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Disconnected Loop Subtraction Methods

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** Speaker*

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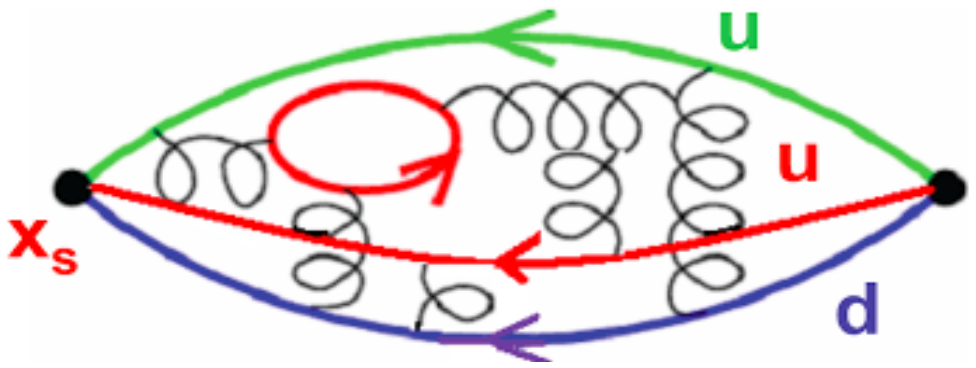
Baylor University PHYSICS

Disconnected Loops

- Disconnected loop effects in many physical quantities
- Hard to evaluate due to many matrix inversions needed to measure all the background fermionic degrees of freedom
- Treat the disconnected quark loops stochastically, through the use of noise vectors to project out operator contributions



Subtraction methods needed in order to reduce the variance of these noisy calculations

Noise Theory

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$$M_{ik}^{-1} = \lim_{N\to\infty}\frac{1}{N}\sum_n^N x_i^{(n)}\eta_k^{*(n)}$$

So we then only have to solve N linear equations to form the matrix inverse

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Expectation value of the trace is invariant under the addition of a traceless matrix

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Goal: Find a traceless matrix $\widetilde{M}^{-1}$ that has off diagonal elements as close to M^{-1} as possible

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Generalizing to any operator Θ

$$Tr(\Theta M^{-1}) = \frac{1}{N} \sum_n^N \eta^{(n)\dagger} \Theta \left(x^{(n)} - \tilde{x}^{(n)}\right) + Tr(\Theta \widetilde{M}^{-1})$$

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HFPOLY

The trace using the HFPOLY subtraction takes the following form:

$$Tr(\Theta M^{-1}) = \frac{1}{N}\sum_{n}^{N}\left(\eta^{(n)\dagger}\Theta\left[x^{(n)} - \tilde{x}_{eig}^{\prime(n)} - \left(\tilde{x}_{poly}^{(n)} - \tilde{x}_{eigpoly}^{\prime(n)}\right)\right]\right) + Tr\left(\Theta\gamma_5\widetilde{M}_{eig}^{\prime -1}\right) + Tr\left(\Theta\widetilde{M}_{poly}^{-1} - \Theta\gamma_5\widetilde{M}_{eigpoly}^{-1}\right)$$

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$$\boxed{\tilde{x}'^{(n)}_{eig} \equiv \gamma_5\widetilde{M}'^{-1}_{eig}\eta^{(n)} = \gamma_5\sum_{q}^{Q}\frac{1}{\lambda'^{(q)}}e'^{(q)}\left(e'^{(q)\dagger}\eta^{(n)}\right)}$$

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$$\tilde{x}^{(n)}_{poly} \equiv \widetilde{M}^{-1}_{poly}\eta^{(n)}$$

Inversion Algorithms

- MINRES-DR(m,k)[1]
  - Calculate the lowest Q eigenpairs of the Hermitian Wilson matrix, $M\gamma_5$, to be used in the HF-type subtraction methods
- GMRES-DR(m,k)[2]
  - Solve the first right hand side, and calculate the lowest Q eigenpairs of the Wilson matrix to be used in the ES subtraction method and projection
- GMRES-Proj[3]
  - Uses the k eigenvectors produced from GMRES-DR to accelerate the convergence of the remaining right hand sides

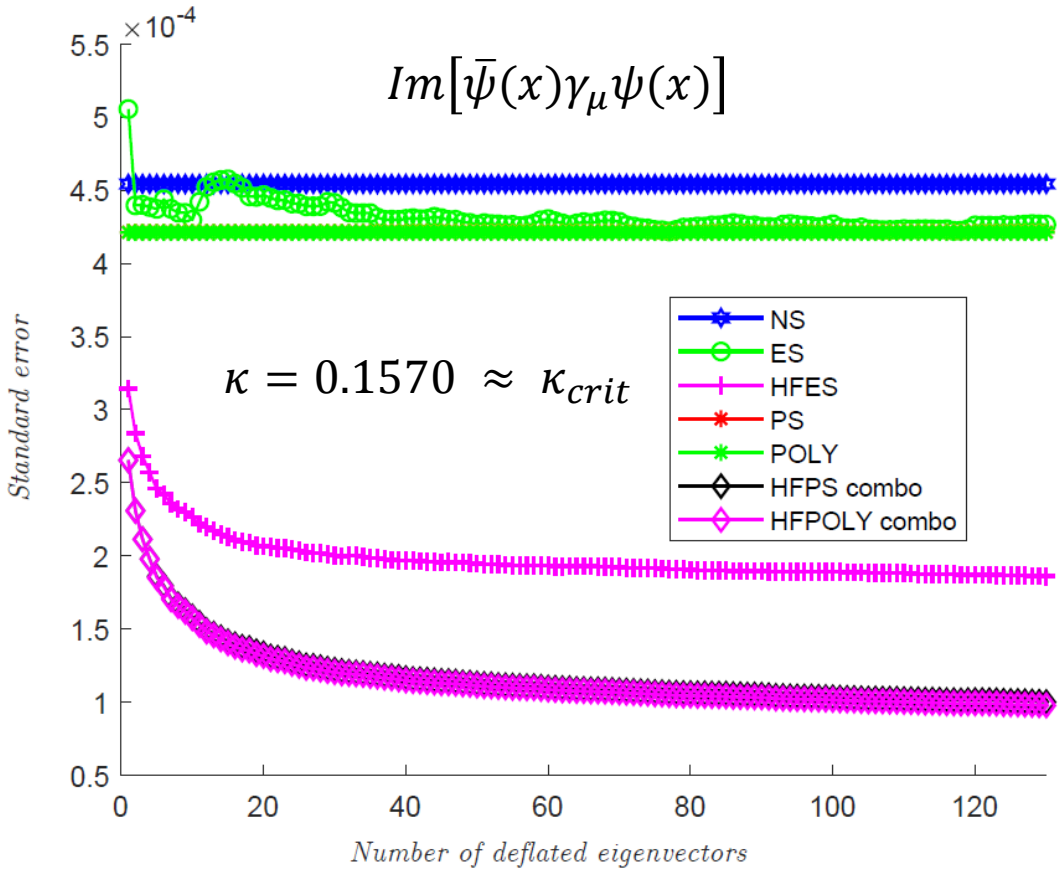
[1] A. Abdel-Raheim et. al., SIAM J. Sci. Comput. 32 (2010) 129
[2] R.B. Morgan, SIAM J. Sci. Comput. 24 (2002) 20
[3] arXiv:0707.0505v1

Standard Error: Quenched

24^3 x 32 lattice
$\beta = 6.0$

Standard error averaged over 10 configurations

Operator: Local Vector



Standard Error: Quenched

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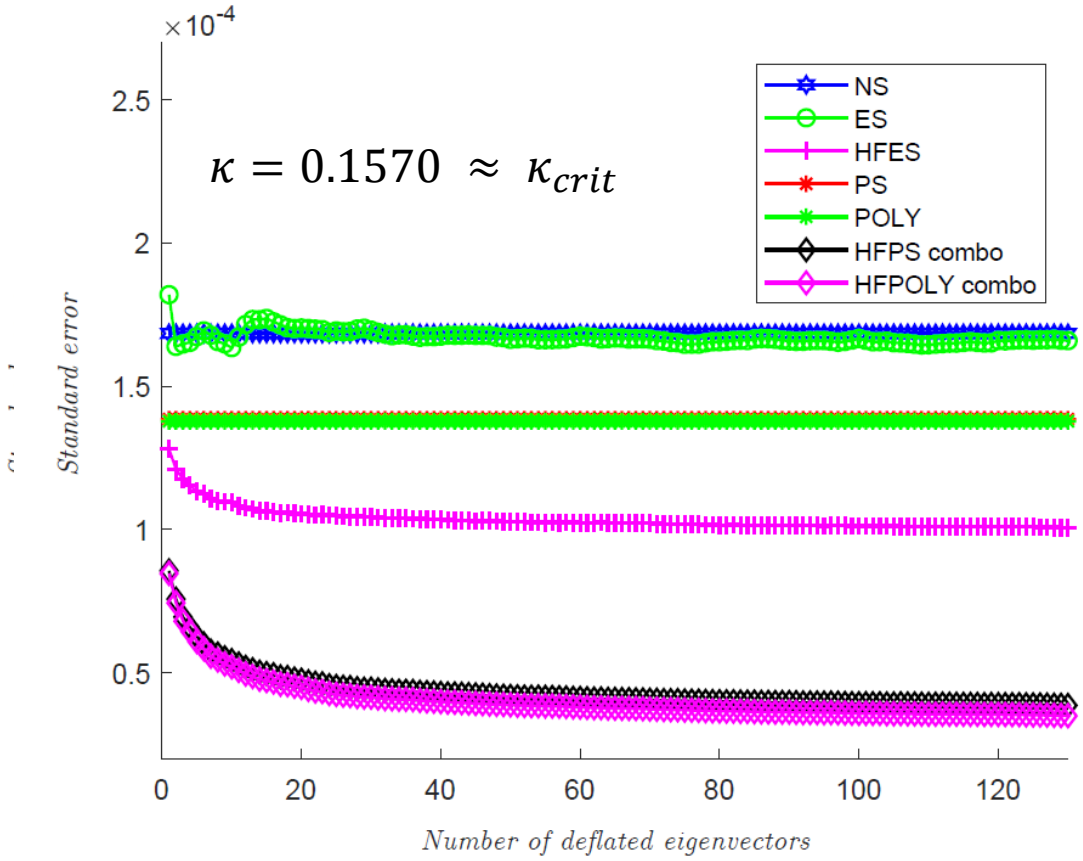
Operator: Point-Split Vector

$$\kappa Im[\bar{\psi}(x+a_\mu)(1+\gamma_\mu)U_\mu^\dagger(x)\psi(x)] - \kappa Im[\bar{\psi}(x)(1-\gamma_\mu)U_\mu(x)\psi(x+a_\mu)]$$



Standard Error: Quenched

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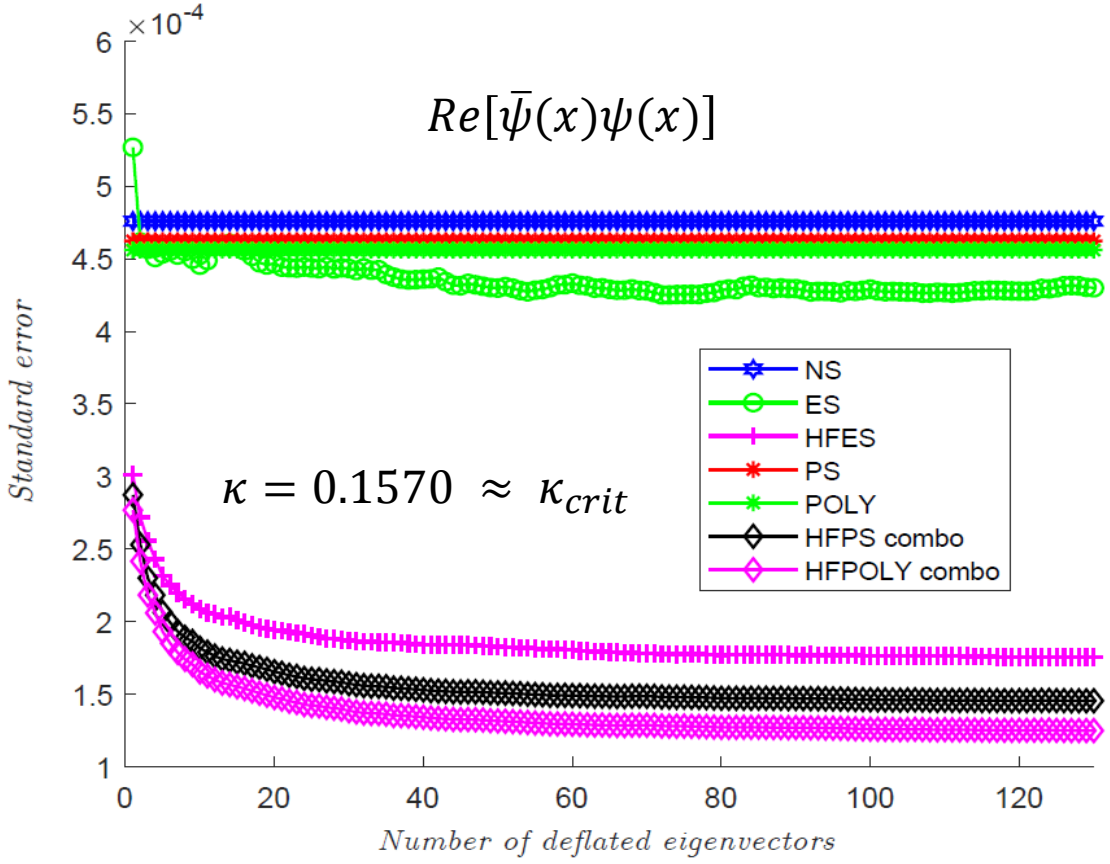
Operator: Scalar



Relative Efficiencies at κ_{crit}

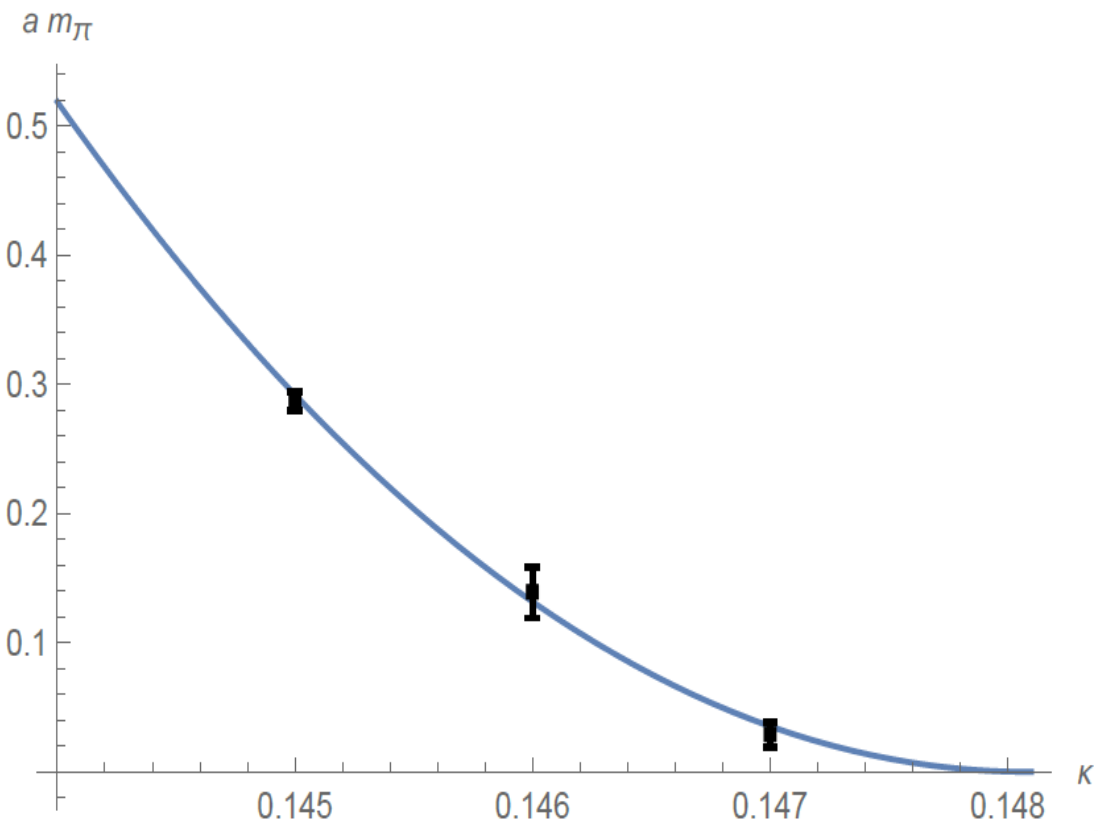
	Scalar		Local		Point-Split	
	vs. NS	**vs. PS**	**vs. NS**	**vs. PS**	**vs. NS**	**vs. PS**
POLY	8.9 %	2.8%	16.4%	0.1%	49.5%	1.1%
HFES	634%	593%	496%	413%	180%	89.2%
HFPS	972%	911%	1970%	1680%	1800%	1180%
HFPOLY	1350%	1270%	2070%	1770%	2200%	1470%

$$\mathrm{R.E} \equiv \left(\frac{1}{\delta y^2} - 1\right) \times 100,$$ where δy^2 is the relative variance

Subtraction Using MILC Configurations

16^3 x 48 lattice
$\beta = 5.8$
$m_\pi = 306.9(5)$ MeV [4]

Analysis of pion correlators over ten configurations determined the value of the hopping parameter to be $\kappa \approx 0.1453$



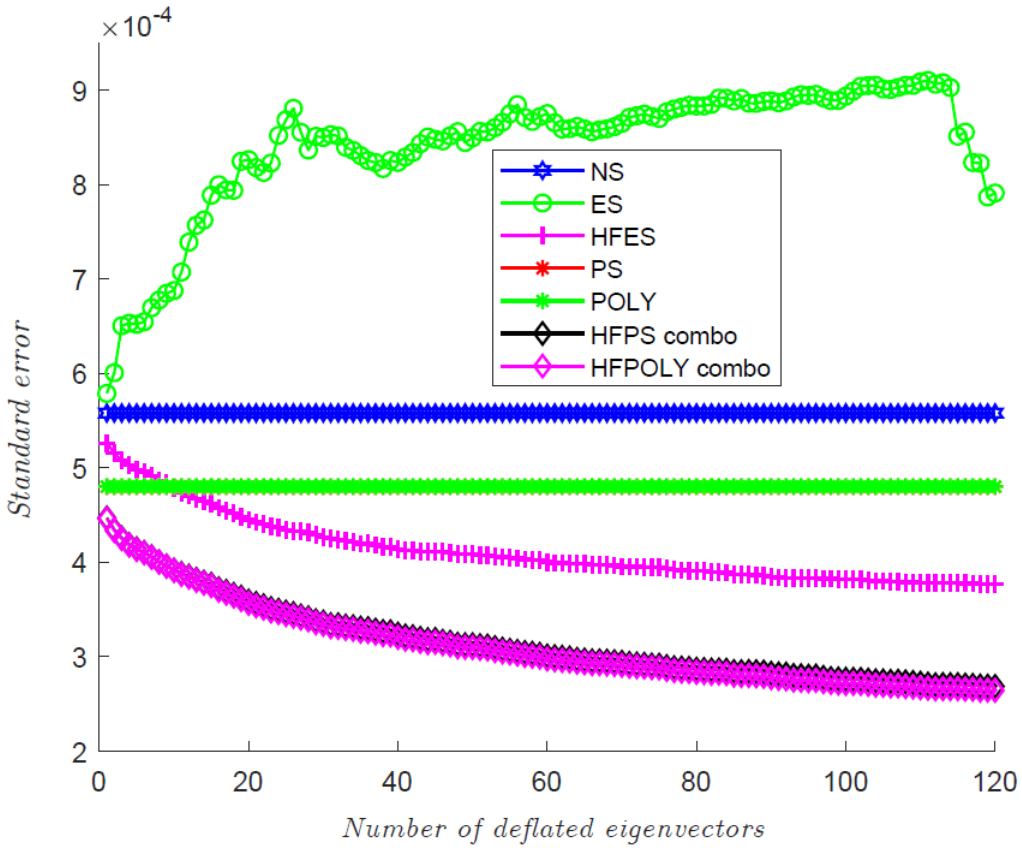
[4] A. Bazavov, et. al., MILC Collaboration, Phys. Rev. D. 87 (2013) 054505

Standard Error: Dynamical

Approximate correspondence to a quenched $\kappa \approx 0.1567$[5]

Standard error averaged over 10 configurations

Operator: Local Vector



[5] S. Cabasino et. al., Phys. Lett. B 258 (1991) 195

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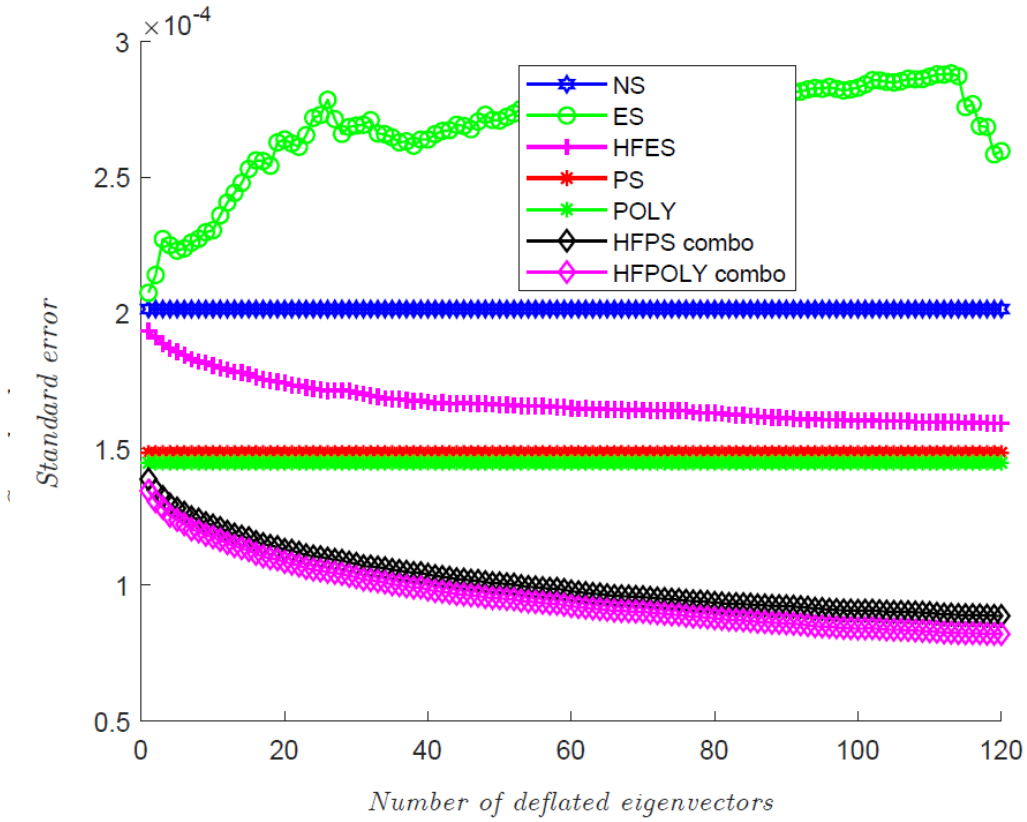
Approximate correspondence to a quenched $\kappa \approx 0.1567$[5]

Standard error averaged over 10 configurations

Operator: Point-Split Vector



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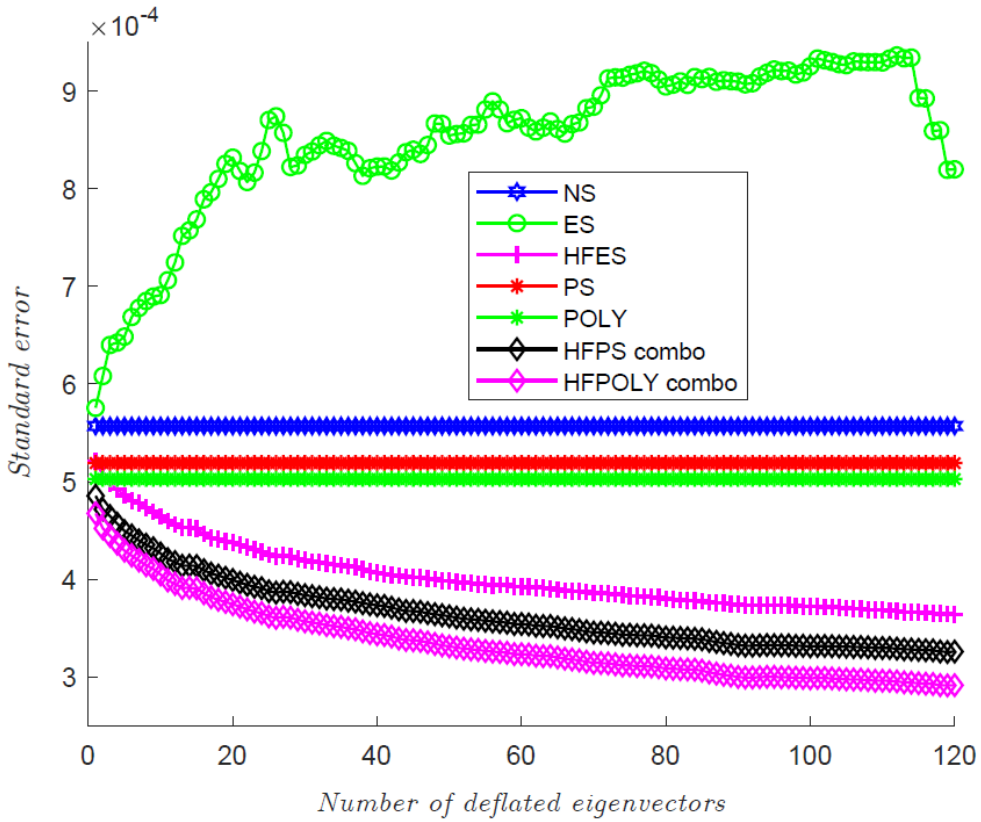

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Relative Efficiencies: Dynamical

	Scalar		Local		Point-Split	
	vs. NS	**vs. PS**	**vs. NS**	**vs. PS**	**vs. NS**	**vs. PS**
POLY	22.8%	6.6%	35.0%	-0.1%	93.4%	5.2%
HFES	134%	104%	120%	62.4%	60.0%	-13.2%
HFPS	192%	153%	332%	220%	417%	181%
HFPOLY	260%	217%	436%	230%	505%	229%

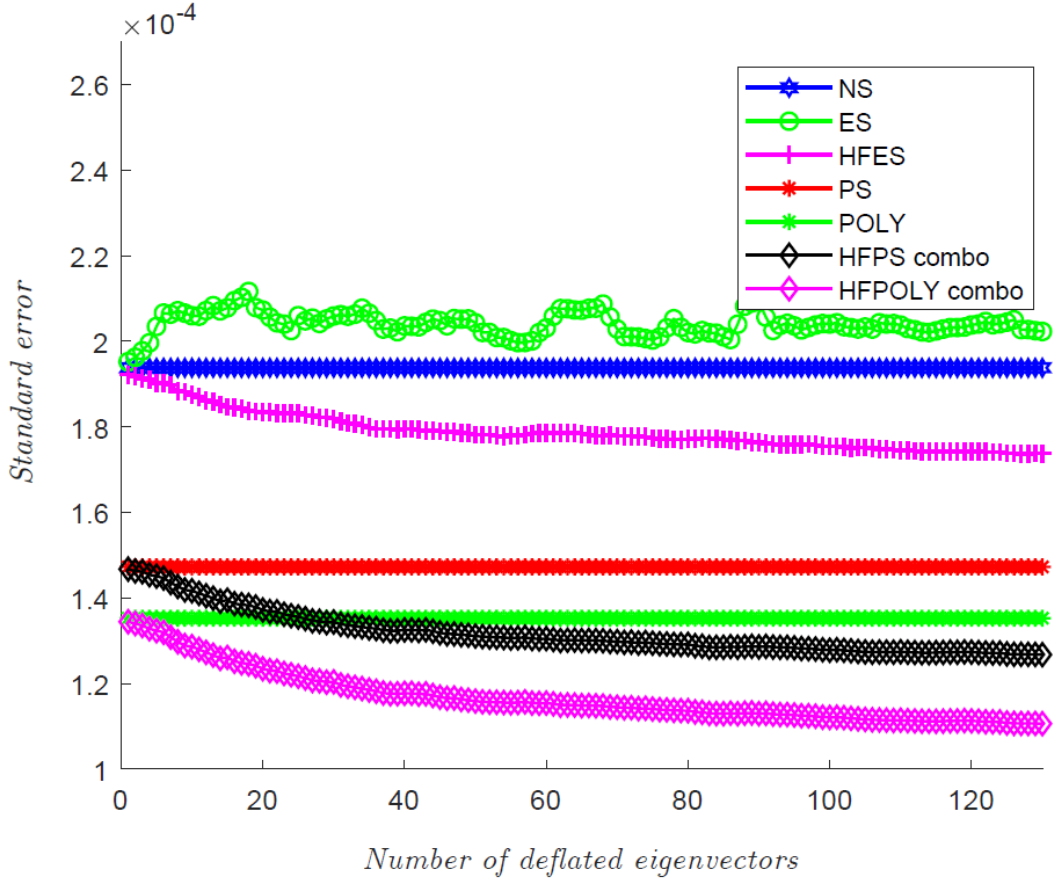
Summary

- Deflation type algorithms using the eigenmodes of the Hermitian Wilson matrix display a large variance reduction in comparison to Perturbative Subtraction near zero quark mass
- Low eigenmode dominance in the local vector and scalar sectors near zero quark mass
- Deflation saturation is achieved at approximately 30 eigenmodes
- As pion masses decreases towards the physical point, we expect even better reduction in the variance due to deflation

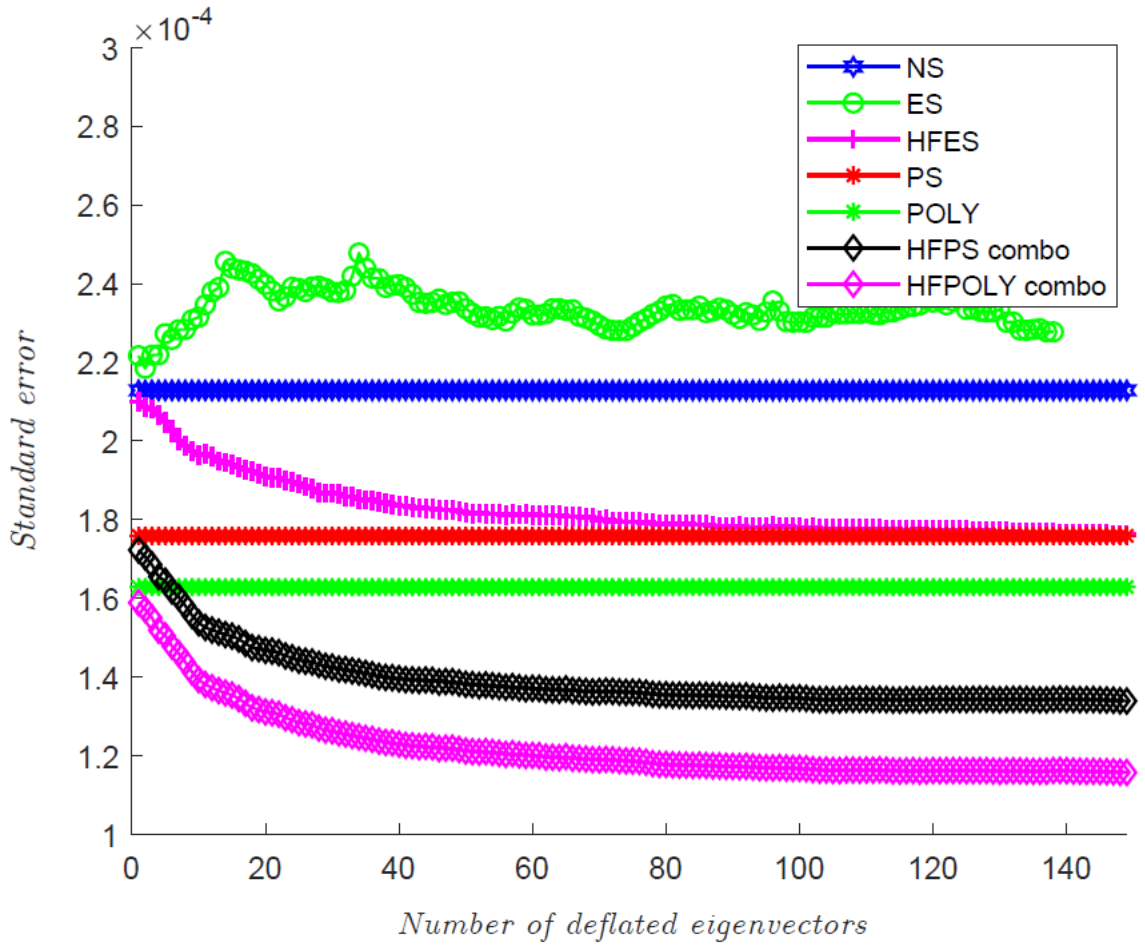
Acknowledgments

Thank you to the Organizing Committee of Lattice 2019 for providing accommodations, Doug Toussaint, Carlton DeTar, Jim Hetrick for their help in obtaining the MILC configurations, and Abdou Abdel-Raheim, Victor Guerrero and Paul Lashomb for their help in this study. This work was partially supported through the Baylor University Research Committee and the Texas Advanced Super Computing Center

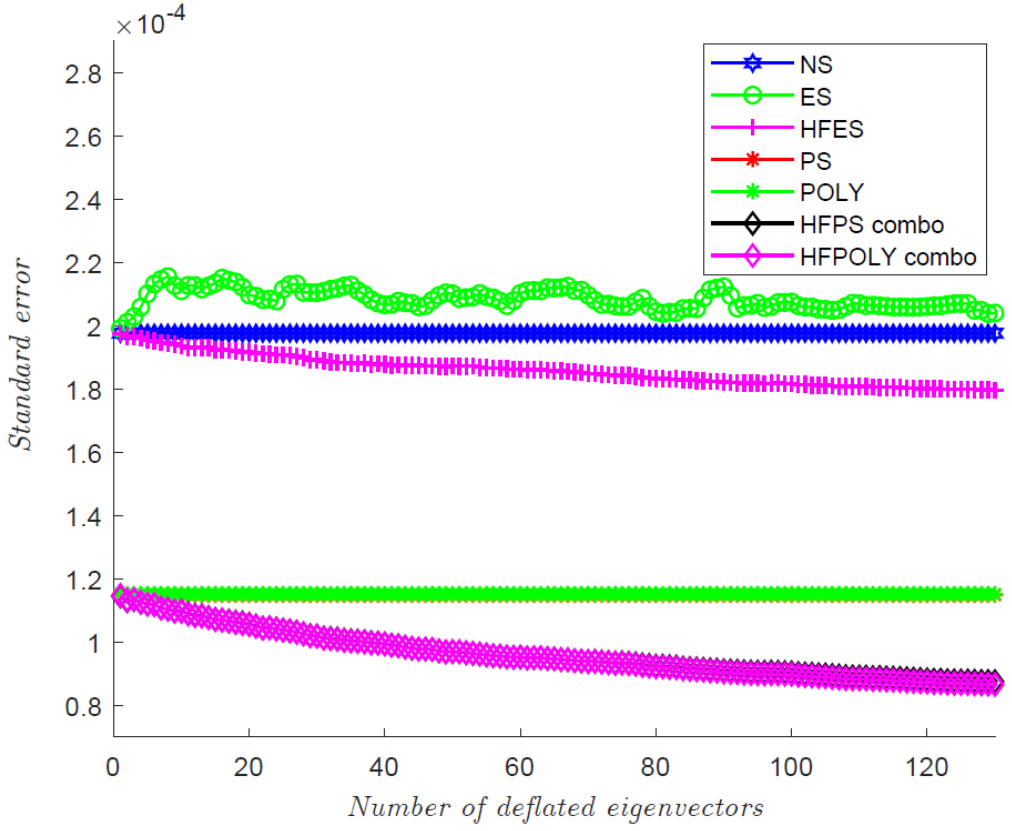
Quenched κ = 0.1550 scalar



Quenched κ = 0.1560 scalar



Quenched κ = 0.1550 local vector



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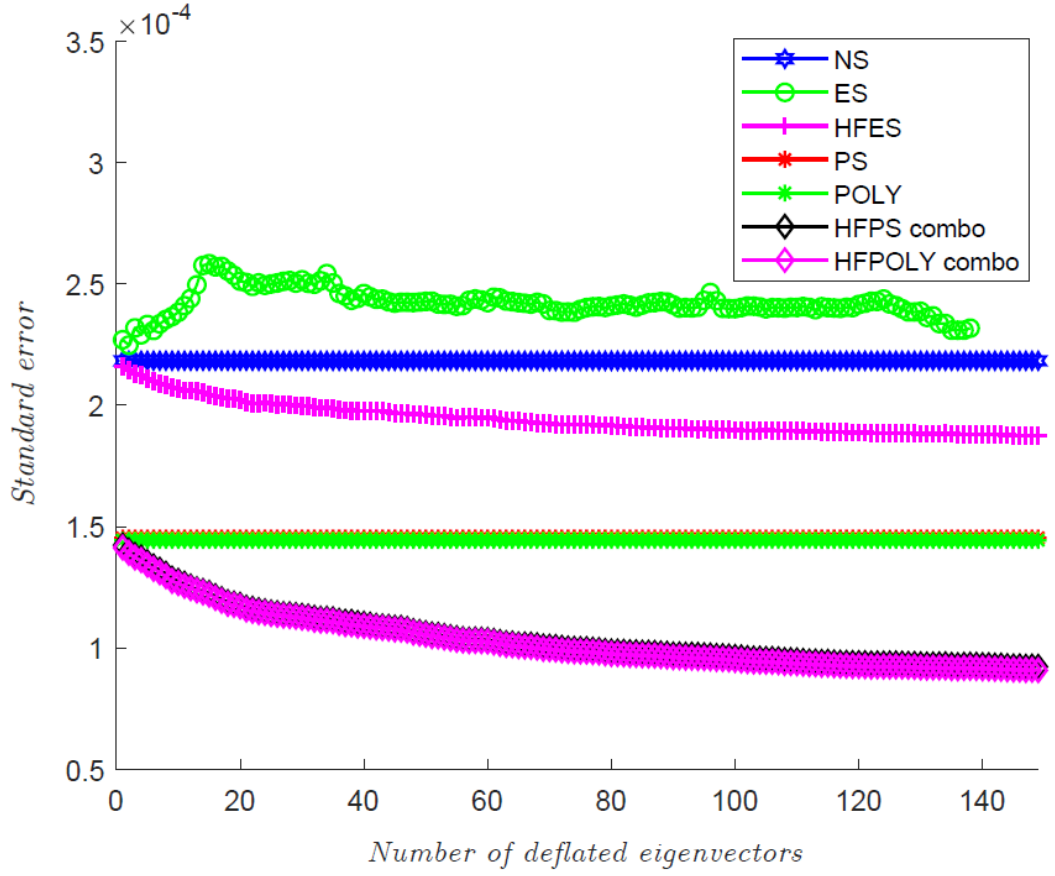
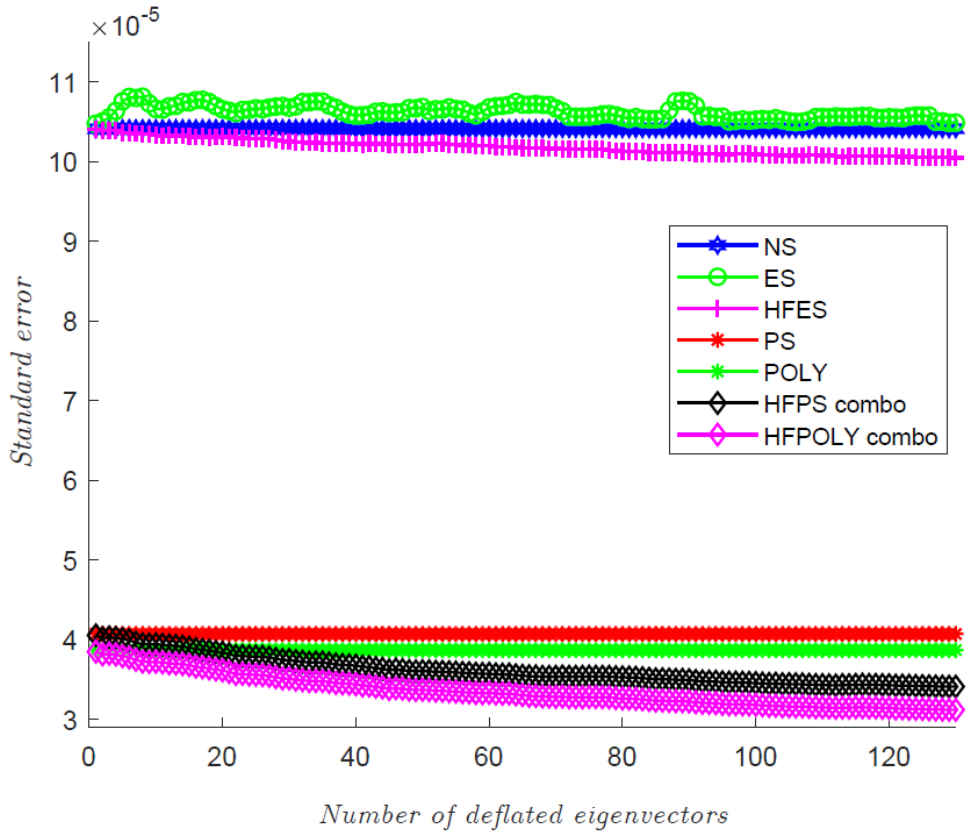


Quenched κ = 0.1550 Point-Split vector



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