$\mathcal{N} = 1$ Supersymmetric SU(3) Gauge Theory with a Twist

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Standard model of particle physics

- describes successfully what we observe at $\lesssim \mathcal{O}(10\,\text{TeV})$

Open questions

- Higgs mass
- dark matter
- unification of forces
- ...

$\Rightarrow$ Not complete
$\Rightarrow$ More fundamental theory?

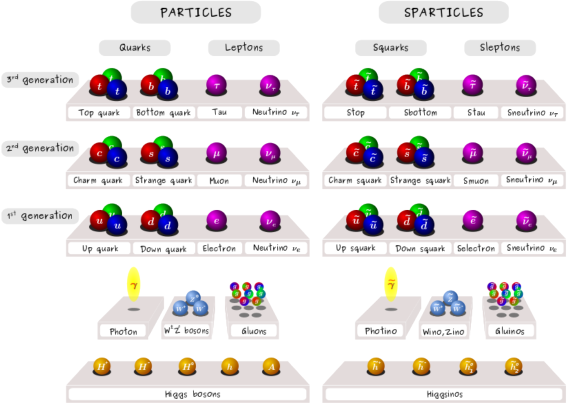
Possible solution

Introduce supersymmetry

Table of content

Minimal Supersymmetric Standard Model

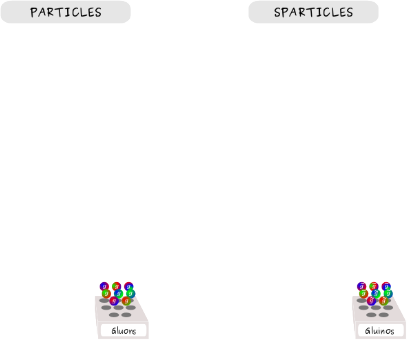
- contains a super partner for every particle



[4] https://web2.ph.utexas.edu/~coker2/index.files/supersymmetry.htm

$\mathcal{N}=1$ Super-Yang-Mills Theory

- contains a super partner for every particle



$\Rightarrow$ important building block of Super-QCD

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$\mathcal{N} = 1$ Super-Yang-Mills Theory

Fields

- Gauge boson (gluon) $A_\mu(x)$ in the adjoint representation
- Super partner (gluino) $\lambda(x)$ is Majorana fermion in the adjoint representation

On-shell Lagrange density

$$\mathcal{L}_{\mathrm{SYM}} = \mathrm{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{i}}{2}\bar{\lambda}\not{D}\lambda \qquad \right)$$

Supersymmetry: Relation between fermionic matter particles and bosonic force particles

$$\delta_\epsilon A_\mu = \mathrm{i}\bar{\epsilon}\gamma_\mu\lambda, \quad \delta_\epsilon\lambda = \mathrm{i}\Sigma_{\mu\nu}F^{\mu\nu}\epsilon$$

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On-shell Lagrange density

$$\mathcal{L}_{\text{SYM}} = \text{tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\text{i}}{2}\bar{\lambda}\not{D}\lambda - \frac{m_{\text{g}}}{2}\bar{\lambda}\lambda\right)$$

Supersymmetry: Relation between fermionic matter particles and bosonic force particles

$$\delta_\epsilon A_\mu = \text{i}\bar{\epsilon}\gamma_\mu\lambda, \quad \delta_\epsilon\lambda = \text{i}\Sigma_{\mu\nu}F^{\mu\nu}\epsilon$$

Softly broken by gluino mass term

Symmetries

Chiral (R) symmetry breaking in $SU(3)$ SYM theory

- Global chiral $U(1)_\mathrm{A}$ symmetry: $\lambda \mapsto \mathrm{e}^{\mathrm{i}\alpha\gamma_5}\lambda$
- Due to anomaly only $\mathbb{Z}_6$ remnant symmetry

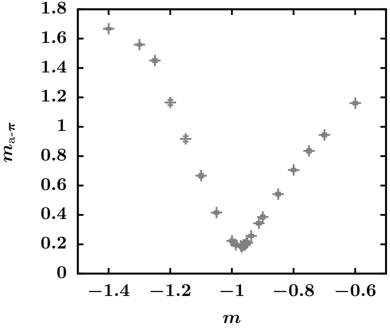
$$\lambda \mapsto \mathrm{e}^{\mathrm{i}\frac{2\pi n}{6}\gamma_5}\lambda \quad \text{with} \quad n \in \{1, \ldots, 6\}$$

- Spontaneously broken to $\mathbb{Z}_2$ symmetry in consequence of gluino condensate $\langle\bar{\lambda}\lambda\rangle \neq 0 \rightarrow 3$ different vacua

The Chiral Limit

$$D_W \quad (x,y) = (4 + m \qquad\qquad)\delta_{x,y} - \frac{1}{2}\sum_{\mu=\pm1}^{\pm4} (\mathbb{1} - \gamma_\mu)\mathcal{V}_\mu(x)\,\delta_{x+\hat{\mu},y}$$

with adjoint representation $[\mathcal{V}_\mu(x)]_{ab} = 2\,\mathrm{tr}\left[\mathcal{U}_\mu^\dagger(x)\,T_a\,\mathcal{U}_\mu(x)\,T_b\right]$



Dirac Operator

$$D_W(x,y) = (4 + m)\delta_{x,y} - \frac{1}{2}\sum_{\mu=\pm 1}^{\pm 4}(\mathbb{1} - \gamma_\mu)\mathcal{V}_\mu(x)\,\delta_{x+\hat{\mu},y}$$

Feature of SYM

- Particular directions of $\mathbb{Z}_6$ symmetry are favored by gluino condensate

Dirac Operator with Twisted Mass

$$D_{\mathrm{W}}^{\mathrm{mtw}}(x,y) = (4 + m + \mathrm{i}\mu\gamma_5)\delta_{x,y} - \frac{1}{2}\sum_{\mu=\pm 1}^{\pm 4}(\mathbb{1} - \gamma_\mu)\mathcal{V}_\mu(x)\,\delta_{x+\hat{\mu},y}$$

Feature of SYM

- Particular directions of $\mathbb{Z}_6$ symmetry are favored by gluino condensate
- Deform lattice action by adding parity-breaking mass μ resembling a twisted mass
- m breaks chiral symmetry explicitly and generates a condensate $\sim \langle\bar{\lambda}\lambda\rangle$
- μ leads to a condensate $\sim \langle\bar{\lambda}\gamma_5\lambda\rangle$

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$\Rightarrow$ Possibility to get closer to chiral symmetry and supersymmetry at finite lattice spacing?

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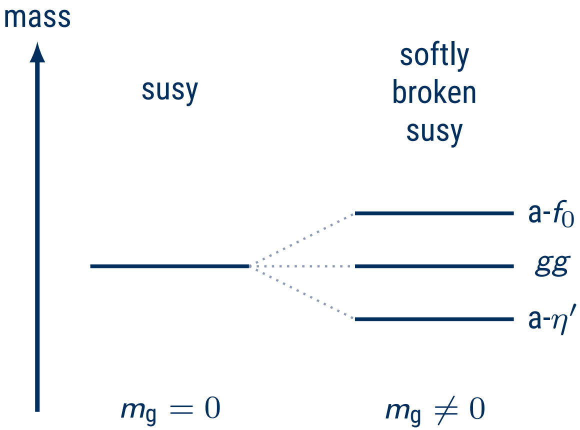
$$= (4 + M\,\mathrm{e}^{\mathrm{i}\alpha\gamma_5})\delta_{x,y} - \frac{1}{2}\sum_{\mu=\pm1}^{\pm4}(\mathbb{1} - \gamma_\mu)\mathcal{V}_\mu(x)\,\delta_{x+\hat{\mu},y}$$

Feature of SYM

- Particular directions of $\mathbb{Z}_6$ symmetry are favored by gluino condensate
- Deform lattice action by adding parity-breaking mass μ resembling a twisted mass
- $m = M\cos(\alpha)$
- $\mu = M\sin(\alpha)$

$\Rightarrow$ Possibility to get closer to chiral symmetry and supersymmetry at finite lattice spacing?

Veneziano-Yankielowicz Multiplet



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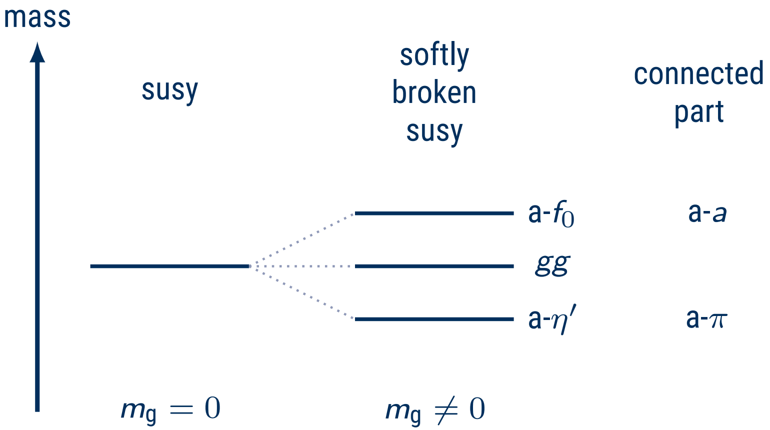

Dirac Operator with Twisted Mass

Numerical Investigations

Spectroscopy

- for different angles $\alpha = \mathtt{arctan}(\mu/m)$
- of connected (unphysical) mesons with good signal-to-noise-ration
- of full (physical) mesons including disconnected contributions
- of gluino-glue

Dirac Operator with Twisted Mass

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Analytical Investigations

Calculation of

- fermionic expectation values of lattice interpolators
- eigenvalues of the free Dirac operator

Fermionic Expectation Values

$$\langle O \rangle = \langle \langle O \rangle_{\mathsf{F}} \rangle_{\mathsf{G}} = \frac{1}{Z} \int \mathcal{D}\mathcal{U}\, \mathrm{e}^{-S_{\mathsf{G}}[\mathcal{U}]}\, \mathcal{D}\lambda\, \mathrm{e}^{-S_{\mathsf{F}}[\lambda,\mathcal{U}]}\, O[\lambda, \mathcal{U}]$$

Twist

$$\begin{aligned}
\lambda &\mapsto \mathrm{e}^{\mathrm{i}\alpha\gamma_5}\lambda \\
\bar{\lambda} &\mapsto \bar{\lambda}\mathrm{e}^{\mathrm{i}\alpha\gamma_5} \\
\bar{\lambda}\lambda &\mapsto \bar{\lambda}\mathrm{e}^{2\mathrm{i}\alpha\gamma_5}\lambda = \bar{\lambda}\lambda\cos(2\alpha) + \mathrm{i}\bar{\lambda}\gamma_5\lambda\sin(2\alpha) \\
\bar{\lambda}\gamma_5\lambda &\mapsto \bar{\lambda}\gamma_5\mathrm{e}^{2\mathrm{i}\alpha\gamma_5}\lambda = \bar{\lambda}\gamma_5\lambda\cos(2\alpha) + \mathrm{i}\bar{\lambda}\lambda\sin(2\alpha)
\end{aligned}$$

Fermionic Expectation Values

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Twist

$$\begin{aligned} \lambda &\mapsto \mathrm{e}^{\mathrm{i}\alpha\gamma_5}\lambda \\ \bar{\lambda} &\mapsto \bar{\lambda}\mathrm{e}^{\mathrm{i}\alpha\gamma_5} \end{aligned}$$

$$\begin{pmatrix} \bar{\lambda}\lambda \\ \mathrm{i}\bar{\lambda}\gamma_5\lambda \end{pmatrix} \mapsto \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ -\sin(2\alpha) & \cos(2\alpha) \end{pmatrix} \begin{pmatrix} \bar{\lambda}\lambda \\ \mathrm{i}\bar{\lambda}\gamma_5\lambda \end{pmatrix}$$

Fermionic Expectation Values

General Observables

$$O_{\text{gen}}(x) = a\,\bar{\lambda}_x \lambda_x + b\,\bar{\lambda}_x \gamma_5 \lambda_x$$

Mesonic Observables

Results

Fermionic Expectation Values

General Observables

Mesonic Observables

	a	b	$\mapsto$	a	b
$O_{\text{a-}f_0}(x) = \bar{\lambda}_x \lambda_x$	1	0		$\cos(2\alpha)$	$\mathrm{i}\sin(2\alpha)$
$O_{\text{a-}\eta'}(x) = \bar{\lambda}_x \gamma_5 \lambda_x$	0	1		$\mathrm{i}\sin(2\alpha)$	$\cos(2\alpha)$

Results

Fermionic Expectation Values

General Observables

Mesonic Observables

Results

- $\langle O_{\text{a-}f_0}(n)\bar{O}_{\text{a-}f_0}(m)\rangle_{\text{F}}(\ 0^\circ\ ) = \langle O_{\text{a-}\eta'}(n)\bar{O}_{\text{a-}\eta'}(m)\rangle_{\text{F}}(90^\circ)$
  $\langle O_{\text{a-}\eta'}(n)\bar{O}_{\text{a-}\eta'}(m)\rangle_{\text{F}}(\ 0^\circ\ ) = \langle O_{\text{a-}f_0}(n)\bar{O}_{\text{a-}f_0}(m)\rangle_{\text{F}}(90^\circ)$
- $\langle O_{\text{a-}\eta'}(n)\bar{O}_{\text{a-}\eta'}(m)\rangle_{\text{F}}(45^\circ) = \langle O_{\text{a-}f_0}(n)\bar{O}_{\text{a-}f_0}(m)\rangle_{\text{F}}(45^\circ)$

Eigenvalues

Goals

- Effect of twist angle α
- $\mathcal{O}(a)$ improvement

Dirac Operator with Mass Twist

$$D_W^{tw}(x,y) = (4 \qquad + M\,\mathrm{e}^{\mathrm{i}\alpha\gamma_5})\delta_{x,y} - \frac{1}{2}\sum_{\mu=\pm1}^{\pm4}\left(\mathbb{1} \qquad - \gamma_\mu\right)\mathcal{V}_\mu(x)\,\delta_{x+\hat{\mu},y}$$

Eigenvalues

Goals

- Effect of twist angle α
- $\mathcal{O}(a)$ improvement $\Rightarrow$ angle φ

Dirac Operator with Double Twist

$$D_{\mathrm{W}}^{\mathrm{dtw}}(x,y) = (4\,\mathrm{e}^{\mathrm{i}\varphi\gamma_5} + M\,\mathrm{e}^{\mathrm{i}\alpha\gamma_5})\delta_{x,y} - \frac{1}{2}\sum_{\mu=\pm 1}^{\pm 4}\left(\mathrm{e}^{\mathrm{i}\varphi\gamma_5} - \gamma_\mu\right)\mathcal{V}_\mu(x)\,\delta_{x+\hat{\mu},y}$$

- Idea based on G. Bergner, T. Kästner, S. Uhlmann, A. Wipf, Annals Phys. 323 (2009)
- General action found in G. Immirzi, K. Yoshida, Nucl. Phys. B210 (1982)

Eigenvalues of Free Dirac Operator

Fermionic Kernel D	Eigenvalues $\lambda^\dagger\lambda$
$\gamma^\mu\partial_\mu + m - \frac{r}{2}\Delta$	$p^2 + m^2 + amrp^2 + \mathcal{O}(a^2)$

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$\gamma^\mu \partial_\mu + M e^{i\alpha\gamma_5} - \frac{r}{2}\Delta$	$p^2 + M^2 + \cancel{aMrp^2 \cos(\alpha)} + \mathcal{O}(a^2)$ (→ 0 for $\alpha = 90°$)

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$\gamma^\mu \partial_\mu + m + \frac{ir}{2}\gamma_5\Delta$	$p^2 + m^2 + \kappa a^2 + \mathcal{O}(a^4)$

- with $\kappa = -\frac{1}{3}\sum_\mu p_\mu^4 + \frac{r^2}{2}\left(\sum_\mu p_\mu^2\right)^2$

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$\gamma^\mu\partial_\mu + M e^{i\alpha\gamma_5} + \frac{r}{2}e^{i\varphi\gamma_5}\Delta$	$p^2 + M^2 + aMrp^2\cos(\alpha-\varphi) + \kappa a^2 + \mathcal{O}(a^3)$ (→ 0 for $\alpha-\varphi=90°$; a^3 → a^4)

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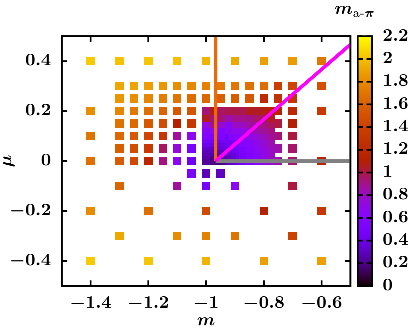
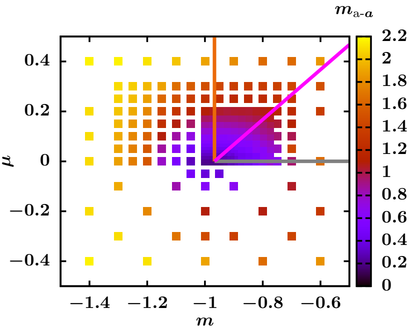
- with $\kappa = -\frac{1}{3}\sum_\mu p_\mu^4 + \frac{r^2}{2}\left(\sum_\mu p_\mu^2\right)^2$
- Mass & Wilson term orthogonal $\Rightarrow \mathcal{O}(a)$ improvement

Connected Correlators

$$D_{\mathrm{W}}^{\mathrm{tw}}(x,y) = (4 + m + \mathrm{i}\mu\gamma_5)\delta_{x,y} - \frac{1}{2}\sum_{\mu=\pm1}^{\pm4}(\mathbb{1} - \gamma_\mu)\mathcal{V}_\mu(x)\,\delta_{x+\mu,y}$$





$8^3 \times 16$

Connected Correlators

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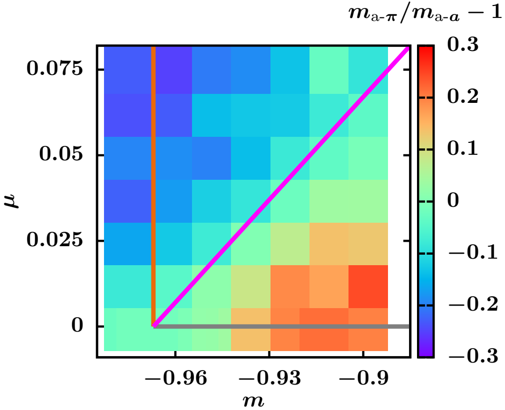

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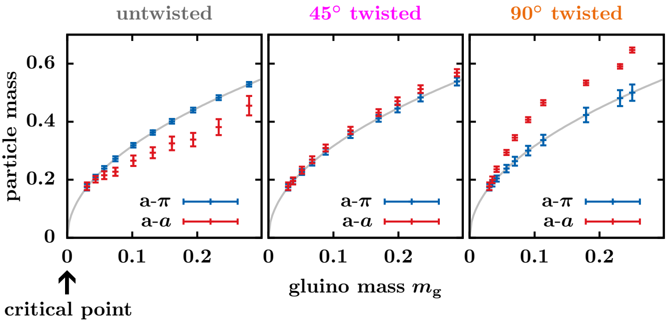

$\Rightarrow$ improvement of the chiral symmetry & supersymmetry at finite lattice spacing may be possible

Mesonic States

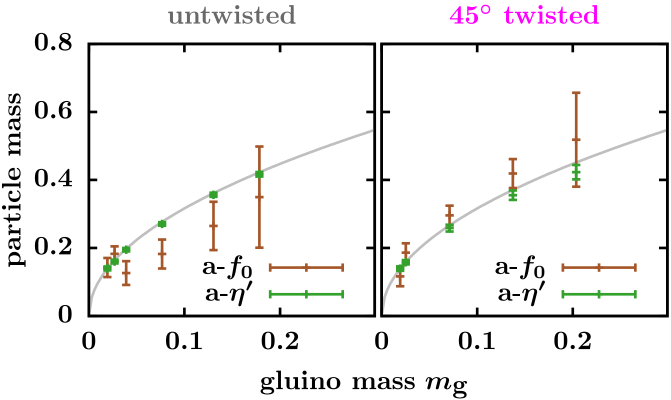

Figure: $8^3 \times 16$, preliminary

Mesonic States & Gluino-Glue

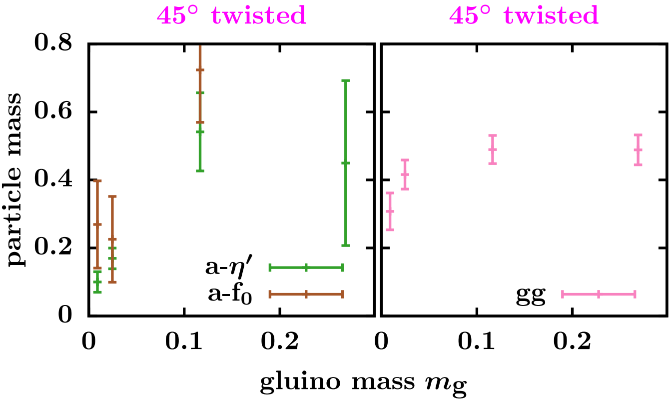

Figure: $16^3 \times 32$, preliminary

Summary

SYM on the lattice

- New analytical investigations
  - Fermionic expectation values
  - Eigenvalues of the free Dirac operator
- Lattice results
  - Fine tuning of bare gluino mass m (and μ) necessary to restore supersymmetry & chiral symmetry
  - Chiral symmetry of multiplet improved at finite lattice spacing with 45° twist
  - Wilson term twist provides an opportunity for $\mathcal{O}(a)$ improvement

Outlook

Summary

SYM on the lattice

Outlook

- Start of SuperMUC New Generation
- More statistics to confirm $m_{\text{a-}\eta'} \approx m_{\text{a-}f_0} \approx m_{gg}$ along $45°$ twist
  $\Rightarrow$ improvement of the susy at finite lattice spacing may be possible
- Spectroscopy at 3 different couplings β for continuum limit

Appendix: pqχpt

SYM

- Only 1 flavor $\Rightarrow$ partially quenched framework

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Pro

- Inclusion of anomalous chiral symmetry
- Masses of pseudoscalars, i.e. a-π, a-η'

Appendix: pqχpt

SYM

- Only 1 flavor $\Rightarrow$ partially quenched framework

Pro

- Inclusion of anomalous chiral symmetry
- Masses of pseudoscalars, i.e. a-π, a-η'

Contra

- No native scalars, i.e. a-a, a-f_0
- No direct access to gauge field for hybrids, i.e. gluino-glue

Appendix: Sign of the Pfaffian

Wilson-Dirac operator $D_\mathrm{W} = D_\mathrm{W}^\mathrm{tw}(\mu=0)$

- is γ_5-Hermitian: $(\gamma_5 D_\mathrm{W})^\dagger = \gamma_5 D_\mathrm{W}$
- is $\mathcal{C}$-Antisymmetric: $(\mathcal{C} D_\mathrm{W})^\mathsf{T} = -\mathcal{C} D_\mathrm{W}$
- $\det(D_\mathrm{W}) \in \mathbb{R}^+$
- $\mathrm{Pf}\,(D_\mathrm{W}) \in \mathbb{R}$

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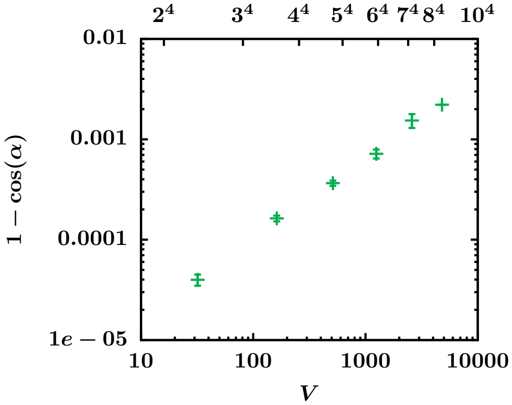
- $\mathrm{Pf}\,(D_W) \in \mathbb{R}$

Twisted Wilson-Dirac operator $D_W^{tw}(\mu\neq 0)$

- in general $\mathrm{Pf}(D_W^{tw}) \in \mathbb{C}$
- in continuum theory $m \to m_{crit},\ \mu \to 0,\ a \to 0$: $\mathrm{Pf}(D_W^{tw}) \in \mathbb{R}$
- at finite lattice spacing: phase of $\mathrm{Pf}(D_W^{tw}) = |\mathrm{Pf}(D_W^{tw})| \cdot e^{i\alpha}$ negligible

Appendix: Sign of the Pfaffian



$m = -0.85,\ \mu = 0.10,\ m_{\text{a-}\pi} \approx 0.70$

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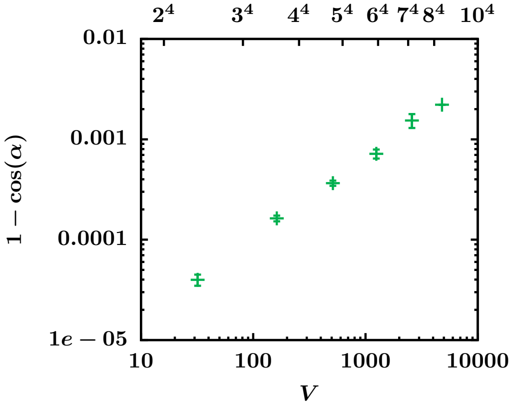

$m = -0.85,\ \mu = 0.10,\ m_{\text{a-}\pi} \approx 0.70$

extrapolated to $16^3 \times 32$:
$1 - \cos(\alpha) < 0.035$