (3) is a renormalization-group-invariant term, so is the left hand. Thus, in order to keep M constant, it is sufficient to keep $\dfrac{\langle \nu(M_0) \rangle}{V}$ constant (in physical units) for each lattice spacing while scaling towards the continuum.

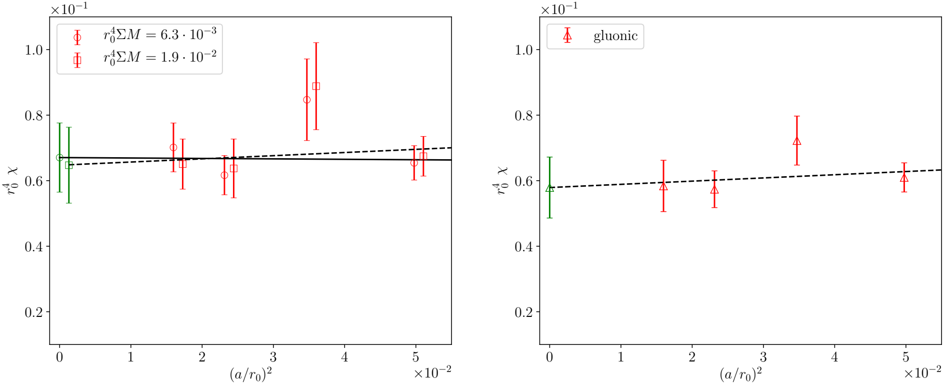
Gluonic + cooling vs spectral projectors measure of χ



Mode numbers



Continuum limit of χ at $T = 0$



This work	Stag. spectral projectors $r_0^4\Sigma M = 6.3 \cdot 10^{-3}$	Stag. spectral projectors $r_0^4\Sigma M = 1.9 \cdot 10^{-2}$	Gluonic + cooling
$r_0^4\chi \cdot 10^2$	6.7(1.1)	6.5(1.2)	5.79(93)

	Overlap operator + index theorem (Del Debbio et al., 2005)	Wilson spectral projectors (Lüscer & Palombi, 2010)
$r_0^4\chi \cdot 10^2$	5.9(3)	6.7(3)

Extension to higher-order cumulants

The staggered spectral projectors method can be generalized to higher-order cumulants of the topological charge distribution. For example, let's see the fourth cumulant:

$$b_2 = -\frac{1}{12}\frac{\langle Q^4\rangle - 3\langle Q^2\rangle^2}{\langle Q^2\rangle}.$$

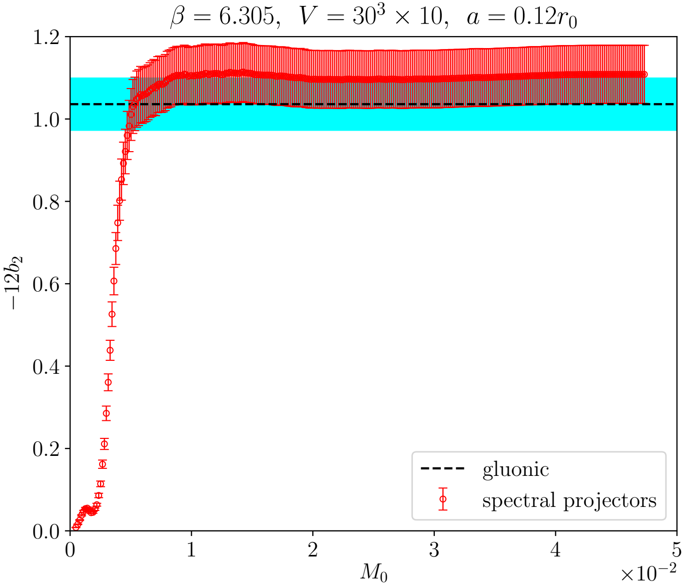
Following what has been done for χ and using the index theorem for staggered fermions:

$$b_2^{\text{stag}} = -\frac{2^{-d}}{12}\frac{Z_S^2}{Z_P^2}\frac{\langle Q_L^4\rangle - 3\langle Q_L^2\rangle^2}{\langle Q_L^2\rangle},$$

Finally, using the spectral expressions for Q_L and Z_S^2/Z_P^2:

$$b_2^{\text{stag}} = -\frac{2^{-d}}{12}\frac{\langle \mathrm{Tr}\{\mathbb{P}_M\}\rangle}{\langle \mathrm{Tr}\{\Gamma_5\mathbb{P}_M\Gamma_5\mathbb{P}_M\}\rangle}\frac{\langle \mathrm{Tr}\{\Gamma_5\mathbb{P}_M\}^4\rangle - 3\langle \mathrm{Tr}\{\Gamma_5\mathbb{P}_M\}^2\rangle^2}{\langle \mathrm{Tr}\{\Gamma_5\mathbb{P}_M\}^2\rangle}$$

Gluonic + cooling vs spectral projectors measure of b_2



Conclusions

Summary

- In this work we present a generalization of the spectral projectors method to the case of staggered fermions.
- We derived, starting from the staggered index theorem, expressions for the topological susceptibility χ and for the quartic coefficient b_2, generalizing the spectral projectors method to higher-order cumulants.
- Spectral projectors yield results which are perfectly in agreement with the standard gluonic determination and with other methods.

Future perspectives

- Extend the study of the finite temperature case and of b_2.
- Extend the application of staggered spectral projectors to full QCD.

Thank you for your attention!

Remarks on θ-dependence

The vacuum energy can be expressed as:

$$E(\theta) = -\frac{1}{V}\log\int[dA]e^{-S[A]+i\theta Q[A]}.$$

Expanding around $\theta = 0$, one gets:

$$E(\theta) = \frac{1}{2}\chi\theta^2\left(1 + \sum_{n=0}^{\infty} b_{2n}\theta^{2n}\right).$$

The coefficients are related to the cumulants of the topological charge distribution:

$$\chi = \frac{\langle Q^2\rangle}{V}, \quad b_{2n} = (-1)^n \frac{2}{(2n+2)!}\frac{\langle Q^{2n+2}\rangle_c}{\langle Q^2\rangle}.$$

Using the staggered index theorem, one obtains for the b_{2n} coefficients:

$$b_{2n}^{\text{stag}} = \frac{2}{(2n+2)!}\frac{(-1)^n}{2^{nd}}\left(\frac{Z_S}{Z_P}\right)^{2n}\frac{\langle Q_L^{2n+2}\rangle_c}{\langle Q_L^2\rangle}$$

where, as already shown,

$$Q_L = \mathrm{Tr}\{\Gamma_5\mathbb{P}_M\}, \quad \left(\frac{Z_S}{Z_P}\right)^{2n} = \left(\frac{\mathrm{Tr}\{\mathbb{P}_M\}}{\mathrm{Tr}\{\Gamma_5\mathbb{P}_M\Gamma_5\mathbb{P}_M\}}\right)^n$$

We also recall that:

$$b_{2n} \underset{T\ll T_C}{\sim} \frac{\bar{b}_{2n}}{N_c^{2n}} + O\left(\frac{1}{N_c^{2n+1}}\right), \qquad b_{2n} \underset{T\gg T_C}{\sim} (-1)^n\frac{2}{(2n+2)!}$$