Quantum Critical Phenomena in an $O(4)$ Fermion Chain

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Outline

Outline for section 1

Motivation: Fermionic lattice models $\xrightarrow{\text{at critical point}}$ Non-linear sigma model with topological terms.

1+1d examples:

Spin Chain	Non-linear sigma model
Heisenberg spin chain	$O(3)$ model with topological terms[1];[2]
$SU(n)$ spin chain	$CP(n-1)$ model with topological terms[3];[4]
Generalized spin chain	Grassmannian model with topological terms[5]

2+1d examples: Competing Néel order and VBS order seperated by quantum critical points which can be described by NLSMs with topological terms[6];[7]

[1] F. D. M. Haldane, 1983, *Phys. Lett.*
[2] I. Affleck and F. D. M. Haldane, 1987, *Phys. Rev.*
[3] I. Affleck, 1988, *Nucl. Phys.*
[4] B. B. Beard et al., 2005, *Phys. Rev. Lett.* arXiv: `hep-lat/0406040` (hep-lat)
[5] I. Affleck, 1985, *Nucl. Phys.*
[6] T. Senthil, 2004, *Science*
[7] T. Senthil and M. P. A. Fisher, 2006, *Phys. Rev.* arXiv: `cond-mat/0510459` (cond-mat)

- The sign problem can be solved very efficiently with the meron-cluster algorithm.
- Hamiltonians that can be studied with the meron-cluster algorithm are restricted.
- However, interesting phase transitions (like those between massive Dirac fermions in 2+1d) still may occur.
- Recent works suggest topological terms may arise at the critical points [8];[9]
- This talk: a model in 1+1d where a topological term is known to arise.

[8] K. Slagle et al., 2015, *Phys. Rev. B* arXiv: 1409.7401 (cond-mat.str-el)
[9] T. Sato et al., 2017, *Phys. Rev. Lett.* arXiv: 1707.03027 (cond-mat.str-el)

Outline for section 2

The Hamiltonian

- A one dimensional fermion lattice with "spin".

$$H = H_0 + H_U \tag{1}$$

-

$$H_0 = -J \sum_{\langle i,j \rangle} H_{\langle i,j \rangle \uparrow} H_{\langle i,j \rangle \downarrow}, \tag{2}$$

where
$H_{\langle i,j \rangle,\sigma} = -(c^\dagger_{i,\sigma} c_{j,\sigma} + c^\dagger_{j,\sigma} c_{i,\sigma}) + 2\left(n_{i,\sigma} - \frac{1}{2}\right)\left(n_{j,\sigma} - \frac{1}{2}\right) - \frac{1}{2}.$

- Hubbard interaction:

$$H_U = U \sum_i \left(n_{i\uparrow} - \frac{1}{2}\right)\left(n_{i\downarrow} - \frac{1}{2}\right) \tag{3}$$

- No sign problem with the meron-cluster algorithm.

Symmetries of H_0

Rewritten in terms of Majorana operators γ_i^μ and $\bar{\gamma}_j^\mu$

$$c_\uparrow = \frac{1}{2}(\gamma^1 - i\gamma^2), \quad c_\uparrow^\dagger = \frac{1}{2}(\gamma^1 + i\gamma^2), \quad n_\uparrow = \frac{1}{2}(-i\gamma^1\gamma^2 + 1) \text{ on even sites}$$
$$c_\uparrow = \frac{1}{2}(\bar{\gamma}^2 + i\bar{\gamma}^1), \quad c_\uparrow^\dagger = \frac{1}{2}(\bar{\gamma}^2 - i\bar{\gamma}^1), \quad n_\uparrow = \frac{1}{2}(-i\bar{\gamma}^1\bar{\gamma}^2 + 1) \text{ on odd sites,} \tag{4}$$

and similarly for spin $\downarrow$ the Hamiltonian takes the form

$$H_0 = -\frac{J}{4}\sum_{\langle i,j\rangle}\prod_{\mu=1}^{4}(1 + i\bar{\gamma}_i^\mu\gamma_j^\mu) \tag{5}$$

- $O(4)$ symmetry generated by

$$\Gamma^{\mu\nu} = i(\sum_{i \text{ even}} \gamma_i^\mu\gamma_i^\nu + \sum_{j \text{ odd}} \bar{\gamma}_j^\mu\bar{\gamma}_j^\nu) \tag{6}$$

- $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$ (spin and charge symmetry).
- Manifest lattice translation symmetry ($\mapsto \mathbb{Z}_2$ chiral symmetry in the continuum limit)

$$H_U = U \sum_i \left(n_{i\uparrow} - \frac{1}{2}\right)\left(n_{i\downarrow} - \frac{1}{2}\right) = -\frac{U}{96}\sum_i \varepsilon_{\mu\nu\rho\sigma}\gamma_i^\mu\gamma_i^\nu\gamma_i^\rho\gamma_i^\sigma \quad (7)$$

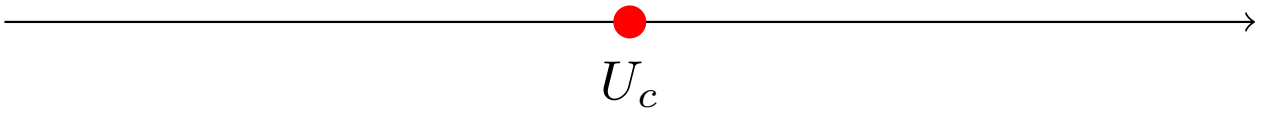
- Breaks the spin $\leftrightarrow$ charge symmetry: $O(4) \to SO(4)$.
- $H_0 + H_U \xrightarrow{U\to\infty}$ Heisenberg spin-$\frac{1}{2}$ chain $\implies$ flows to the $k = 1$ WZW model[10];[11]

$$S_0 + S_{\text{WZW}} = \frac{1}{4\lambda^2}\int d^2x\, \text{tr}(\partial_\mu g^{-1}\partial^\mu g) + \frac{k}{48\pi^2}\int \text{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg).$$

where g is a spacetime dependent group element in $SU(2)$. The second integration is over a 3-disk whose boundary is the spacetime.

Massive phase with $\mathbb{Z}_2$ chiral symmetry spontaneously broken

Critical phase with emergent $k = 1$ WZW term in the IR



[10] I. Affleck and F. D. M. Haldane, 1987, *Phys. Rev.*
[11] R. Shankar and N. Read, 1990, *Nucl. Phys.*

Outline for section 3

The hopping term:

$$H_{\text{hop}} = \sum_{\alpha} \int dx \left(-\psi^{\dagger}_{\alpha,L}(x) i \frac{d}{dx} \psi_{\alpha,L}(x) + \psi^{\dagger}_{\alpha,R}(x) i \frac{d}{dx} \psi_{\alpha,R}(x) \right).$$

	Chiral preserving	Chiral breaking
Quadratic	$S^i_{L,R}$, $Q^i_{L,R}$	$M^0 = \bar{\psi}_\alpha \sigma^3 \psi^\alpha$ M^i with $M^3 = \bar{\psi}_\alpha \psi^\alpha$
Quartic	$S^1_L S^1_L$, $S^2_L S^2_L$	$S^i_L S^i_R = \frac{1}{4}(M_0 M^0 - M_i M^i) + 1$ $Q^i_L Q^i_R = \frac{1}{4}(3 M_0 M^0 + M_i M^i)$

Symmetry considerations:

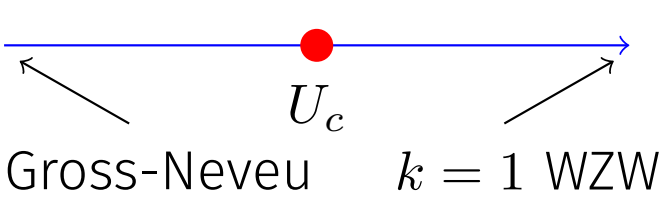
1. $O(4)$ symmetry: S^i and Q^i rotate as two vectors.
2. lattice translation symmetry ($\mathbb{Z}_2$ chiral symmetry): M^μ flip sign.
3. Particle-hole symmetry ($\psi_1 \to \psi_1^\dagger$): $S^i \leftrightarrow Q^i$. The only possibility: $S^i_L S^i_R + Q^i_L Q^i_R = M_0 M^0 + 1$

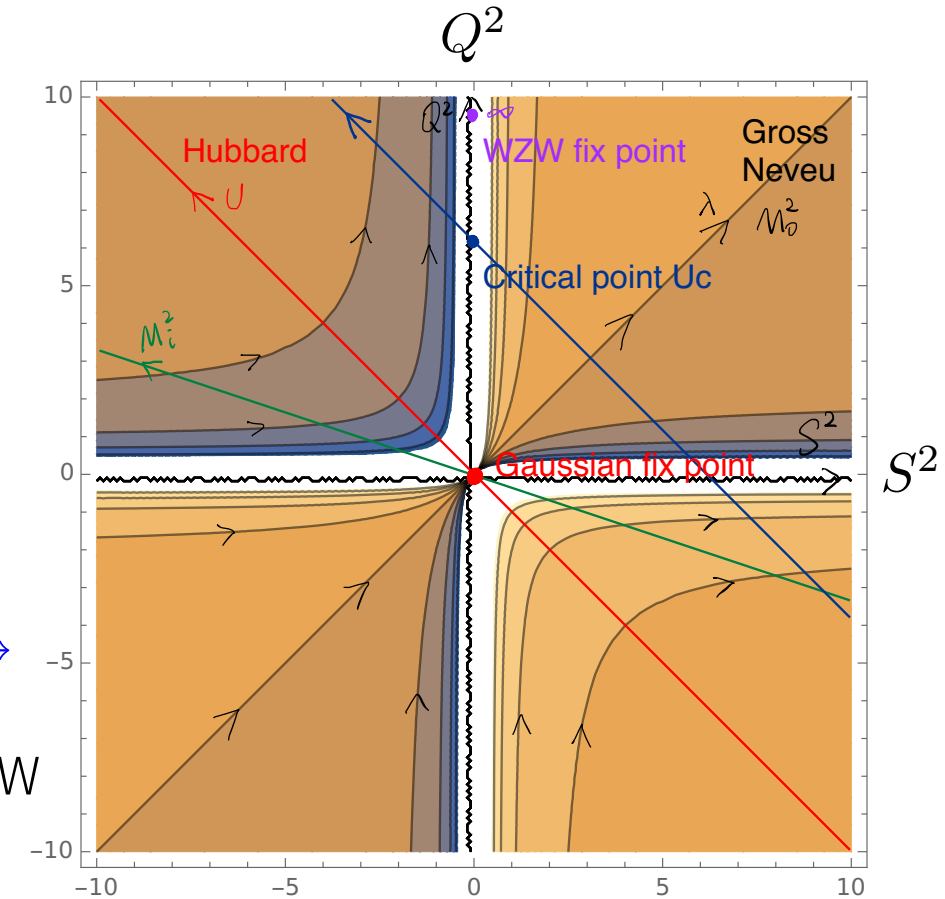
$$
\begin{aligned}
S &= \int d^2x\, \bar{\psi}_\alpha \gamma^\mu i\partial_\mu \psi^\alpha(x) - \frac{\lambda}{4} M_0 M^0 \\
&= \int d^2x\, \bar{\psi}_\alpha \gamma^\mu i\partial_\mu \psi^\alpha(x) - \frac{\lambda}{4} (\bar{\psi}_\alpha \sigma^3 \psi^\alpha)^2 \\
&\xrightarrow{\text{chiral rotation}} \int d^2x\, \bar{\psi}_\alpha \gamma^\mu i\partial_\mu \psi^\alpha(x) + \frac{\lambda}{4} (\bar{\psi}_\alpha \psi^\alpha)^2 \\
&= \int d^2x\, \bar{\psi}_\alpha \gamma^\mu i\partial_\mu \psi^\alpha + \phi \bar{\psi}_\alpha \psi^\alpha - \frac{1}{\lambda}\phi^2 \qquad (8)
\end{aligned}
$$

$\mathbb{Z}_2$ chiral symmetry (lattice translation symmetry): $\psi_\alpha \to \sigma^3 \psi_\alpha$, $\phi \to -\phi$.

At low energy, $\langle \phi \rangle = \pm \Lambda e^{1/2 - \frac{\pi}{\lambda}}$ gets a VEV and the $\mathbb{Z}_2$ symmetry is broken spontaneously.

$\langle \phi \rangle$ is a physical scale $\implies \beta(\lambda) = -\frac{\lambda^2}{\pi}$.





Flow diagram in the $S_L^i S_R^i$ - $Q_L^i Q_R^i$ plane

Outline for section 4

$$Z = \mathrm{tr}\, e^{-\beta H} = \sum_{\{C\}} W[C] \quad (9)$$

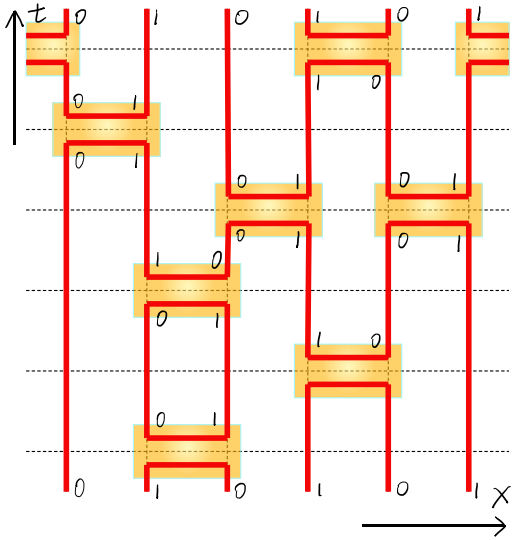


Illustration of a configuration and its loop-cluster structure[13]

Meron-cluster $\implies N_h/2 + N_t$ is even.

[13] Credit to Shailesh Chandrasekharan.

- Dimer susceptibility (order parameter):

$$\chi_D = \int dt \sum_i \langle D_0(0) D_i(t) \rangle, \tag{10}$$

where

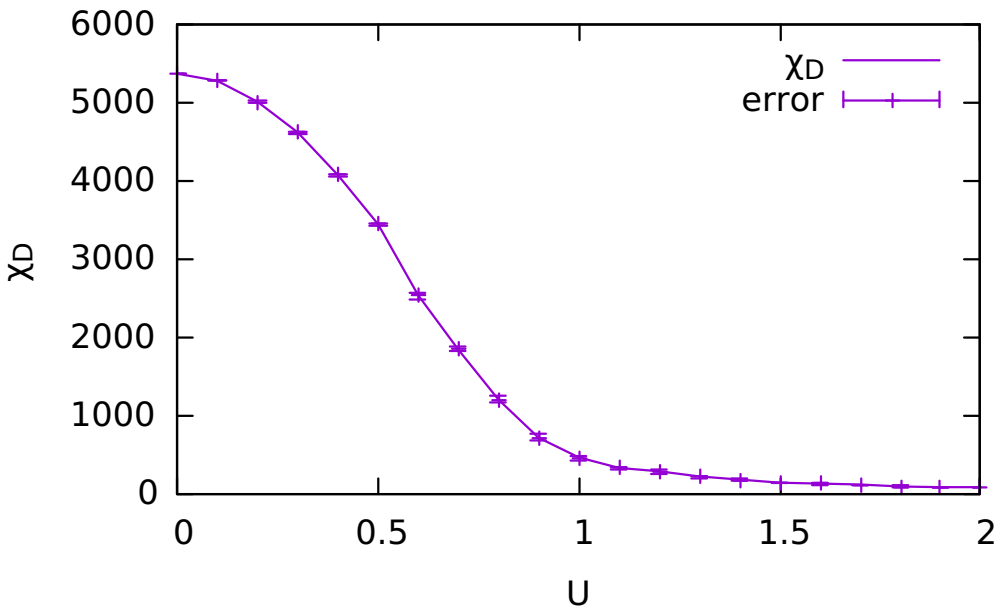
$$D_i(t) = H_{0,2i}(t) - H_{0,2i+1}(t). \tag{11}$$

- Spin susceptibility:

$$\chi_S = \int dt \sum_i \langle S_0^+(0) S_i^-(t) \rangle, \tag{12}$$

where

$$S_i^+(t) = c_{i\uparrow}^\dagger(t) c_{i\downarrow}(t), \quad S_i^-(t) = c_{i\downarrow}^\dagger(t) c_{i\uparrow}(t). \tag{13}$$

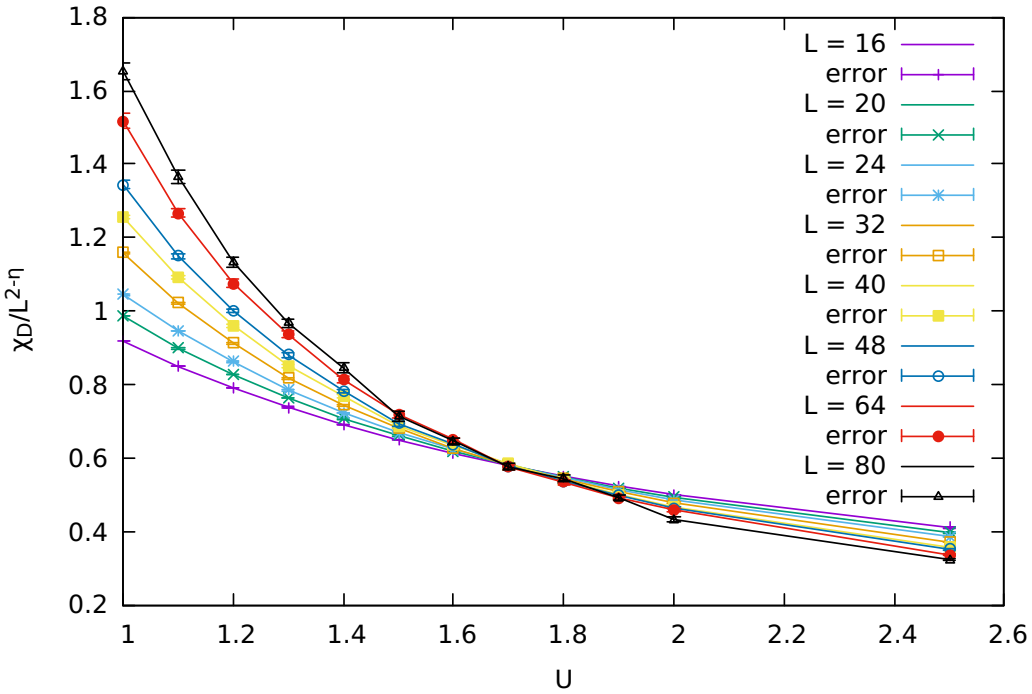


χ_D as a function of U at L = 192

Finite size scaling:

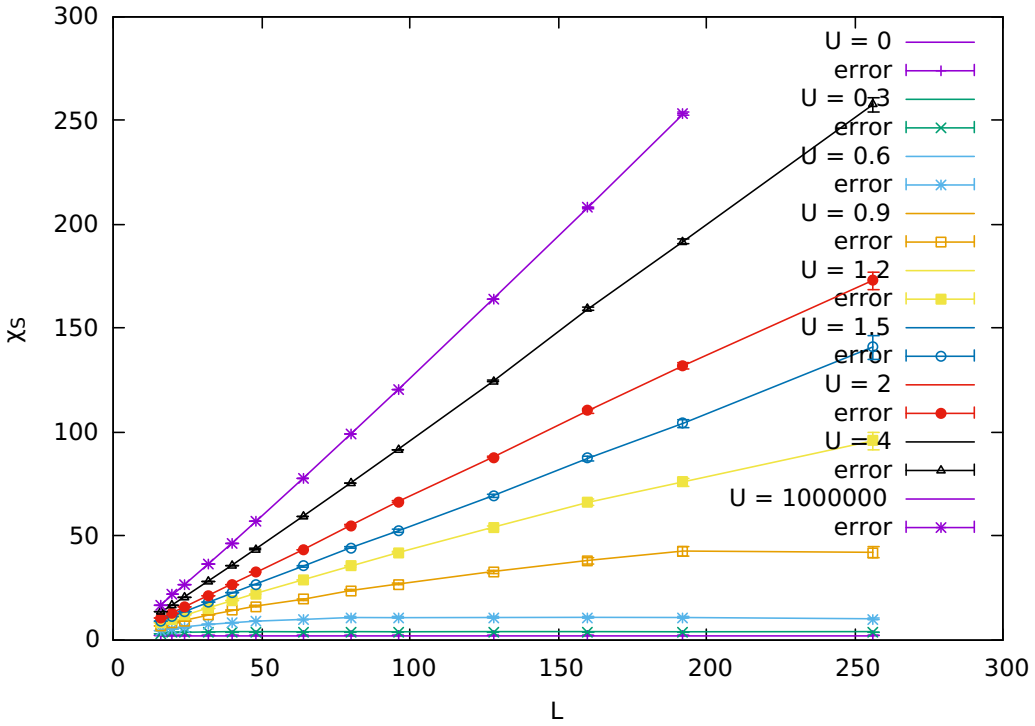
$$\frac{\chi_D}{L^{2-\eta}} = \tilde{\chi}(L^{\frac{1}{\nu}} u), \tag{14}$$

where $u = \frac{U - U_c}{U_c}$.



$\chi_D/L^{2-\eta}$ as a function of U at $\eta = 1.0$

Massive to massless transition in the spin sector.



χ_S as a function of L

Outline for section 5

Conclusions

- Using the meron-cluster algorithm, we studied the phase transition between a massive phase with spontaneously broken chiral symmetry and a critical phase with a topological term.
- Near the critical point ($U = 1.7$),
  - the dimer susceptibility has scaling dimension $1.0028(72)$,
  - the spin susceptibility has scaling dimension $0.9994(18)$.

  This agrees with the emergent $\mathfrak{so}(4) \cong \mathfrak{su}_L(2) \times \mathfrak{su}_R(2)$ symmetry in $k = 1$ WZW model.
  - $g = D + i\vec{S} \cdot \vec{\sigma}$ forms a vector representation of $SO(4)$,
  - The original spin $\mathfrak{su}(2)$ symmetry sits at diagonal.

Further Questions

- What is the relationship between the $N = 2$ Gross-Neveu model and the $\mathbb{CP}^3$ model at $\theta = \pi$?
  - (Almost) identical lattice Hamiltonians;
  - Spontaneous $\mathbb{Z}_2$ symmetry breaking and dynamical mass generation;
  - Identical one-loop β function;
- What sort of phases does our model describe in 2+1d? What sort of topological terms will emerge at the quantum critical point?

Thank You