Quantum Critical Phenomena in an $O(4)$ Fermion Chain

Motivation
Induce topological terms in non-linear sigma models
Apply the meron-cluster algorithm

Our model

Analysis of the Quantum Critical Point
Relevant operators and symmetries
The Gross-Neveu model
Phase diagram and flow diagram

The Algorithm and Results
The meron-cluster approach
Monte Carlo analysis

Conclusions

Further questions

Hanqing Liu
(In collaboration with Shailesh Chandrasekharan and Ribhu Kaul)

1Department of Physics, Duke University
2Department of Physics and Astronomy, University of Kentucky

June 21, 2019
1 Motivation
   - Induce topological terms in non-linear sigma models
   - Apply the meron-cluster algorithm

2 Our model

3 Analysis of the Quantum Critical Point
   - Relevant operators and symmetries
   - The Gross-Neveu model
   - Phase diagram and flow diagram

4 The Algorithm and Results
   - The meron-cluster approach
   - Monte Carlo analysis

5 Conclusions
   - Conclusions
   - Further questions
1 Motivation

- Induce topological terms in non-linear sigma models
- Apply the meron-cluster algorithm

2 Our model

3 Analysis of the Quantum Critical Point

4 The Algorithm and Results

5 Conclusions
Motivation: Fermionic lattice models at critical point
Non-linear sigma model with topological terms.

1+1d examples:

<table>
<thead>
<tr>
<th>Spin Chain</th>
<th>Non-linear sigma model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heisenberg spin chain</td>
<td>$O(3)$ model with topological terms$^{1,2}$</td>
</tr>
<tr>
<td>$SU(n)$ spin chain</td>
<td>$CP(n - 1)$ model with topological terms$^{3,4}$</td>
</tr>
<tr>
<td>Generalized spin chain</td>
<td>Grassmannian model with topological terms$^{5}$</td>
</tr>
</tbody>
</table>

2+1d examples: Competing Néel order and VBS order separated by quantum critical points which can be described by NLSMs with topological terms$^{6,7}$

---

The sign problem can be solved very efficiently with the meron-cluster algorithm.

Hamiltonians that can be studied with the meron-cluster algorithm are restricted.

However, interesting phase transitions (like those between massive Dirac fermions in 2+1d) still may occur.

Recent works suggest topological terms may arise at the critical points \(^8;^9\)

This talk: a model in 1+1d where a topological term is known to arise.

---


Outline for section 2

1. Motivation
2. Our model
3. Analysis of the Quantum Critical Point
4. The Algorithm and Results
5. Conclusions
A one dimensional fermion lattice with “spin”.

\[ H = H_0 + H_U \]  

(1)

\[ H_0 = -J \sum_{\langle i, j \rangle} H_{\langle i, j \rangle \uparrow} H_{\langle i, j \rangle \downarrow}, \]  

(2)

where

\[ H_{\langle i, j \rangle, \sigma} = -(c_{i, \sigma}^{\dagger} c_{j, \sigma} + c_{j, \sigma}^{\dagger} c_{i, \sigma}) + 2 \left( n_{i, \sigma} - \frac{1}{2} \right) \left( n_{j, \sigma} - \frac{1}{2} \right) - \frac{1}{2}. \]

Hubbard interaction:

\[ H_U = U \sum_i \left( n_{i \uparrow} - \frac{1}{2} \right) \left( n_{i \downarrow} - \frac{1}{2} \right) \]  

(3)

No sign problem with the meron-cluster algorithm.
Rewritten in terms of Majorana operators $\gamma_i^\mu$ and $\bar{\gamma}_j^\mu$

$$
c_{\uparrow} = \frac{1}{2}(\gamma^1 - i\gamma^2), \quad c_{\uparrow}^\dagger = \frac{1}{2}(\gamma^1 + i\gamma^2), \quad n_{\uparrow} = \frac{1}{2}(-i\gamma^1\gamma^2 + 1) \quad \text{on even sites}
$$

$$
c_{\uparrow} = \frac{1}{2}(\bar{\gamma}^2 + i\bar{\gamma}^1), \quad c_{\uparrow}^\dagger = \frac{1}{2}(\bar{\gamma}^2 - i\bar{\gamma}^1), \quad n_{\uparrow} = \frac{1}{2}(-i\bar{\gamma}^2 + 1) \quad \text{on odd sites},
$$

and similarly for spin $\downarrow$ the Hamiltonian takes the form

$$
H_0 = -\frac{J}{4} \sum_{\langle i, j \rangle} \prod_{\mu=1}^{4} (1 + i\bar{\gamma}_i^\mu \gamma_j^\mu) \quad (5)
$$

- $O(4)$ symmetry generated by

$$
\Gamma^{\mu\nu} = i(\sum_{i \text{ even}} \gamma_i^\mu \gamma_i^\nu + \sum_{j \text{ odd}} \bar{\gamma}_j^\mu \bar{\gamma}_j^\nu) \quad (6)
$$

- $so(4) \cong su(2) \times su(2)$ (spin and charge symmetry).

- Manifest lattice translation symmetry ($\rightarrow \mathbb{Z}_2$ chiral symmetry in the continuum limit)
Adding a Hubbard Interaction

\[ H_U = U \sum_i \left( n_i^\uparrow - \frac{1}{2} \right) \left( n_i^\downarrow - \frac{1}{2} \right) = -\frac{U}{96} \sum_i \varepsilon_{\mu\nu\rho\sigma} \gamma_i^\mu \gamma_i^\nu \gamma_i^\rho \gamma_i^\sigma \quad (7) \]

- Breaks the spin ↔ charge symmetry: \( O(4) \rightarrow SO(4) \).
- \( H_0 + H_U \xrightarrow{U \to \infty} \) Heisenberg spin-\( \frac{1}{2} \) chain \( \iff \) flows to the \( k = 1 \) WZW model\textsuperscript{10,11}

\[ S_0 + S_{WZW} = \frac{1}{4\lambda^2} \int d^2x \text{tr}(\partial_\mu g^{-1} \partial^\mu g) + \frac{k}{48\pi^2} \int \text{tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg). \]

where \( g \) is a spacetime dependent group element in \( SU(2) \). The second integration is over a 3-disk whose boundary is the spacetime.

Massive phase with \( \mathbb{Z}_2 \)

**Critical phase with emergent** \( k = 1 \) WZW term in the IR

\[ U_c \]

\textsuperscript{10}I. Affleck and F. D. M. Haldane, 1987, Phys. Rev.

\textsuperscript{11}R. Shankar and N. Read, 1990, Nucl. Phys.
1 Motivation

2 Our model

3 Analysis of the Quantum Critical Point
   - Relevant operators and symmetries
   - The Gross-Neveu model
   - Phase diagram and flow diagram

4 The Algorithm and Results

5 Conclusions
The hopping term:

\[ H_{\text{hop}} = \sum_{\alpha} \int dx \left( -\psi_{\alpha,L}(x)i \frac{d}{dx} \psi_{\alpha,L}(x) + \psi_{\alpha,R}(x)i \frac{d}{dx} \psi_{\alpha,R}(x) \right) \].

<table>
<thead>
<tr>
<th>Chiral preserving</th>
<th>Chiral breaking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadratic</td>
<td></td>
</tr>
<tr>
<td>( S_{L,R}^i, Q_{L,R}^i )</td>
<td>( M^0 = \psi_\alpha \sigma^3 \psi_\alpha )</td>
</tr>
<tr>
<td>( M^i ) with ( M^3 = \bar{\psi} \psi_\alpha )</td>
<td></td>
</tr>
<tr>
<td>Quartic</td>
<td></td>
</tr>
<tr>
<td>( S_L^1 S_L^2, S_L^2 S_L^2 )</td>
<td>( S_{L,R}^i S_{L,R}^i = \frac{1}{4} (M_0 M^0 - M_i M^i) + 1 )</td>
</tr>
<tr>
<td>( Q_{L,R}^i Q_{L,R}^i = \frac{1}{4} (3M_0 M^0 + M_i M^i) )</td>
<td></td>
</tr>
</tbody>
</table>

Symmetry considerations:

1. \( O(4) \) symmetry: \( S^i \) and \( Q^i \) rotate as two vectors.
2. Lattice translation symmetry (\( \mathbb{Z}_2 \) chiral symmetry): \( M^\mu \) flip sign.
3. Particle-hole symmetry (\( \psi_1 \rightarrow \psi_1^\dagger \)): \( S^i \leftrightarrow Q^i \). The only possibility: \( S_L^i S_R^i + Q_L^i Q_R^i = M_0 M^0 + 1 \)
Quantum Critical Phenomena in an $O(4)$ Fermion Chain

Motivation
Induce topological terms in non-linear sigma models
Apply the meron-cluster algorithm

Our model

Analysis of the Quantum Critical Point
Relevant operators and symmetries

The Gross-Neveu model

Phase diagram and flow diagram

The Algorithm and Results
The meron-cluster approach
Monte Carlo analysis

Conclusions
Conclusions
Further questions

The Gross-Neveu Model

\[
S = \int d^2 x \bar{\psi}_\alpha \gamma^\mu i \partial_\mu \psi^\alpha (x) - \frac{\lambda}{4} M_0 M^0
\]

\[
= \int d^2 x \bar{\psi}_\alpha \gamma^\mu i \partial_\mu \psi^\alpha (x) - \frac{\lambda}{4} (\bar{\psi}_\sigma^3 \psi^\alpha)^2
\]

\[
\xrightarrow{\text{chiral rotation}} \int d^2 x \bar{\psi}_\alpha \gamma^\mu i \partial_\mu \psi^\alpha (x) + \frac{\lambda}{4} (\bar{\psi}_\alpha \psi^\alpha)^2
\]

\[
= \int d^2 x \bar{\psi}_\alpha \gamma^\mu i \partial_\mu \psi^\alpha + \phi \bar{\psi}_\alpha \psi^\alpha - \frac{1}{\lambda} \phi^2
\]

(8)

$\mathbb{Z}_2$ chiral symmetry (lattice translation symmetry):
$\psi_\alpha \rightarrow \sigma^3 \psi_\alpha, \phi \rightarrow -\phi$.

At low energy, $\langle \phi \rangle = \pm \Lambda e^{1/2-\frac{\pi}{\lambda}}$ gets a VEV and the $\mathbb{Z}_2$ symmetry is broken spontaneously.

$\langle \phi \rangle$ is a physical scale $\Longrightarrow \beta(\lambda) = -\frac{\lambda^2}{\pi}$.
Quantum Critical Phenomena in an $O(4)$ Fermion Chain

**Motivation**
Induce topological terms in non-linear sigma models
Apply the meron-cluster algorithm

**Our model**

**Analysis of the Quantum Critical Point**
Relevant operators and symmetries
The Gross-Neveu model
Phase diagram and flow diagram

**The Algorithm and Results**
The meron-cluster approach
Monte Carlo analysis

**Conclusions**
Conclusions
Further questions

---

**Phase Diagram and Flow Diagram**

Flow diagram in the $S_L^i S_R^i - Q_L^i Q_R^i$ plane

- Massive phase
- Critical phase
- Gross-Neveu
- $k = 1$ WZW

- Hubbard
- WZW fix point
- Gaussian fix point
- Critical point $U_c$
Outline for section 4

1 Motivation

2 Our model

3 Analysis of the Quantum Critical Point

4 The Algorithm and Results
   - The meron-cluster approach
   - Monte Carlo analysis

5 Conclusions
Quantum Critical Phenomena in an $O(4)$ Fermion Chain

**Motivation**
Induce topological terms in non-linear sigma models
Apply the meron-cluster algorithm

**Our model**

**Analysis of the Quantum Critical Point**
Relevant operators and symmetries
The Gross-Neveu model
Phase diagram and flow diagram

**The Algorithm and Results**
The meron-cluster approach
Monte Carlo analysis

**Conclusions**
Conclusions
Further questions

---

The Meron-cluster Approach

$$Z = \text{tr} \ e^{-\beta H} = \sum_{\{C\}} W[C] \quad (9)$$

Illustration of a configuration and its loop-cluster structure$^{13}$

Meron-cluster $\Rightarrow N_h/2 + N_t$ is even.

---

$^{13}$Credit to Shailesh Chandrasekharan.
Observables

- Dimer susceptibility (order parameter):
  \[ \chi_D = \int dt \sum_i \langle D_0(0)D_i(t) \rangle, \]  
  where 
  \[ D_i(t) = H_{0,2i}(t) - H_{0,2i+1}(t). \]  

- Spin susceptibility:
  \[ \chi_S = \int dt \sum_i \langle S^+_0(0)S^-_i(t) \rangle, \]  
  where 
  \[ S^+_i(t) = c_{i\uparrow}(t)c_{i\downarrow}(t), \quad S^-_i(t) = c_{i\downarrow}(t)c_{i\uparrow}(t). \]
Quantum Critical Phenomena in an $O(4)$ Fermion Chain

Motivation
Induce topological terms in non-linear sigma models
Apply the meron-cluster algorithm

Our model

Analysis of the Quantum Critical Point
Relevant operators and symmetries
The Gross-Neveu model
Phase diagram and flow diagram

The Algorithm and Results
The meron-cluster approach
Monte Carlo analysis

Conclusions
Conclusions
Further questions

Monte Carlo Results 1

$\chi_D$ as a function of $U$ at $L = 192$
Finite size scaling:

\[ \frac{\chi_D}{L^{2-\eta}} = \tilde{\chi}(L^{\frac{1}{\nu}}u), \]

where \( u = \frac{U-U_c}{U_c} \).

\[ \chi_D / L^{2-\eta} \text{ as a function of } U \text{ at } \eta = 1.0 \]
Monte Carlo Results 3

Massive to massless transition in the spin sector.

$\chi_S$ as a function of $L$
Quantum Critical Phenomena in an $O(4)$ Fermion Chain

Motivation
- Induce topological terms in non-linear sigma models
- Apply the meron-cluster algorithm

Our model

Analysis of the Quantum Critical Point
- Relevant operators and symmetries
- The Gross-Neveu model
- Phase diagram and flow diagram

The Algorithm and Results
- The meron-cluster approach
- Monte Carlo analysis

Conclusions
- Conclusions
- Further questions
Using the meron-cluster algorithm, we studied the phase transition between a massive phase with spontaneously broken chiral symmetry and a critical phase with a topological term.

Near the critical point \( \mathcal{U} = 1.7 \),
- the dimer susceptibility has scaling dimension \( 1.0028(72) \),
- the spin susceptibility has scaling dimension \( 0.9994(18) \).

This agrees with the emergent \( \mathfrak{so}(4) \cong \mathfrak{su}_L(2) \times \mathfrak{su}_R(2) \) symmetry in \( k = 1 \) WZW model.
- \( g = D + i \vec{S} \cdot \vec{\sigma} \) forms a vector representation of \( SO(4) \),
- The original spin \( \mathfrak{su}(2) \) symmetry sits at diagonal.
Further Questions

- What is the relationship between the $\mathcal{N} = 2$ Gross-Neveu model and the $\mathbb{CP}^3$ model at $\theta = \pi$?
  - (Almost) identical lattice Hamiltonians;
  - Spontaneous $\mathbb{Z}_2$ symmetry breaking and dynamical mass generation;
  - Identical one-loop $\beta$ function;

- What sort of phases does our model describe in 2+1d? What sort of topological terms will emerge at the quantum critical point?
Thank You