$Sp(2N)$ Yang-Mills towards large-N

Ed Bennett[a], **Jack Holligan**[a], Deog Ki Hong[b], Jong-Wan Lee[b], David Lin[c], Biagio Lucini[a], Maurizio Piai[a], Davide Vadacchino[d]

[a] *Swansea University, UK*

[b] *Pusan National University, South Korea*

[c] *Institute of Physics, National Chiao-Tung University, Taiwan*

[d] *INFN, Pisa, Italy*

37th Lattice Symposium
21st June 2019

Outline

Symplectic Groups

$Sp(2N) = \{M \in SU(2N) : \Omega^\dagger M \Omega = M^*\}$ where

$$\Omega = \left[\begin{array}{cc} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{array} \right] \tag{1}$$

and $\mathbb{1}_N$ is the $N \times N$ identity matrix.

Symplectic Groups

$Sp(2N) = \{M \in SU(2N) : \Omega^\dagger M \Omega = M^*\}$ where

$$\Omega = \begin{bmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{bmatrix} \quad (1)$$

and $\mathbb{1}_N$ is the $N \times N$ identity matrix.

In this case, M can be written in block form:

$$M = \begin{bmatrix} A & B \\ -B^* & A^* \end{bmatrix} \quad (2)$$

s.t. $A^\dagger A + B^\dagger B = \mathbb{1}_N$ and $A^T B = B^T A$.

N.B. in this notation, the number of colours $N_c = 2N$.

Why Symplectic Groups?

- Pure gauge theories based on the groups $SU(N)$, $SO(N)$ or $Sp(2N)$ produce the same physics in the large-N_c limit. As yet, $Sp(2N)$ are the only groups left unstudied.

Why Symplectic Groups?

- Pure gauge theories based on the groups $SU(N)$, $SO(N)$ or $Sp(2N)$ produce the same physics in the large-N_c limit. As yet, $Sp(2N)$ are the only groups left unstudied.
- In addition, the symplectic groups with two fundamental fermions (*see Jong-Wan Lee's talk, Friday 14:20*) are a potential building block in composite Higgs models[1]. Such a theory could provide a solution to the naturalness problem.

Throughout this talk, we will refer to fermions and gauge bosons as quarks and gluons respectively for concreteness. In addition, we work on a hypercubic, toroidal lattice with lattice-spacing a, total length L and $N_{s/t}$ the number of lattice sites in the spatial/temporal direction respectively ($L = N_s a = N_t a$).

---

[1] *$Sp(4)$ gauge theory on the lattice: towards $SU(4)/Sp(4)$ composite Higgs and beyond*, Bennett et. al., 2017

Lattice Regularisation

We regularise the $Sp(2N)$ Yang-Mills theory on a 4-dimensional Euclidean lattice. If the theory has bare coupling g and a lattice link from site x in the $\hat{\mu}$ direction is denoted by $U_\mu(x)$ then the Wilson action is

$$S = \frac{\beta}{2N} \sum_x \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{tr}\left[\mathbb{1} - U_{\mu\nu}(x)\right] \tag{3}$$

where

$$\beta = \frac{4N}{g^2} \tag{4}$$

and $U_{\mu\nu}(x)$ is the plaquette:

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x). \tag{5}$$

Observables

The particle states in pure Yang-Mills are bound states of gluons known as “glueballs”. The masses of same can be determined by the variational method.

Observables

The particle states in pure Yang-Mills are bound states of gluons known as "glueballs". The masses of same can be determined by the variational method.

In addition to glueballs, pure Yang-Mills theories contain confining flux tubes. If two infinitely massive static quarks are separated by a large distance r, the interaction potential between them is given by

$$V(r) = \sigma r \tag{6}$$

with σ being the string tension. This is another observable we can measure with lattice computations.

Both the glueball spectrum and the string tension require a non-perturbative treatment of pure $Sp(2N)$ Yang-Mills.

Variational Method

When we move to the Euclidean lattice, the continuous rotational symmetry of the former is broken to the discrete cubic symmetry of the latter. Thus the operators we use to construct glueball states on the lattice, $\Phi(t)$, must lie in irreducible representations (irreps) of the cubic group.

Variational Method

When we move to the Euclidean lattice, the continuous rotational symmetry of the former is broken to the discrete cubic symmetry of the latter. Thus the operators we use to construct glueball states on the lattice, $\Phi(t)$, must lie in irreducible representations (irreps) of the cubic group.

These operators are gauge invariant, path ordered, products of link variables. A single operator at spatial coordinate, $\underline{n}$, is given by

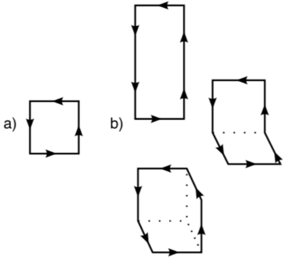
$$\phi_{\mathcal{C}}(\underline{n}, t) \quad = \quad \mathrm{tr}\left(\hat{\mathcal{P}} \prod_{\mathcal{C}} U_l\right) \tag{7}$$

with $\mathcal{C}$ begin a closed path on the lattice at a fixed time slice. The path is chosen such that the operator transforms in a specific irrep of the cubic group. Its average over the total number of spatial sites, N_s^3, is

$$\phi_{\mathcal{C}}(t) \quad = \quad \frac{1}{N_s^3} \sum_{\underline{n}} \phi_{\mathcal{C}}(\underline{n}, t). \tag{8}$$

Variational Method

Examples of such operators used in the determination of the mass spectrum are



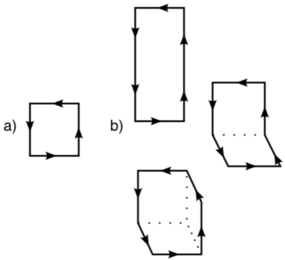
Each of the paths shown on the left lies on a single time-slice.

$R^P = A1^{\pm}$, $A2^{\pm}$, $E^{\pm}$, $T1^{\pm}$ and $T2^{\pm}$.

The pseudo-reality of the $Sp(2N)$ groups guarantees that charge conjugation is positive.

Variational Method

Examples of such operators used in the determination of the mass spectrum are



Each of the paths shown on the left lies on a single time-slice.

$R^P = A1^{\pm}$, $A2^{\pm}$, $E^{\pm}$, $T1^{\pm}$ and $T2^{\pm}$.

The pseudo-reality of the $Sp(2N)$ groups guarantees that charge conjugation is positive.

Taking the correlator of two of these operators and normalising them to its value at $t = 0$ gives:

$$C_{\mathcal{C}}(t) = \frac{\langle \phi_{\mathcal{C}}(0)\phi_{\mathcal{C}}(t)\rangle}{\langle \phi_{\mathcal{C}}(0)\phi_{\mathcal{C}}(0)\rangle} = \frac{\sum_n |\langle n|\phi(0)|0\rangle|^2 e^{-m_n t}}{\sum_n |\langle n|\phi(0)|0\rangle|^2} \approx |c_1|^2 e^{-m_1 t} \qquad (9)$$

where $m_1 = \min\{m_n\}$.

Variational Method

$$C_{\mathcal{C}}(t) \approx |c_1|^2 e^{-m_1 t} \tag{10}$$

Problems:

- The signal decays exponentially while background statistical noise remains constant.

Variational Method

$$C_{\mathcal{C}}(t) \approx |c_1|^2 e^{-m_1 t} \tag{10}$$

Problems:

- The signal decays exponentially while background statistical noise remains constant.
- As we approach the continuum limit, the operators decrease in size while the glueball maintains a physical volume. Thus, the operators have less and less resemblance to the glueballs we are trying to study.

Variational Method

$$C_{\mathcal{C}}(t) \approx |c_1|^2 e^{-m_1 t} \tag{10}$$

Problems:

- The signal decays exponentially while background statistical noise remains constant.
- As we approach the continuum limit, the operators decrease in size while the glueball maintains a physical volume. Thus, the operators have less and less resemblance to the glueballs we are trying to study.
- In the continuum limit, the operators shrink and are dominated by UV fluctuations causing them to overlap evenly on all states.

Variational Method

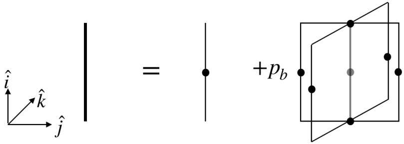
$$C_{\mathcal{C}}(t) \approx |c_1|^2 e^{-m_1 t} \tag{10}$$

Problems:

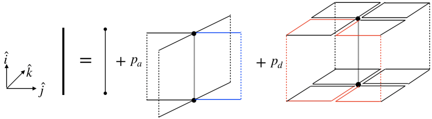
- The signal decays exponentially while background statistical noise remains constant.
- As we approach the continuum limit, the operators decrease in size while the glueball maintains a physical volume. Thus, the operators have less and less resemblance to the glueballs we are trying to study.
- In the continuum limit, the operators shrink and are dominated by UV fluctuations causing them to overlap evenly on all states.
- Can we make sure that $|c_1|$ is large enough to detect the signal?

Variational Method

We can solve these three issues with the use of blocking[2] and improved smearing[3].



Blocking to extend the physical size of the operators.



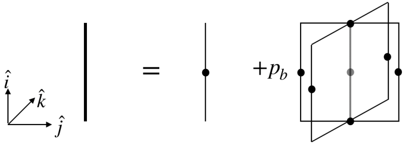
Smearing to average out UV fluctuations.

[2] *An improved method for lattice glueball calculations*, M. Teper, 1987.

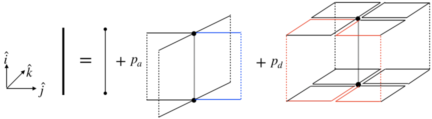
[3] *Glueballs and k-strings*, Lucini, Teper & Wegner, 2004.

Variational Method

We can solve these three issues with the use of blocking[2] and improved smearing[3].



Blocking to extend the physical size of the operators.



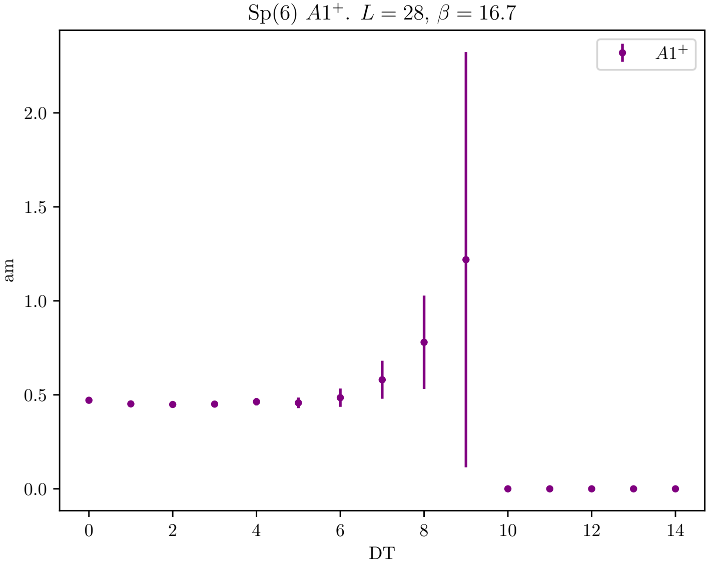
Smearing to average out UV fluctuations.

We look for a plateau in the measurements of the mass at different t values. This means that the signal is still strong enough to rise above background noise and is receiving minimal contributions from higher mass states.

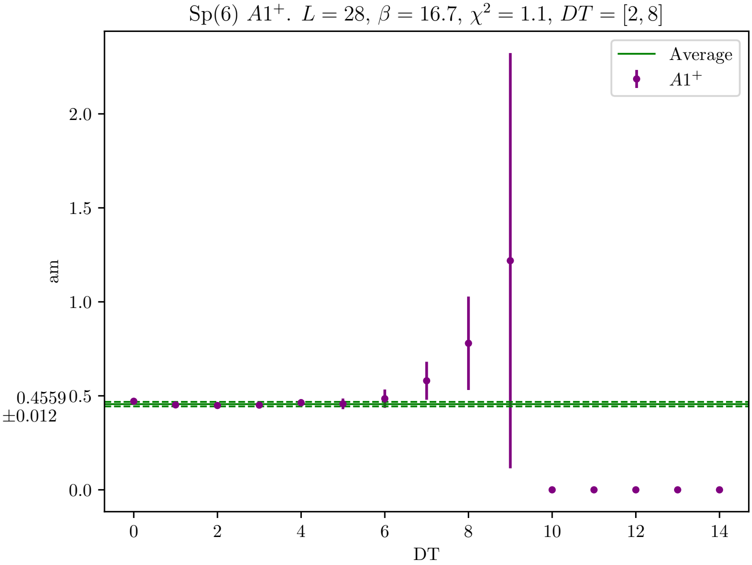
---

[2] *An improved method for lattice glueball calculations*, M. Teper, 1987.
[3] *Glueballs and k-strings*, Lucini, Teper & Wegner, 2004.

Plateau example



Plateau example



String Tension

By the same method, we can measure the mass of the torelon (a stringy flux tube) and, thus, determine the string tension, σ.

String Tension

By the same method, we can measure the mass of the torelon (a stringy flux tube) and, thus, determine the string tension, σ.

We are familiar with the definition of tension as mass per unit length. However, we must account for finite size effects. The string tension is related to the torelon masses (at NLO) via:

$$m(L) = \sigma L - \frac{\pi}{3L} - \frac{\pi^2}{18L^3}\frac{1}{\sigma}. \tag{11}$$

A single Polyakov Loop at a fixed time slice that extends across the whole toroidal lattice can be treated as a single closed string of circumference L.

Inputs

For $N_c = 2, 4, 6$ and 8:

- 10,000 thermalisation sweeps to equilibrate the lattice.
- A single update consists of 4 over-relaxations and 1 heat-bath on each lattice link of the entire lattice.
- Save the lattice configuration every tenth update to allow for decorrelation until we have a total of 20,000 configurations.
- Repeat until we have done this for at least four values of (L, β) for each N_c.
- With each Markov Chain, perform the Variational Method to determine the mass spectrum of lattice glueballs.
- Extrapolate each glueball channel to the continuum limit.

Finally, extrapolate each channel to large-N_c.

Glueball Results

When extrapolating to the continuum limit, we use the formula

$$\frac{m(a)}{\sqrt{\sigma}} = \frac{m(0)}{\sqrt{\sigma}} + c_1 a^2 \sigma \tag{12}$$

where a is the lattice spacing.

Glueball Results

When extrapolating to the continuum limit, we use the formula

$$\frac{m(a)}{\sqrt{\sigma}} = \frac{m(0)}{\sqrt{\sigma}} + c_1 a^2 \sigma \tag{12}$$

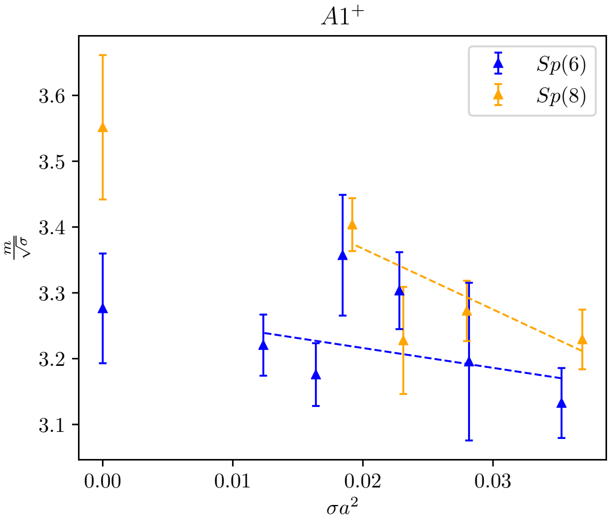
where a is the lattice spacing.

Similarly, when extrapolating to large-N_c from the continuum results, we use the formula

$$\frac{m(N_c)}{\sqrt{\sigma}} = \frac{m(\infty)}{\sqrt{\sigma}} + \frac{c_2}{N_c}. \tag{13}$$

N.B. that equation (13) is not the same as the one used in $SU(N_c)$ extrapolations.

Glueball Results (Sample, Preliminary)



Glueball Results (Preliminary)

$m(R^P)/\sqrt{\sigma}$					
	$Sp(2)$	$Sp(4)$[4]	$Sp(6)$	$Sp(8)$	$Sp(\infty)$
$A1^+$	3.767(27)	3.557(52)	3.276(83)	3.55(11)	3.28(13)
$A1^{+*}$	5.730(76)	6.05(4)	5.98(19)	6.05(16)	6.310(84)
$A1^-$	6.057(55)	5.74(19)	5.299(46)	5.44(12)	4.98(10)
$A1^{-*}$	7.769(62)	7.81(8)	7.54(12)	7.601(92)	7.60(12)
$A2^+$	7.30(15)	7.91(17)	7.57(12)	7.05(31)	7.71(35)
$A2^-$	8.45(20)	9.42(40)	9.01(38)	9.05(47)	9.49(34)
E^+	5.344(70)	5.02(16)	4.970(89)	4.91(16)	4.770(20)
E^-	7.199(59)	6.61(13)	6.431(91)	6.290(32)	5.992(14)
$T1^+$	5.534(94)	5.070(91)	4.989(46)	4.836(18)	4.607(41)
$T1^-$	7.205(49)	6.872(89)	6.70(15)	6.42(17)	6.40(12)
$T2^+$	8.69(18)	8.73(30)	7.946(60)	8.12(35)	7.59(19)
$T2^-$	9.51(20)	9.50(37)	9.24(21)	8.74(35)	8.96(24)

Table 1: Continuum and $N_c \to \infty$ Glueball Masses.

[4] *$Sp(4)$ gauge theory on the lattice: towards $SU(4)/Sp(4)$ composite Higgs and beyond*, Bennett et. al., 2017

$Sp(2N)$ vs $SU(N)$ (Preliminary)

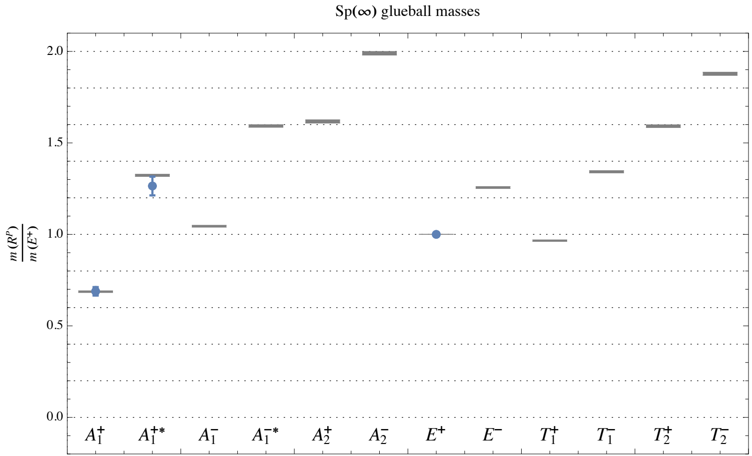


Figure 1: Glueball masses as a fraction of $m(E^+)$. Blue dots: $SU(\infty)$. Grey bars: $Sp(\infty)$.

Casimir Scaling

With these new results, we can test the conjecture[5]

$$\eta(0^{++}) \equiv \frac{m_{0^{++}}^2}{\sigma} \cdot \frac{C_2(F)}{C_2(A)} = 5.41(10) \text{ (in 3+1 dimensions)} \tag{14}$$

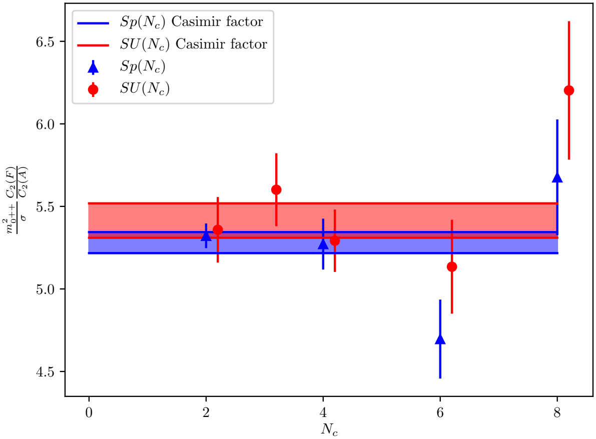
where η is a universal constant. $C_2(F)$ and $C_2(A)$ are the fundamental and adjoint Casimir operators respectively:

$$\frac{C_2(F)}{C_2(A)} = \begin{cases} \frac{N_c^2-1}{2N_c^2} & SU(N_c) \\ \frac{N_c-1}{2(N_c-2)} & SO(N_c) \\ \frac{N_c+1}{2N_c+4} & Sp(N_c) \end{cases} \tag{15}$$

In all cases, this fraction approaches $\frac{1}{2}$ in the large-N_c limit.

[5] *Casimir scaling and Yang-Mills glueballs*, Deog Ki Hong, et. al., 2017

Casimir Scaling for 0^{++} (Preliminary)



$SU(N_c)$: $\eta = 5.41(10)$
$Sp(N_c)$: $\eta = 5.281(64)$

Conclusion and Next Steps

- We have examined the glueball spectrum for $Sp(2N)$ groups extrapolating to both the continuum and large-N_c.
- Large-N_c limits in both $SU(N_c)$ and $Sp(N_c)$ show good agreement at this stage.
- We are also exploring the second excited states of $A1^+$ and $A1^-$ as well as the first excited states of E^+ and $T1^+$. (Preliminary plots contained in appendix.)
- The $Sp(N_c)$ data provides further evidence for the conjecture of Casimir Scaling.
- We plan to examine the phase transition between confinement and deconfinement in the Symplectic groups.

More Glueball Results (Preliminary)

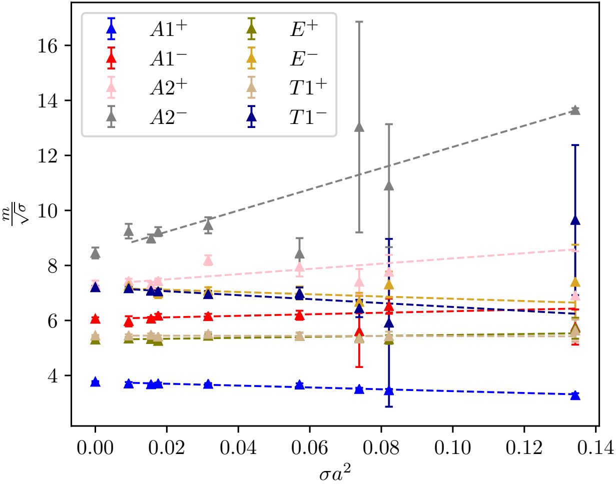


Figure 2: $Sp(2)$ continuum extrapolations.

More Glueball Results (Preliminary)

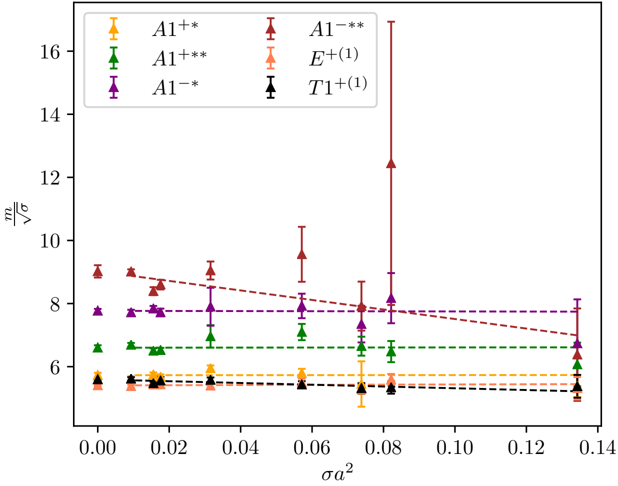


Figure 3: $Sp(2)$ continuum extrapolations.

More Glueball Results (Preliminary)

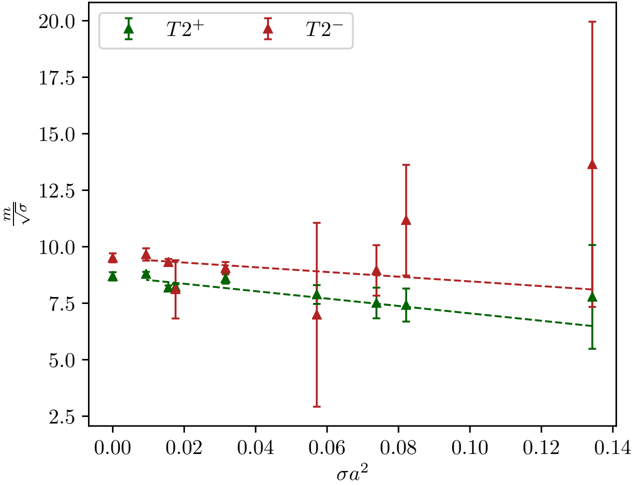


Figure 4: $Sp(2)$ continuum extrapolations.

More Glueball Results (Preliminary)

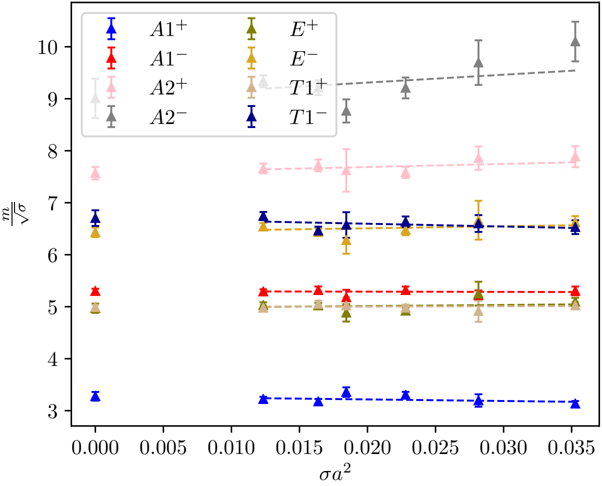


Figure 5: $Sp(6)$ continuum extrapolations.

More Glueball Results (Preliminary)

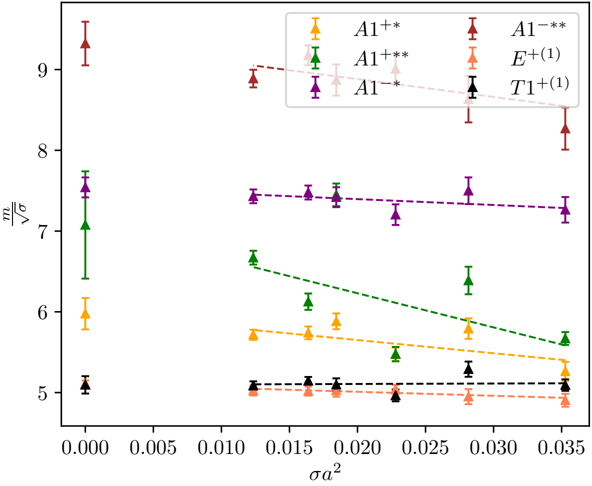


Figure 6: $Sp(6)$ continuum extrapolations.

More Glueball Results (Preliminary)

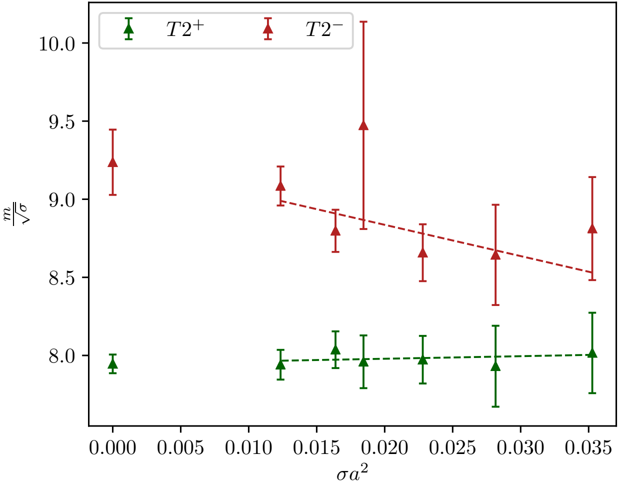


Figure 7: $Sp(6)$ continuum extrapolations.

More Glueball Results (Preliminary)

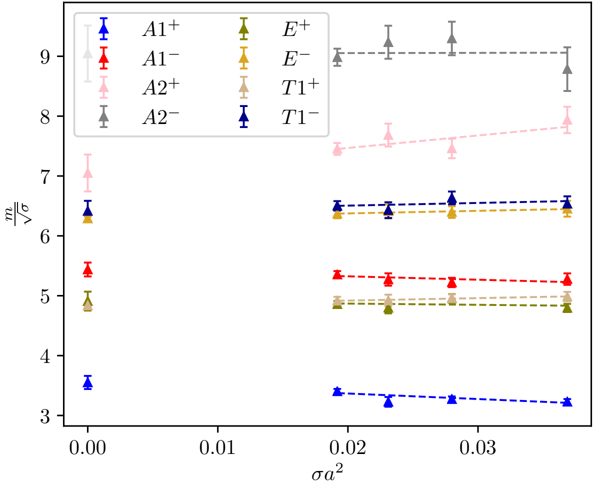


Figure 8: $Sp(8)$ continuum extrapolations.

More Glueball Results (Preliminary)

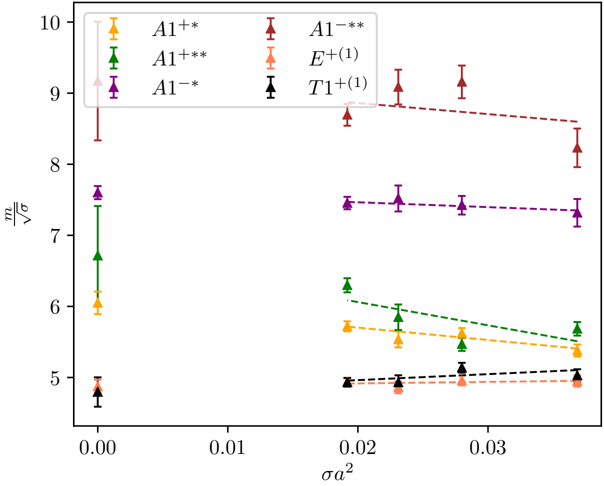


Figure 9: $Sp(8)$ continuum extrapolations.

More Glueball Results (Preliminary)

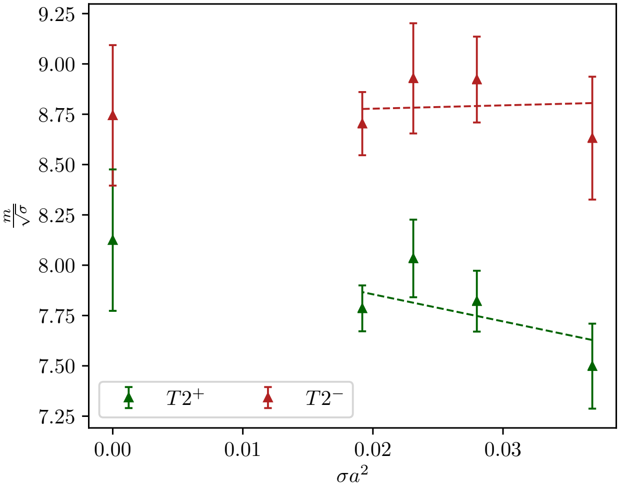


Figure 10: $Sp(8)$ continuum extrapolations.

More Glueball Results (Preliminary)

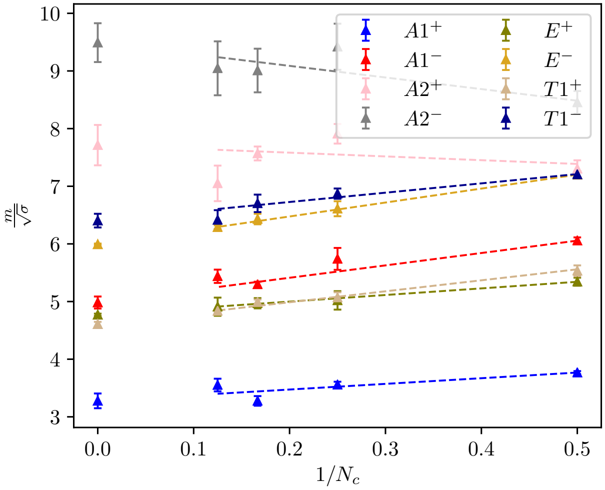


Figure 11: large-N_c extrapolations.

More Glueball Results (Preliminary)

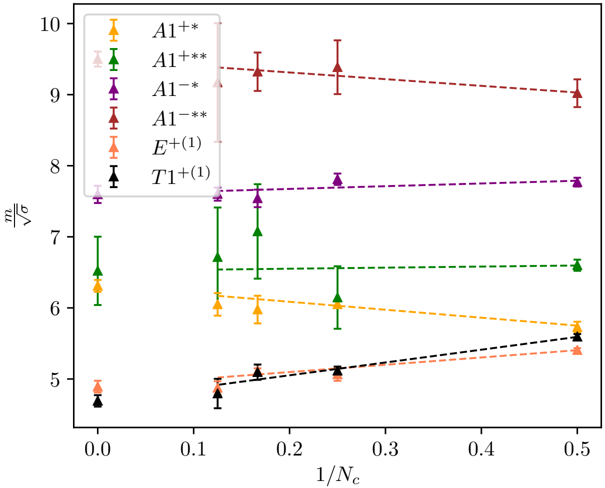


Figure 12: large-N_c extrapolations.

More Glueball Results (Preliminary)

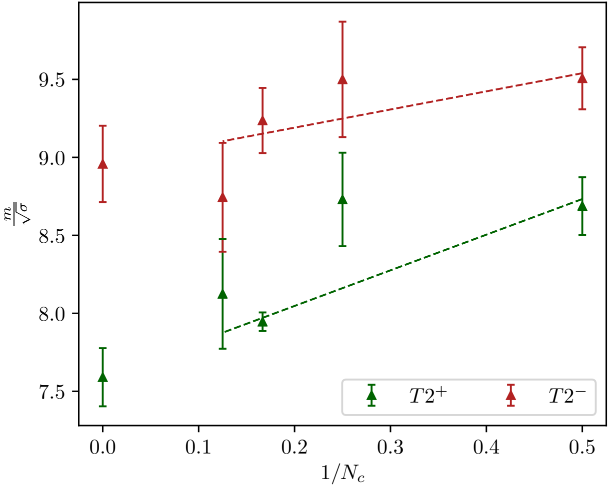


Figure 13: large-N_c extrapolations.