How to extract the "Abelian" part of double-winding Wilson loop

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- It is known that the naive Abelian projection fails when we apply it to Wilson loops in higher representations and double-winding Wilson loops.
- Recently we found that the string tension extracted from Wilson loops in a higher representation is reproduced by using the highest weight part of Abelian Wilson loops instead of naive Abelian Wilson loops. (A. Shibata, R. M., S. Kato, K.-I. Kondo, PoS(LTTICE2018)256)
- However, for double-winding Wilson loops, correctly-behaving "Abelian" operators are not known.

In this talk, I would like to propose a more general way to extract an Abelian part of a loop operator which would behave correctly.

Outline

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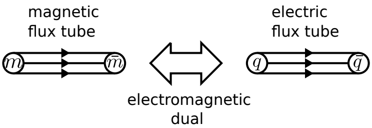
1. Introduction
2. Abelian loop operators which would behave correctly
3. Behavior of the proposed operators on double-winding contours

Dualsuperconductivity picture

- The dualsuperconductor picture is a promising scenario for quark confinement proposed by Nambu, 't Hooft and Mandelstam, in which magnetic monopoles play an important role.
- In this scenario, the QCD vacuum is considered as a dualsuperconductor.
- QCD strings formed between color charges are considered as electromagnetic dual of Abrikosov vortices.



Ordinary superconductivity is the result of condensation of Cooper pairs. Therefore we can suppose that, in order to be a dualsuperconductor, condensation of magnetic monopoles have to occur.

Abelian projection

Frequently we define magnetic monopoles by using Abelian projection. We consider $SU(N)$ gauge theories for simplicity. In this method, we factorize link variables $U_l \in SU(N)$ into Abelian link variables $u_l \in U(1)^{N-1}$ and the off-diagonal parts U_l^{off} in a fixed gauge as

$$U_l = u_l U_l^{\mathrm{off}}.$$

An Abelian link variable u_l is defined as the diagonal element of the gauge group which maximizes

$$\mathrm{Re}\,\mathrm{tr}(u_l U_l^{\dagger}),$$

for each link. This means that an Abelian link variable is the best diagonal approximation to a link variable. This decomposition is performed in a fixed gauge. The MA gauge where the functional

$$\sum_{l} \sum_{i=1}^{N-1} \mathrm{tr}(U_l H_i U_l^{\dagger} H_i),$$

(H_i is a Cartan generator) is maximized is frequently used. The magnetic current is defined by applying the DeGrand-Touissaint procedure to this Abelian link variables.

A numerical test of the dualsuperconductor picture

It was checked numerically that the magnetic-monopole contributions to Wilson loops reproduce the correct string tension extracted from the Wilson loops by using Abelian projection as follows.

1. The Abelian Wilson loop is defined by using Abelian link variables u_l as

$$\mathrm{tr} \prod_{l \in C} u_l .$$

   It was checked that the average of the Abelian Wilson loop reproduces the string tension for the full Wilson loop in SU(2) (Suzuki-Yotsuyanagi, 1990) and in SU(3) (Stack-Tucker-Wensley, 2002), which is called the Abelian dominance.

2. The monopole contributions is extracted from Abelian Wilson loops by applying the DeGrand-Tousaint procedure.
   It was checked that the monopole part of the Wilson loop reproduces the string tension in SU(2) (Suzuki-Yotsuyanagi, 1990) and in SU(3) (Stack-Tucker-Wensley, 2002), which is called the monopole dominance.

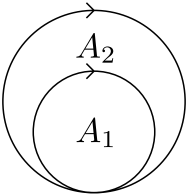
Double-winding Wilson loops

In the $SU(2)$ pure gauge theory, it was numerically checked that double-winding Wilson loops obey the difference of area law, i.e.,

$$\langle \mathrm{tr}(W(C_1)W(C_2)) \rangle \sim e^{-\sigma(A_2 - A_1)},$$

where $W(C_i)$ is the untraced Wilson loop whose contour is C_i.

Recently, it has been discussed that naive Abelian Wilson loops cannot reproduce this behavior of double-winding Wilson loops.
Greensite and Höllwieser, Phys. Rev. D91, 054509 (2015).



In this talk I'd like to give a way to avoid this problem.

Outline

The Haar-measure-corrected Abelian Wilson loop

Firstly we consider the single-winding Wilson loop in higher representations. Recently we have proposed correctly-behaving Abelian loop operators in higher representation, which is the highest weight part of Abelian Wilson loops. Today we propose another operator which behaves correctly, and then modify it to reproduce the behavior of double-winding Wilson loops.
Now we consider $SU(2)$ gauge theory for simplicity.

Let an untraced Abelian Wilson loop defined on a loop C be $W^{\mathrm{Abel}}(C) = \mathrm{diag}(e^{i\theta_C}, e^{-i\theta_C})$. Then we propose the operator

$$2\sin^2\theta_C \, \mathrm{tr}_J \, W^{\mathrm{Abel}}(C),$$

where tr_J denotes the trace in the spin-J representation, i.e.,

$$\mathrm{tr}_J \, W^{\mathrm{Abel}}(C) := e^{-2iJ\theta_C} + e^{-2i(J-1)\theta_C} + \cdots + e^{2iJ\theta_C}.$$

We call this operator as the Haar-measure-corrected Abelian Wilson loop because the factor $2\sin^2\theta_C$ corresponds to the difference of the Haar measure between $SU(2)$ and $U(1)$.

Reason why we consider the operator behaves correctly

The reason why we consider this operator behaves correctly is as follows. First we write the expectation value of a Wilson loop and an Abelian Wilson loop by using distribution function of the untraced Wilson loop and that of the Abelian Wilson loop as

$$\langle \mathrm{tr}_J W(C) \rangle = \int_{SU(2)} dW \, P(W; C) \, \mathrm{tr}_J W \tag{1}$$

$$\langle \mathrm{tr}_J W^{\mathrm{Abel}}(C) \rangle = \int_{U(1)} dw \, P^{\mathrm{Abel}}(w; C) \, \mathrm{tr}_J w, \tag{2}$$

where dW is the Haar measure on $SU(2)$, dw is the Haar measure on $U(1)$, and the distribution functions are defined as

$$P(W; C) := \int DU \, \delta_{SU(2)}(W - \prod_{l \in C} U_l) e^{-S[U]}$$

$$P_{\mathrm{Abel}}(w : C) := \int DU \, \delta_{U(1)}(w - \prod_{l} u_l) e^{-S[U]}.$$

Because $P(W; C)$ is a class function, i.e., $P(W) = P(gWg^{-1}), \forall g \in SU(2)$, $P(W)$ can be written as a function of the eigenvalues $e^{\pm i\theta}$ of W and then

$$P(W; C) = P(\theta; C).$$

Then Eqs. (1) and (2) reduce to

$$\langle \mathrm{tr}_J W(C) \rangle = \frac{1}{\pi} \int_0^{\pi} d\theta\, 2\sin^2\theta P(\theta; C) \sum_{k=0}^{J} e^{2i(k-J)\theta} \tag{3}$$

$$\langle \mathrm{tr}_J W^{\mathrm{Abel}}(C) \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta\, P^{\mathrm{Abel}}(\theta; C) \sum_{k=0}^{J} e^{2i(k-J)\theta}. \tag{4}$$

If $P(\theta; C)$ and $P^{\mathrm{Abel}}(\theta : C)$ behave similarly, the difference between Eq. (3) and Eq. (4) is only the factor $2\sin^2\theta$.

Therefore, we guess from this that if we modify the operator including the difference of the measure as

$$\langle 2\sin^2\theta_C\, \mathrm{tr}_J W^{\mathrm{Abel}}(C) \rangle = \frac{1}{\pi} \int_0^{\pi} d\theta\, 2\sin^2\theta P^{\mathrm{Abel}}(\theta; C) \sum_{k=0}^{J} e^{2i(k-J)\theta}$$

we obtain the correct behavior.

Relation to the highest-weight part

It has been already checked to some extent that the Haar-measure-corrected Abelian Wilson loop behaves correctly. This is because we can relate the Haar-measure-corrected Abelian Wilson loop and the highest-weight part $\widetilde{W}_J^{\mathrm{Abel}}(C)$ of the Abelian Wilson loop, which is defined as

$$\widetilde{W}_J^{\mathrm{Abel}}(C) := \frac{1}{2}(e^{2iJ\theta_C} + e^{-2iJ\theta_C}),$$

and it has been numerically shown that this behaves similarly to the original Wilson loop in the spin-J representation

$$\langle \widetilde{W}_J^{\mathrm{Abel}}(C) \rangle \sim \langle \mathrm{tr}_J\, W(C) \rangle$$

when $J = 1$ in the previous studies. Here the symbol "$\sim$" means that the decay factors of area-law fall off in both sides have similar values.

The operator relation is

$$2\sin^2\theta\, \mathrm{tr}_J\, W^{\mathrm{Abel}} = \widetilde{W}_J^{\mathrm{Abel}} - \widetilde{W}_{J+1}^{\mathrm{Abel}}$$

By taking the average we obtain (A: the minimal area surrounded by C)

$$\begin{aligned}\langle 2\sin^2\theta\, \mathrm{tr}_J\, W_J^{\mathrm{Abel}} \rangle &= \langle \widetilde{W}_J^{\mathrm{Abel}} \rangle - \langle \widetilde{W}_{J+1}^{\mathrm{Abel}} \rangle \\ &\sim C_1 e^{-\sigma_J A} + C_2 e^{-\sigma_{J+1} A} \\ &\sim e^{-\sigma_J A} \qquad (\because \sigma_{J+1} > \sigma_J).\end{aligned}$$

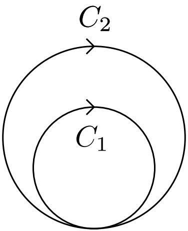
Extension to double-winding Wilson loops

For double-winding contours, the straightforward modification of the operator is as follows.

$$\langle 2\sin^2\theta_1 2\sin^2\theta_2 [W^{\rm Abel}(C_1)]^a{}_b [W^{\rm Abel}(C_2)]^c{}_d\rangle \sim \langle [W(C_1)]^a{}_b [W(C_2)]^c{}_d\rangle \,,$$

where the untraced Abelian Wilson loop on a contour C_i is parametrized as

$$W^{\rm Abel}(C_i) = {\rm diag}(e^{i\theta_i}, e^{-i\theta_i}).$$



Outline

Behavior of the proposed operators on double-winding contours

Let us estimate the average of the Haar-measure-corrected Abelian double-winding Wilson loops. Let the untraced Abelian Wilson loop on a contour C_i be $\mathrm{diag}(e^{i\theta_i}, e^{-i\theta_i})$. Then the operator we consider is

$$2\sin^2\theta_1 2\sin^2\theta_2 \cos(\theta_1+\theta_2).$$

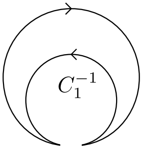
This operator is decomposed as

$$\begin{aligned}
&2\sin^2\theta_1 2\sin^2\theta_2 \cos(\theta_1+\theta_2) \\
&= -\cos(\theta_2-\theta_1) + \frac{5}{4}\cos(\theta_1+\theta_2) + \frac{1}{4}\cos(3(\theta_1+\theta_2)) \\
&+ \frac{1}{4}\cos(3\theta_2-\theta_1) + \frac{1}{4}\cos(\theta_2-3\theta_1) - \frac{1}{2}\cos(3\theta_1+\theta_2) - \frac{1}{2}\cos(\theta_1+3\theta_2).
\end{aligned}$$

Here we expect

$$\langle\cos(\theta_2-\theta_1)\rangle \sim e^{-\sigma_{\mathrm{fund}}(A_2-A_1)},$$

This gives the difference of area law. As shown in the next slide, we expect that the other terms decreases faster.



Assumption

In order to estimate the average of other terms, we assume factorization for non-intersecting coplanar Abelian Wilson loops as

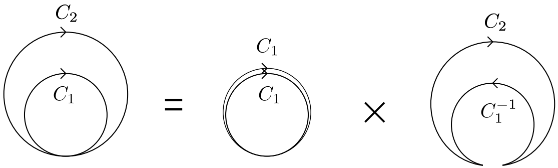
$$\langle [W^{\mathrm{Abel}}(\widetilde{C}_1)]^a{}_b [W^{\mathrm{Abel}}(\widetilde{C}_2)]^c{}_d \rangle \sim \langle [W^{\mathrm{Abel}}(\widetilde{C}_1)]^a{}_b \rangle \, \langle [W^{\mathrm{Abel}}(\widetilde{C}_2)]^c{}_d \rangle \,,$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are non-intersecting coplanar loops and the symbol "$\sim$" means that the decay factors of the area-law fall off on both sides have similar values.

By using this assumption we estimate, for example, the second term as

$$\langle \cos(\theta_1 + \theta_2) \rangle = \langle e^{i\theta_1} e^{i\theta_2} \rangle = \langle e^{2i\theta_1} e^{i(\theta_2 - \theta_1)} \rangle \sim \langle e^{i2\theta_1} \rangle \, \langle e^{i(\theta_2 - \theta_1)} \rangle$$
$$\sim e^{-\sigma_{\mathrm{adj}} A_1 - \sigma_{\mathrm{fund}}(A_2 - A_1)},$$

where the last similarity is because the operator $e^{i2\theta_1}$ is actually the highest weight part in the adjoint representation.



Similarly we estimate the other terms as

$$\langle \cos(3(\theta_1+\theta_2)) \rangle \sim e^{-\sigma_{[J=3]}A_1 - \sigma_{[J=3/2]}(A_2-A_1)}$$
$$\langle \cos((3\theta_2-\theta_1)) \rangle \sim e^{-\sigma_{\rm adj}A_1 - \sigma_{[J=3/2]}(A_2-A_1)}$$
$$\langle \cos((\theta_2-3\theta_1)) \rangle \sim e^{-\sigma_{\rm adj}A_1 - \sigma_{\rm fund}(A_2-A_1)}$$
$$\langle \cos(3\theta_1+\theta_2) \rangle \sim e^{-\sigma_{[J=2]}A_1 - \sigma_{\rm fund}(A_2-A_1)}$$
$$\langle \cos(\theta_1+3\theta_2) \rangle \sim e^{-\sigma_{[J=2]}A_1 - \sigma_{[J=3/2]}(A_2-A_1)}$$

Thus the terms other than $-\langle \cos(\theta_1-\theta_2) \rangle$ decrease exponentially with A_1 with A_2-A_1 held constant, and therefore for sufficiently large A_1 we obtain the difference of area law,

$$\begin{aligned}
&\langle 2\sin^2\theta_1 2\sin^2\theta_2 \cos(\theta_1+\theta_2) \rangle \\
&= -\langle \cos(\theta_2-\theta_1) \rangle + \frac{5}{4}\langle \cos(\theta_1+\theta_2) \rangle + \frac{1}{4}\langle \cos(3(\theta_1+\theta_2)) \rangle \\
&+ \frac{1}{4}\langle \cos(3\theta_2-\theta_1) \rangle + \frac{1}{4}\langle \cos(\theta_2-3\theta_1) \rangle - \frac{1}{2}\langle \cos(3\theta_1+\theta_2) \rangle - \frac{1}{2}\langle \cos(\theta_1+3\theta_2) \rangle \\
&\sim e^{\sigma_{\rm fund}(A_2-A_1)}.
\end{aligned}$$

Summary

- We propose a way to extract "Abelian" part of loop operators which could apply to Wilson loops in higher representations and double-winding Wilson loops.
- Because of the operator relation between the proposed operators and the highest-weight part of Abelian Wilson loops, the proposed operators reproduce the correct area-law behavior of Wilson loops in higher representation.
- Under an assumption, we estimate the average of the proposed operators defined on double winding contours, which gives the difference-of-area law.