FORMULATING LATTICE FIELD FOR A QUANTUM COMPUTER*

*The accidental discovery of a Quantum Algorithm for Lattice Gauge theories: Circa 1998:

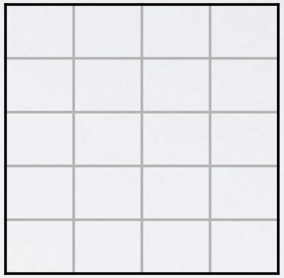
Rich Brower — Boston University
Lattice June 19, 2019 Wuhan China

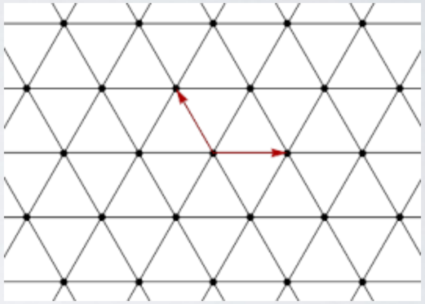
ABSTRACT

1. The quantum link (or QCD abacus) Hamiltonian was introduced as a classical algorithm representing both gauge and matter fields by single bit fermion operators in an extra dimension.
2. This formalism is recast for quantum computing, as a Hamiltonian in Minkowski space for real time Qubit simulations.
3. The advantages of pseudo-fermions to implement the Jordan Wigner transformation and the Trotter expansion in local gauge invariant local kernels is discussed.
4. For U(1) compact QED the kernels on a triangular lattice are defined and a Qubit circuit implementation given to test on existing hardware.

Universality == Many equivalent LFT

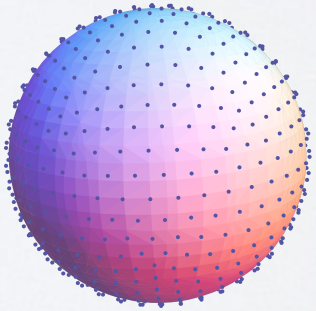
- Different space-time + field discretizations define exactly the same continuum quantum field theory

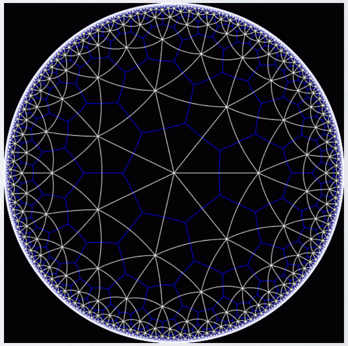




- *eg. Lattice different lattice give identical c = 1/2 CFT — square, triangle or spherical lattice!*
- *Fields: Continuum phi 4th field and single bit Ising fields are equivalent*

$$s \in \pm 1 \iff \phi_x \in \mathbb{R}$$





- *Bosonic Sine Gordon Theory = Fermionic Thirring Theory*
- *N = 4 SUSY is dual AdS Gravity*

WARM UP: O(3) HEISENBERG SPIN MODEL

Classical O(3) MODEL

$$Z = \int dS \, \exp[\frac{1}{g^2} \sum_{\langle x,y \rangle} \vec{S}_x \cdot \vec{S}_y]$$

Quantum antiferromagnet O(3) MODEL

$$\to Tr[e^{-\frac{\beta}{g^2} \sum_{x,\mu} \vec{\sigma}_x \cdot \vec{\sigma}_{x+\mu}}]$$

Warm up with Fermionic Quantum Operator

$$\vec{S}_x \to a_i^\dagger(x) \sigma^{ij} a_j(x) \quad \text{or} \quad \hat{S}^{ij}(x) = a_i^\dagger(x) a_j(x)$$

$$\{a_i(x), a_j^\dagger(x)\} = \delta_{ij}\delta_{xy} \quad , \quad \{a_i^\dagger(x) a_j^\dagger(x)\} = 0 \quad , \quad \{a_i(x), a_j(x)\} = 0$$

$$Z = \mathrm{Tr}\exp(-\beta \hat{H}). \qquad \hat{H} = \sum_{\langle x,y \rangle} Tr[\hat{S}_x \hat{S}_y] \quad , \quad Tr[\hat{S}_x] = 0$$

Global Rotation: $\vec{J} = \sum_x Tr[\vec{\sigma}\hat{S}_x] \implies [\vec{J}, \hat{H}] = 0$

Local Fermion No: $\hat{F}_x = Tr[\hat{S}_x] \implies [\hat{F}_x, \hat{H}] = 0$

B.B. Beard and U-J Wiese

https://arxiv.org/abs/cond-mat/9602164

KOGUT SUSKIND HAMILTONIAN

$$H = \frac{g^2}{2} \sum_{\langle x, x+\mu \rangle} (Tr[E_L^2(x,\mu)] + Tr[E_R^2(x,\mu)]) - \frac{1}{2g^2} \sum_{\Box} Tr[U_\Box + U_\Box^\dagger]$$

Example: U(1) Abelian P/Q symplectic operators in Q-basis are

$$E = i\frac{d}{d\theta} \quad , \quad U = exp[i\theta]$$

U(N) generalization of Gauge Algebra is

$$E^{ij} \equiv \lambda_\alpha^{ij} E^\alpha \quad \Longrightarrow \quad [E^\alpha, E^\beta] = 2if^{\alpha\beta\gamma}E^\alpha$$

$$[E_L, U] = -E_L U \quad , \quad [E_R, U] = -U E_R$$

$$E_R = U^\dagger E_L U \quad , \quad [U, U^\dagger] = 0 \qquad UU^\dagger = 1$$

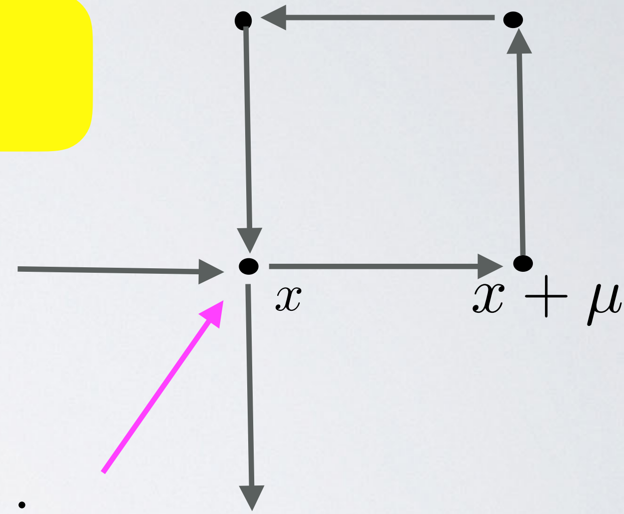
OPERATOR QUANTUM GAUGE LINK

On each link (x,x + μ) introduce $2N_c$complex fermion

$a_, a_i^\dagger$ right(+) moving and $b_j, b_j^\dagger$ left(-) moving fluxon

LINK:

$$a_i \longrightarrow b_j^\dagger$$



$$U_{ij}(x, x+\mu) \quad \to \quad \hat{U}_{ij} = a_i(x)\, b_j^\dagger(x+\mu)$$

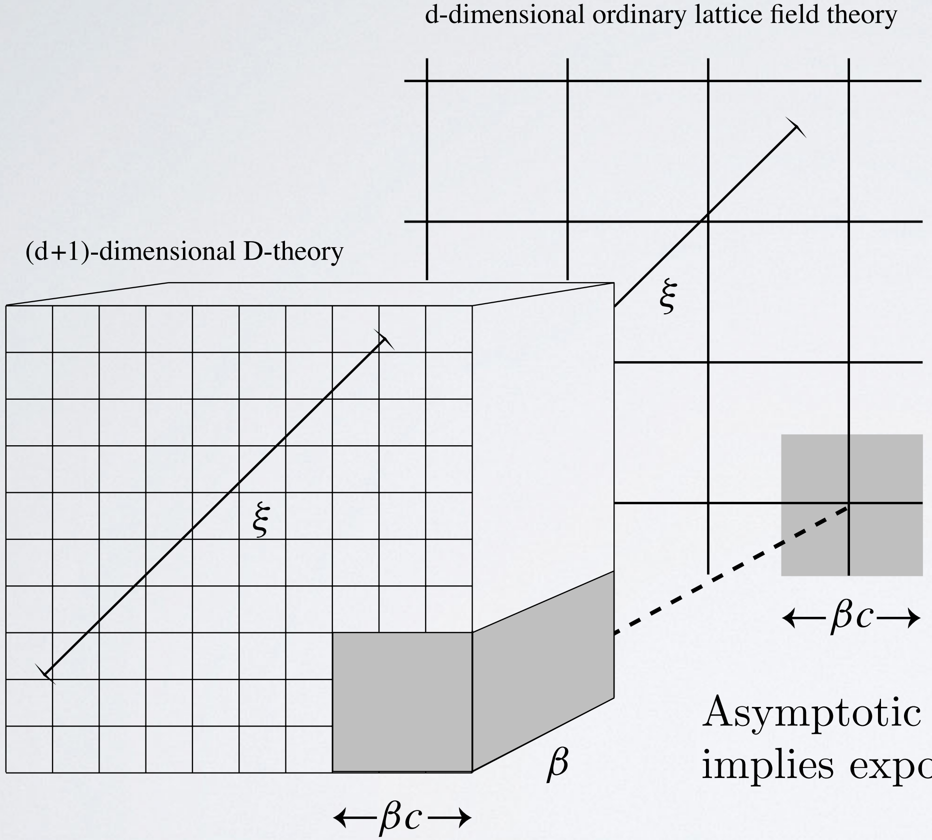
$$\{a_i, a_j^\dagger\} = \delta_{ij} \qquad \{b_i, b_j^\dagger\} = \delta_{ij}$$

Local Gaue Operators $\quad \Omega_{ij}(x) = a_i^\dagger(x) a_j(x) + \cdots$

For SU(3) QCD have SU(6) per Link Lie Algebra $\quad \begin{bmatrix} a_i a_j^\dagger & a_i b_j^\dagger \\ b_i a_j^\dagger & b_i b_j^\dagger \end{bmatrix}$

Hilbert Space is a large qubit array of color vectors in 4d space-time

D-THEORY: FERMIONS FOR GAUGE, SCALAR AND DIRAC LATTICE FIELDS!



$$Z = \mathrm{Tr}\exp(-\beta \hat{H}).$$

Original Motivation: MAKE
Easy Bosonic into Hard Fermionic
Theories may have smart cluster (aka worm) algorithms?

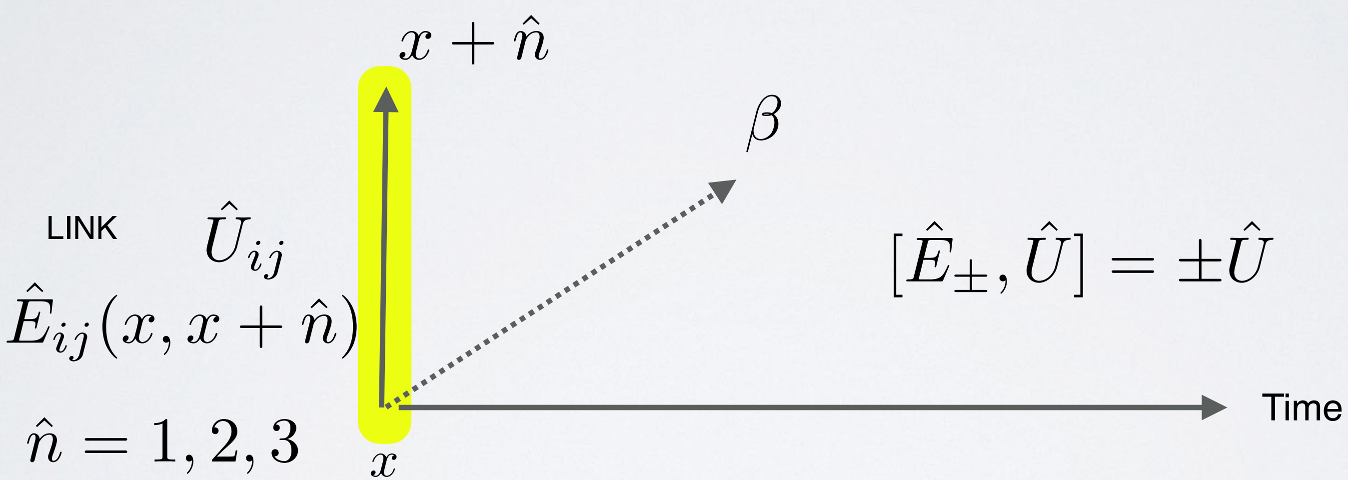
$$\xi = 1/am \sim \exp[c\beta/g^2]$$

Asymptotic Freedom of 2d spin and 4d gauge implies exponential dimensional reduction as $\beta \to \infty$

see Square-Lattice Heisenberg Antiferromagnet at Very Large Correlation Lengths
B. B. Beard, R. J. Birgeneau, M. Greven, and U.-J. Wiese Phys. Rev. Lett. **80**, 1742 (1998)

QC MINKOWSKI DYAMICS

- Change 4-th axis to real Time for Hamiltonian.
- Let short 5-th axis be field space (like "Domain Wall Fermion flavors)

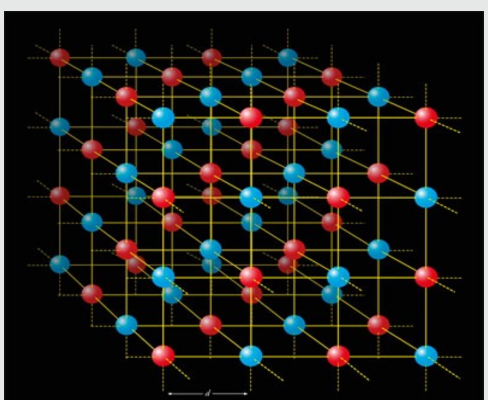


$$\hat{H}_0 = -\frac{1}{4g^2}\sum_{\Box} \mathrm{Tr}[\hat{U}_\Box + \hat{U}^\dagger_\Box] + g^2 \sum_{\langle x,y\rangle} Tr[\hat{E}^2_+(x,y) + \hat{E}^2_-(x,y] + \text{quarks}$$

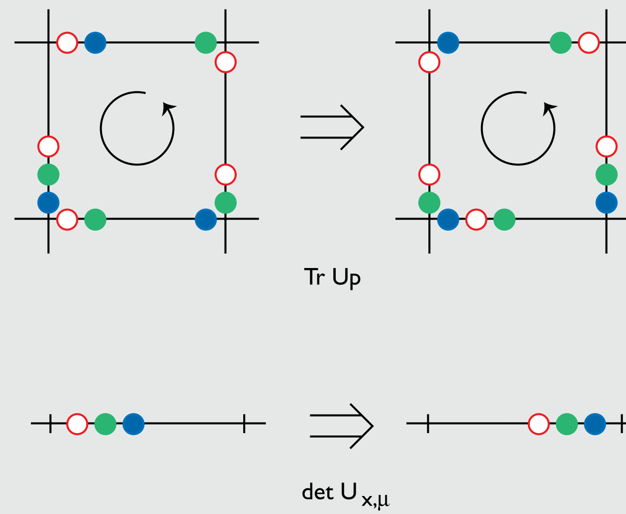
Quantum Links: THE QCD ABACUS

$$\hat{H} = \beta \sum_{x,\mu\neq\nu} \mathrm{Tr}[\hat{U}_{x,\mu}\hat{U}_{x+\hat{\mu},\nu}\hat{U}^{\dagger}_{x+\hat{\nu},\mu}\hat{U}^{\dagger}_{x,\nu}] + \sum_{x,\mu} [\det\hat{U}_{x,\mu} + \det\hat{U}^{\dagger}_{x,\mu}]$$

circa 2400 b.c Abacus

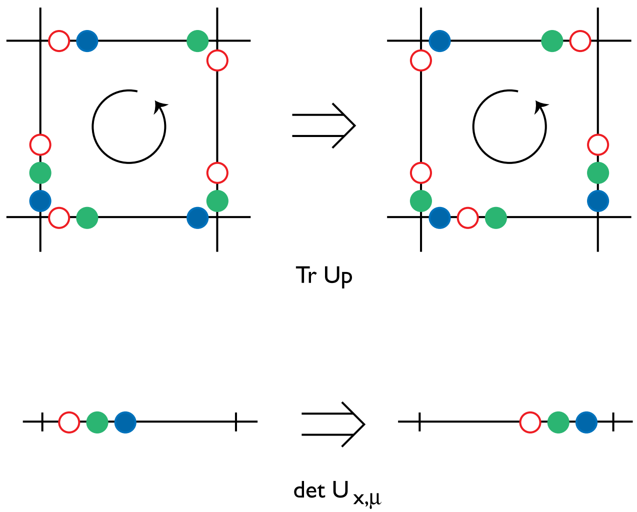


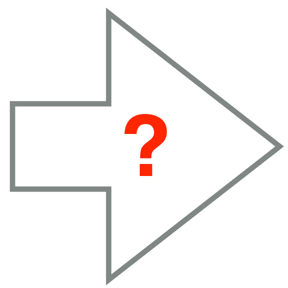
circa 20xx a.d.



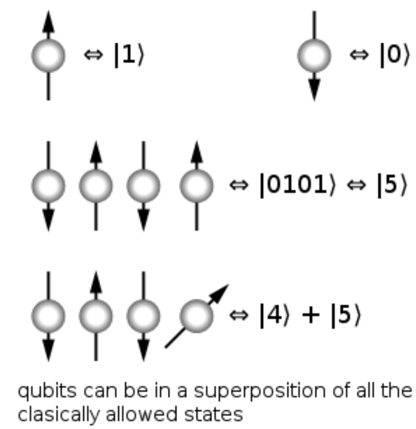
From Bits to Qubits ?

QCD Abacus*





Fermionic Qubit Algorithm ?



* See THE QCD ABACUS: A New Formulation for Lattice Gauge Theories R. C. Brower https://arxiv.org/abs/hep-lat/9711027 Lecture at "APCTP-ICTP Joint International Conference '97 on Recent Developments in Non-perturbative Method" May, 1997, Seoul, Korea. MIT Preprint CTP 2693.

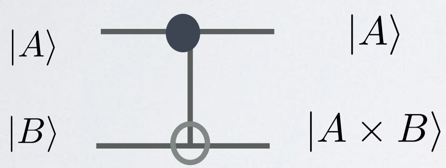
FERMIONIC QUBIT GATES

Each Fermion $a^\dagger a + a a^\dagger = 1$ $\quad aa = a^\dagger a^\dagger = 0$

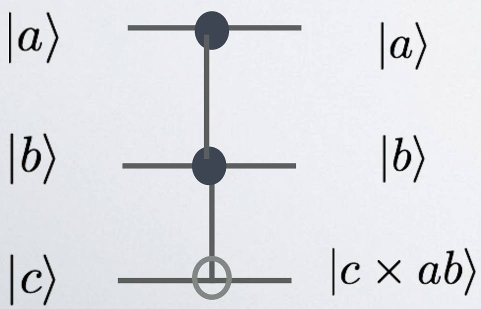
On Each qubit $a^\dagger(\alpha|1\rangle + \beta|0\rangle) = \beta|1\rangle$ $\quad a(\alpha|1\rangle + \beta|0\rangle) = \alpha|0\rangle$

$$a^\dagger + a \implies X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad (a + a^\dagger)^2 - 1 \implies H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$(a^\dagger + a)/i \implies Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad (2a^\dagger a - 1) \implies Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$$(1 - a^\dagger a) + a^\dagger a(b + b^\dagger) \implies \text{ContrNOT}$$



$$(1 - a^\dagger a b^\dagger b) + a^\dagger a b^\dagger b(c^\dagger + c) \implies \text{Toffoli}$$

Classical Reversible Computing Conserving Energy R. Landauer IBM J journal of Research and Development, vol. 5, pp. 183-191, 1961

WHAT ABOUT ANTI-SYMMETRIC FERMIONIC FOCK SPACE?

Step #1: PARA-STATISTICS:
D-Theory only require anti-commutator within each a's and b's set

0

$$[a_i^\dagger a_j, a_p^\dagger b_q] = [a_i^\dagger a_j, a_p^\dagger]\, b_q + a_p^\dagger [a_i^\dagger a_j, b_q]$$

$$\Longrightarrow \quad [E_L, U] = -E_L\, U$$

same for a –> b

Step #2; Jordan-Wigner:
Apply to Locally to each set of 3 a's and b's.

$$a_1^\dagger = \sigma_1^+ \quad , \quad a_2^\dagger = -\sigma_1^z \sigma_2^+ \quad , \quad a_3^\dagger = \sigma_1^z \sigma_2^z \sigma_3^+$$

same for a –> b

PARA-STATISTICS & JORDAN-WIGNER TO THE RESCUE

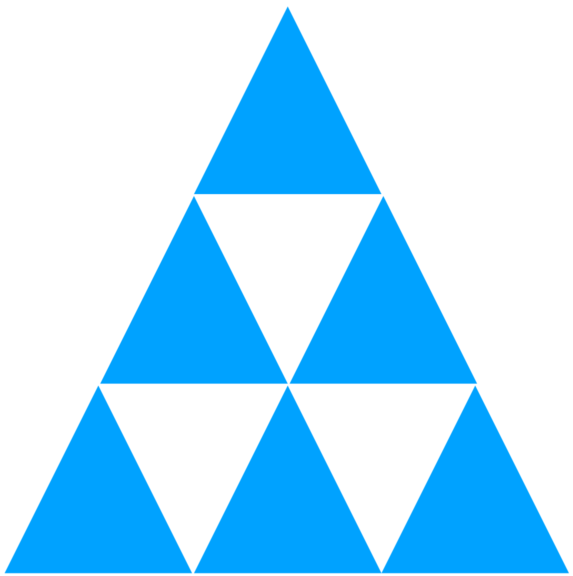
STARTING TO TEST REAL TIME QUBIT ALGORITHM FOR U(1) QUANTUM LINK GAUGE THEORY

Joint Work with D. Berenstein (UCSB), Cameron Valier (BU) & H. Kawai (BU)

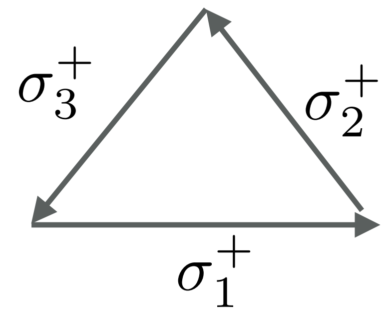
$$\hat{H} = \frac{e^2}{2} \sum_{links,s} (\sigma^z_s + \sigma^z_{s+1})^2 + \alpha \sum_{links,s} [\sigma^+_s \otimes \sigma^-_{s+1} + \sigma^-_s \otimes \sigma^+_{s+1}]$$
$$- \frac{1}{2e^2} \sum_{\triangle,s} [\sigma^+_s \otimes \sigma^+_s \otimes \sigma^+_s + \sigma^-_s \otimes \sigma^-_s \otimes \sigma^-_s]$$

Few very simple kernels in Trotter factorization into Gauge invariant Unitary operators with very few Qubit width

Choose 2 + 1 on U(1) Hamiltonian on a Triangular spacial lattice



To evaluate lattice: alternate between two coloring of triangles.



+ h.c

Total: about 15-20 consecutive gate operations (coherence time) per qubit per Trotter step

Estimate of current machines: 3 Trotter steps on Lattice

Extra dimension builds local field rep. from XYZ ferromagnetic chain

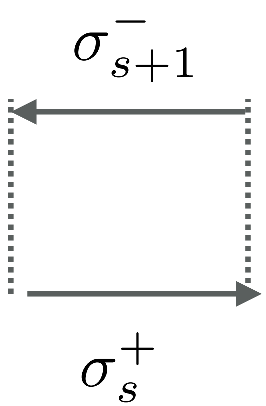
Using

$$\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+ = \frac{1}{2}(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y)$$

$$(\sigma^z \otimes 1 + 1 \otimes \sigma^z)^2 = 2(1 + \sigma^z \otimes \sigma^z)$$

The each links between two triangle are coupled by 2 Qubit ferromagnetic interaction operator to align them.

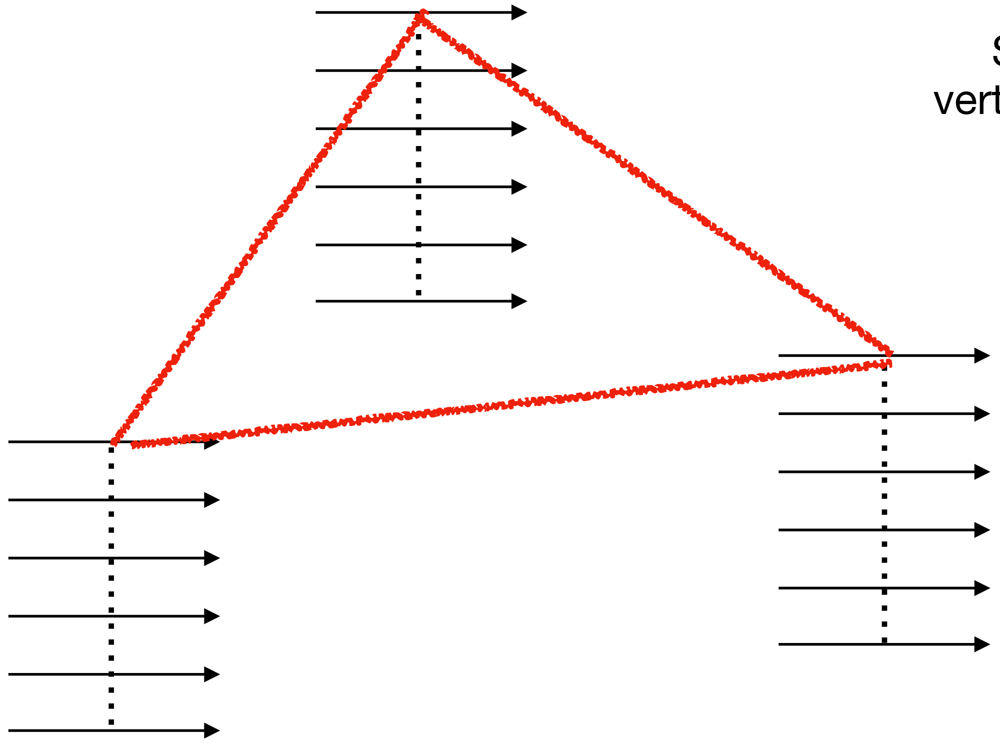
$$H_{align} \simeq -\alpha_{align} \sum (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y) - \beta_{align} \sum \sigma^z \otimes \sigma^z$$



XY couoling of gauge fixed extra dimension squares!

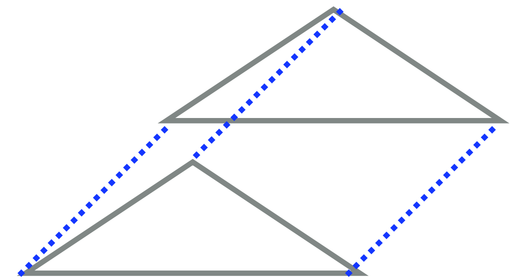
E^2 coupling term

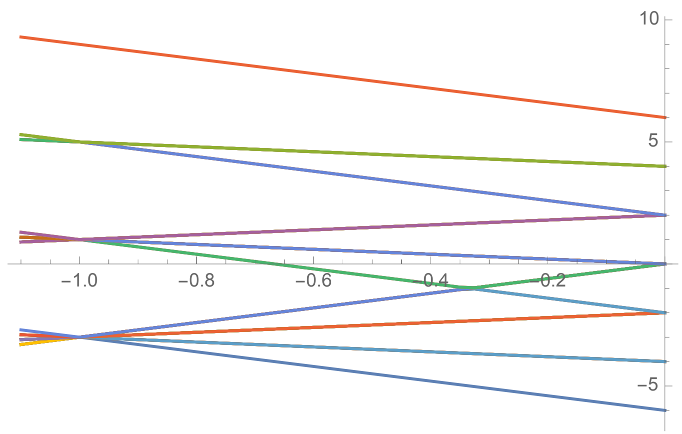
Plaquette



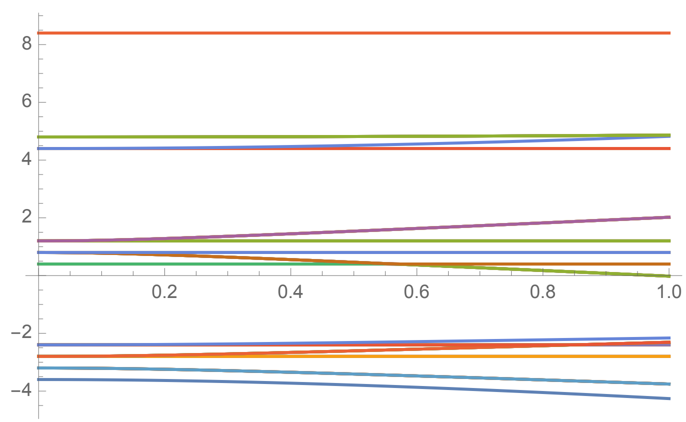
Still gauge invariant if broken vertically: few operations per qubit. (Not all to all)

Parameter fitting Two Triangle couple Hamiltonian needing 6 Qubits & eigenvalues of 64x64 matrices





Just XXZ piece: need to avoid level crossings, close to XXX is better



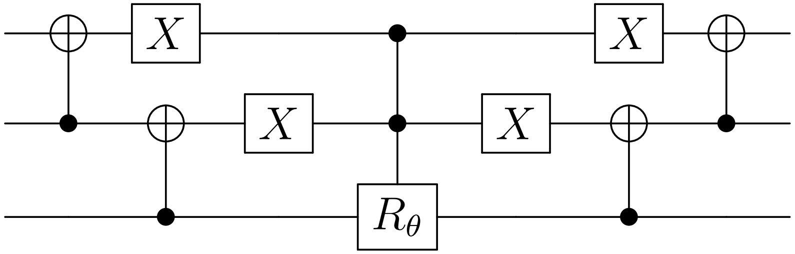
Together with plaquette operator: Gap persists: suggests simulation will not be too polluted by UV

$$\mathcal{H}_\triangle \propto \sigma^+ \otimes \sigma^+ \otimes \sigma^+ + \sigma^- \otimes \sigma^- \otimes \sigma^-$$

$$U_\triangle(t) = \exp(-itH_\triangle)$$

This is a rotation on a 2-plane of 8 dimensional Hilbert space (+++) rotating into (- - -).

Can be written in terms of a double control gate after some bit flips which need to be undone

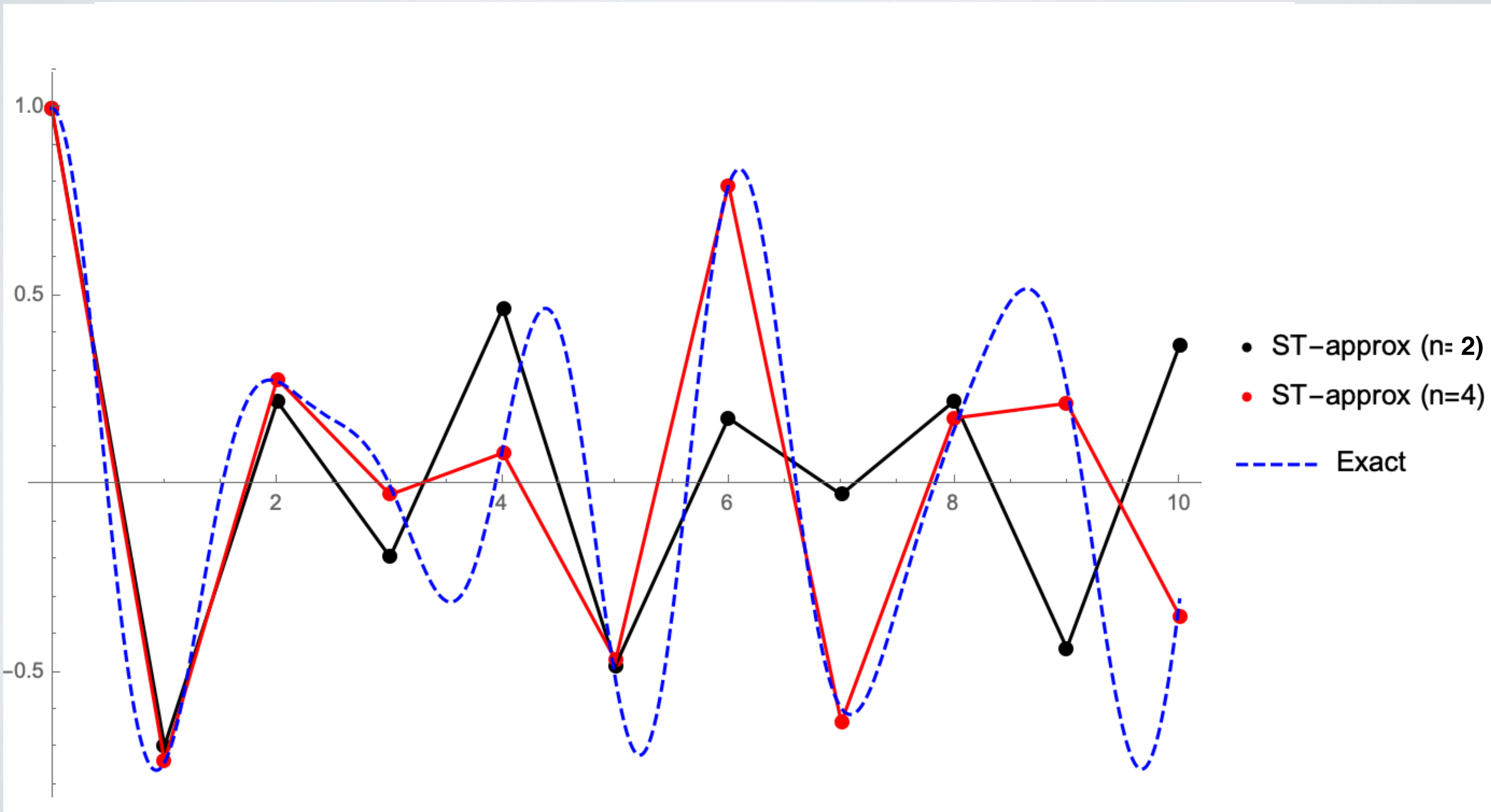


Depending on details of architecture: it can take anywhere between
5 computing cycles and 20 (depth).

For experts: May be done efficiently with ancillas if Toffoli gates available.

TROTTER VS EXACT 2 COUPLED TRIANGLES

$$Real\langle\Psi(0)|U(t)|\Psi(0)\rangle \quad \text{vs} \quad Real\langle\Psi(0)|[U_{xy}(t/n)U_E(t/n)U_\triangle(t/n)]^n|\Psi(0)\rangle$$



CONCLUSIONS/FUTURE STEPS

- **Can you bosonize the quark fields?**
- **Can you demonstrate truncated field yield correct continuum limit**
- **What is the complexity of kernels for SU(3) QCD?**
- **Algorithmic experiments in small systems:**

BACK UP SLIDES

SOME HISTORICAL REFERENCES ON EUCLIDEAN QUANTUM LINKS (AKA D-THEORY)

- *D. Horn, Finite Matrix Model with Continuous Local Gauge Inv. Phys. Lett. B100 (1981)*
- *P. Orland, D. Rohrlich, Lattice Gauge Magnets: Local Isospin From Spin Nucl. Phys. B338 (1990) 647*
- *S. Chandrasekharan, U-J Wiese Quantum links models: A discrete approach to gauge theories Nucl. Phys. B492 (1997)*
- *R. C. Brower, S. Chandrasekharan, U-J Wise , QCD as quantum link model, Phys. Rev D 60 (1999).*
- *R. C. Brower, The QCD Abacus: APCTP-ICPT Conference, Seoul, Korea, May (1997)*
- *R. C. Brower, S. Chandrasekharan, U-J Wiese, D-theory: Field quantization … discrete variable Nucl. Phys. B (2004)*

BUT WILSON'S LGT CAN NOT DO IT ALL

- **Problem 1: LGT on curved Riemann Manifolds*:**
  e.g. CFT on boundary of Anti-de-Sitter space.
  Quantum Finite Elements should solve this?

- **Problem 2: Chiral Gauge Theories#:**
  Many be topological methods in extra dimension?

- **Problem 3 : Minkowski Space is exponential hard @:**
  e.g.real time jets, scattering, sign problem of baryon chemical pot at finite T etc.

*https://arxiv.org/abs/1610.08587 Dirac Fermions on Simplicial Manifold
https://arxiv.org/abs/1803.08512 Phi 4th on Riemannian Manifold

https://arxiv.org/pdf/1809.11171 A non-Perturbative Definition of the Standard Model ?

@ This workshop?

Oak Ridge National Laboratory's 200 petaflop supercomputer

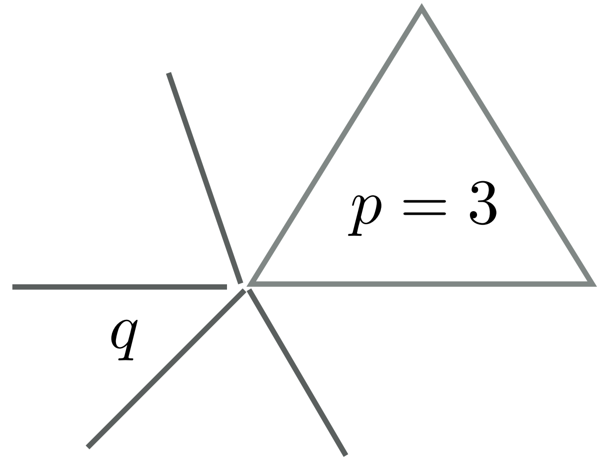
"Lattice Gauge Theory Machine" 200,000,000,000,000,000 Floats/sec
9,216 IBM POWER9 CPUs and 27648 NVIDIA GPUs
Each GPU has 5120 Cores and total of 580,608,000,000,000 transistors

UNIVERSALITY: Tools & Tricks

1. *Lattice: Discretization: (Graphs & Simplicial Structure)*
    - Subgroups of Space-Time Isometries
2. *"Field Truncation" (Group Algebra & IR expansion)*
    - Discrete Manifolds for Global/Gauge/Spinors
3. **"Dualities" (Hodge star & Dim Reduction)**
    - Gauge/Spin & Ads/CFT & Gauge/Gravity etc

EQUILATERAL TRIANGULATION

Triangle case p = 3



Preserves Discrete
Subgroup of Isometries

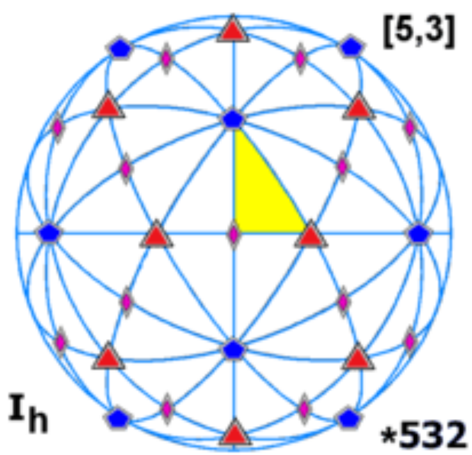
$$\frac{1}{p} + \frac{1}{q} > 1/2 \qquad \text{de Sitter} \quad \mathbb{S}^2 \qquad \text{vertex} \quad q = 3, 4, 5$$

$$\frac{1}{p} + \frac{1}{q} = 1/2 \qquad \text{flat} \quad \mathbb{T}^2 \qquad \text{vertex} \quad q = 6$$

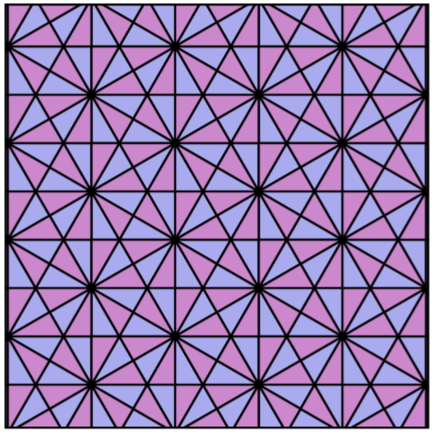
$$\frac{1}{p} + \frac{1}{q} < 1/2 \qquad \text{Hyperbolic} \quad \mathbb{A}dS^2 \qquad \text{vertex} \quad q = 7, 8, 9, \cdots$$

DISCRETE ISOMETRIES & THE TRIANGLE GROUP

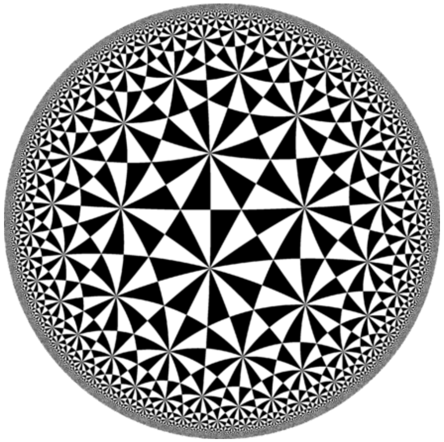
$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} \begin{cases} > \pi & \text{Postive curvature} \\ = \pi & \text{Zero curvature} \\ < \pi & \text{Negative Curvature} \end{cases}$$



$(2, 3, 5)$
120 element
Icosahedral in O(3)



$(2, 3, 6)$
Triangle Lattice
on Euclidean $\mathbb{R}^2$

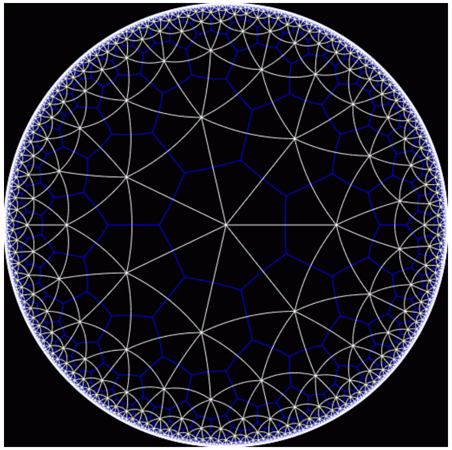


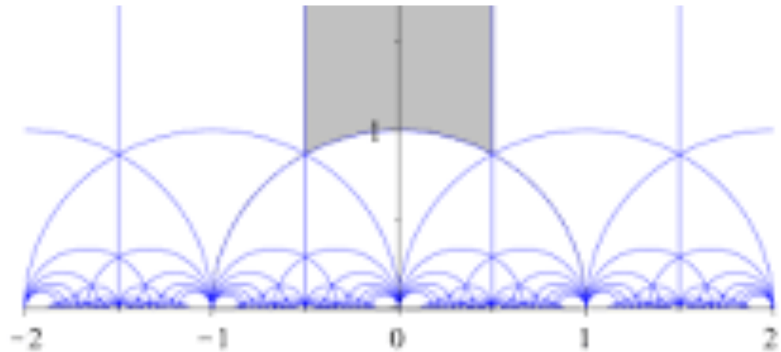
$(2, 3, 7)$
Subgroup of Modular
Group on $\mathbb{H}^2$

https://en.wikipedia.org/wiki/(2,3,7)_triangle_group

Hyperbolic (e.g. Poincare Disk) and Global AdS

q = 7



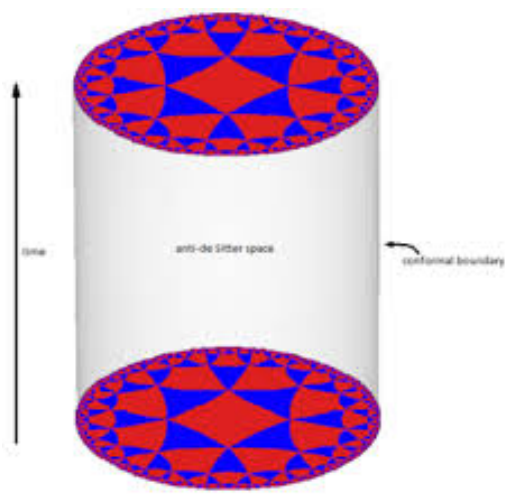


Triangle Group Tessellation: Preserve Finite subgroup of the Modular Group

1/2 + 1/3 + 1/q < 1

$$z \to \frac{az+b}{cz+d} \quad ad - bc = 1$$

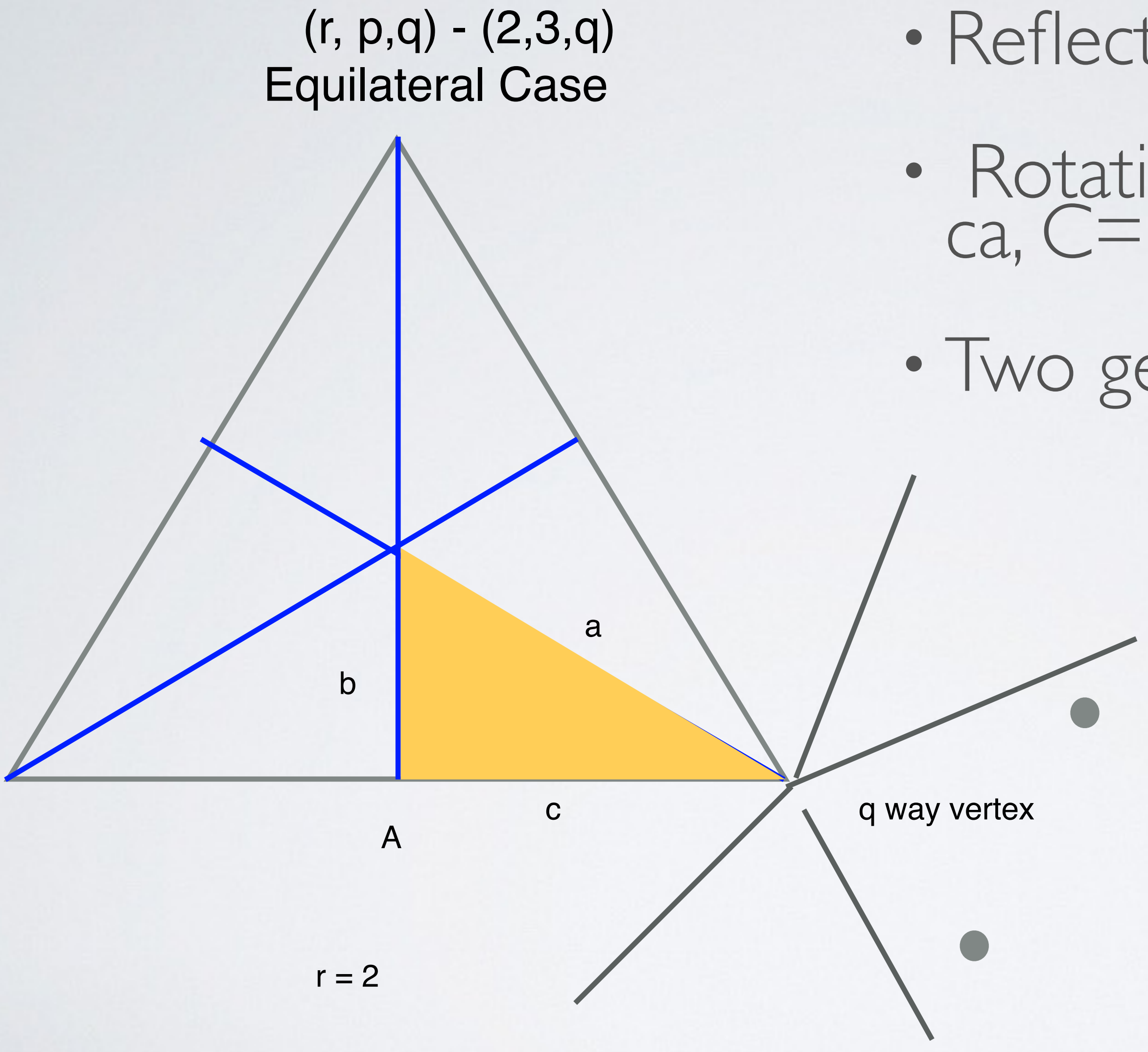
$$a, b, c, d \in \mathbb{Z} \; mod \; q$$



Are these Tessellation "Tensor Networks" ?
YES: See Daniel Harlow's Slide from Wednesday

Can we do QC lattice Field Theories in AdS?
Classical YES /QC Maybe

Triangle Group Tiling

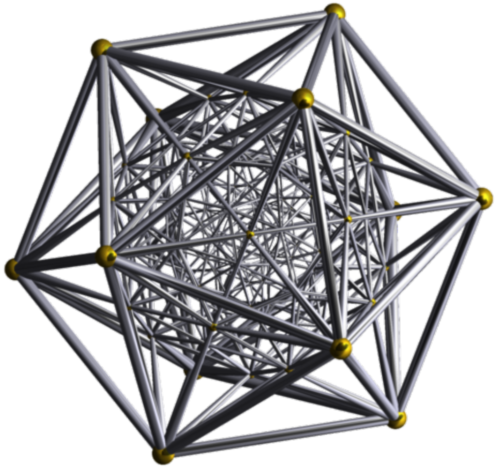


- Reflection: a, b, c
- Rotations: A = bc, B = ca, C= AB =ba
- Two generators

$$\Delta(p,q,r) = \{a,b,c | a^2 = b^2 = c^2 = A^p = B^q = C^r\}$$

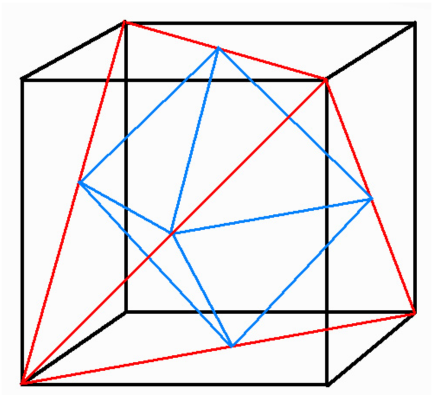
3 Spheres and 4D Radial Simplicial Lattices

$$\mathbb{S}^3 \Longrightarrow \mathbb{R} \times \mathbb{S}^3$$



Aristotle' s 2% Error!

$$(2\pi - 5ArcCos[1/3])/(2\pi) = 0.0204336$$



Fast Code Domains of
Regular 3D Grids on Refinement

The full symmetry group of the 600-cell is the Weyl group of H_4. This is a group of order 14400. It consists of 7200 rotations and 7200 rotation-reflections. The rotations form an invariant subgroup of the full symmetry group.

FROM QULINKS TO QUBITS

» ***Universally equivalent lattice fields theories***

- "Exact" Symmetries respected as well as possible

» ***Discretize Theory on any computer:***

- Classical: Discrete Lattice for space time (1st quantization)
- Quantum (Quibts): Discrete Fields (2nd quantization)

» ***Euclidean to Minkowski: Fermionic Bits to Qubits.***

» ***Test on IMB Q***

- All gauge kernel for U(1) pure gauge theory.