QED$_{\infty}$ in muon $g - 2$, hadron spectroscopy, and beyond

Luchang Jin
University of Connecticut / RIKEN BNL Research Center

Thomas Blum (UConn / RBRC), Norman Christ (Columbia), Masashi Hayakawa (Nagoya), Xu Feng (Peking U), Taku Izubuchi (BNL / RBRC), Chulwoo Jung (BNL), Christoph Lehner (Regensburg), Cheng Tu (UConn)

and

RBC-UKQCD collaborations

June 20, 2019

37th international conference on lattice field theory (LATTICE 2019)
Hilton Hotel Wuhan Riverside, Wuhan, China
The RBC & UKQCD collaborations

**BNL and BNL/RBRC**
- Yasumichi Aoki (KEK)
- Taku Izubuchi
- Yong-Chull Jang
- Chulwoo Jung
- Meifeng Lin
- Aaron Meyer
- Hiroshi Ohki
- Shigemi Ohta (KEK)
- Amarjit Soni

**University of Connecticut**
- Bigeng Wang
- Tianle Wang
- Yidi Zhao

**Edinburgh University**
- Peter Boyle
- Luigi Del Debbio
- Felix Erben
- Vera Gülpers
- Tadeusz Janowski
- Julia Kettle
- Michael Marshall
- Fionn Ó hÓgáin
- Antonin Portelli
- Tobias Tsang
- Andrew Yong
- Azusa Yamaguchi

**Columbia University**
- Ryan Abbot
- Norman Christ
- Duo Guo
- Christopher Kelly
- Bob Mawhinney
- Masaaki Tomii
- Jiqun Tu

**Masashi Hayakawa (Nagoya)**

**KEK**
- Julien Frison

**University of Liverpool**
- Nicolas Garron

**MIT**
- David Murphy

**Peking University**
- Xu Feng

**University of Regensburg**
- Christoph Lehner (BNL)

**University of Southampton**
- Nils Asmussen
- Jonathan Flynn
- Ryan Hill
- Andreas Jüttner
- James Richings
- Chris Sachrajda

**Stony Brook University**
- Jun-Sik Yoo
- Sergey Syritsyn (RBRC)
1. **Introduction**

2. QED$_\infty$ in Muon $g - 2$
   
   Long-distance contribution to the HLbL in position space from the $\pi^0$ pole

3. QED$_\infty$ in hadron spectroscopy (X. Feng and L. Jin 2018)
   
   Pion mass splitting with the infinite volume reconstruction method

4. Conclusion
Muon $g - 2$: experiments

\[ \vec{\mu} = -g \frac{e}{2m} \vec{s} \]

World Average dominated by BNL

\[ a_\mu = (11659208.9 \pm 6.3) \times 10^{-10} \]

In comparison, for electron

\[ a_e = (11596521.8073 \pm 0.0028) \times 10^{-10} \]

- Fermilab E989 (0.14 ppm) Almost 4 times more accurate then the previous experiment.

- J-PARC E34 also plans to measure muon $g - 2$ with similar precision.
## Muon $g - 2$: theory

<table>
<thead>
<tr>
<th></th>
<th>$a_\mu \times 10^{10}$</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED incl. 5-loops</td>
<td>11658471.9 ± 0.0</td>
<td>Aoyama, et al, 2012</td>
</tr>
<tr>
<td>Weak incl. 2-loops</td>
<td>15.4 ± 0.1</td>
<td>Gnendiger et al, 2013</td>
</tr>
<tr>
<td>HVP</td>
<td>692.5 ± 2.7</td>
<td><em>Talk: C. Lehner (Mon 14:20)</em></td>
</tr>
<tr>
<td>HVP NLO&amp;NNLO</td>
<td>−8.7 ± 0.1</td>
<td>C. Lehner et al (RBC-UKQCD), 2018</td>
</tr>
<tr>
<td>HLbL</td>
<td>10.3 ± 2.9</td>
<td>FJ17</td>
</tr>
</tbody>
</table>

|                      |                         | Hadronic Models, “Consensus”    |
| Standard Model       | 11659181.3 ± 4.0        |                                 |
| Experiment           | 11659208.9 ± 6.3        | E821, The $g - 2$ Collab. 2006  |
| Difference (Exp-SM)  | 27.6 ± 7.5              |                                 |

- More than 3 standard deviations due to mistake in the highly sophisticated perturbative calculation, inaccuracy of $e^+e^- \rightarrow \text{hadrons}$ experiments, incorrect hadronic model, or new physics?
Dispersive approach with $e^+e^- \rightarrow \text{hadrons}$ experimental data is very successful for HVP.

HLbL is much more complicated. Anyway, lots of works have been done in this direction.

- Pion pole, pion box, rescattering: G. Colangelo, M. Hoferichter, B. Kubis, M. Procura, P. Stoffer.
- Light by light scattering sum rule: L. Dai, M. Pennington.
- Schwinger’s sum rule: F. Hagelstein, V. Pascalutsa.
Confinement of quarks*

Kenneth G. Wilson

Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 12 June 1974)

A mechanism for total confinement of quarks, similar to that of Schwinger, is defined which requires the existence of Abelian or non-Abelian gauge fields. It is shown how to quantize a gauge field theory on a discrete lattice in Euclidean space-time, preserving exact gauge invariance and treating the gauge fields as angular variables (which makes a gauge-fixing term unnecessary). The lattice gauge theory has a computable strong-coupling limit; in this limit the binding mechanism applies and there are no free quarks. There is unfortunately no Lorentz (or Euclidean) invariance in the strong-coupling limit. The strong-coupling expansion involves sums over all quark paths and sums over all surfaces (on the lattice) joining quark paths. This structure is reminiscent of relativistic string models of hadrons.

- Discrete lattice usually corresponds to hard cut off in momentum space.
- Hard cut off regularization is said to break gauge invariance, while lattice does not.
- Position space formulation can provide new insight.
## Muon g − 2: theory

<table>
<thead>
<tr>
<th></th>
<th>$a_{\mu}$</th>
<th>$\times 10^{10}$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED incl. 5-loops</td>
<td>11658471.9</td>
<td>± 0.0</td>
<td>Aoyama, et al, 2012</td>
</tr>
<tr>
<td>Weak incl. 2-loops</td>
<td>15.4</td>
<td>± 0.1</td>
<td>Gnendiger et al, 2013</td>
</tr>
<tr>
<td>HVP</td>
<td>692.5</td>
<td>± 2.7</td>
<td></td>
</tr>
<tr>
<td>HVP NLO&amp;NNLO</td>
<td>−8.7</td>
<td>± 0.1</td>
<td>FJ17</td>
</tr>
<tr>
<td>HLbL</td>
<td>7.4</td>
<td>± 6.6</td>
<td>Talk: T. Blum (Tue 16:30)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Lattice QCD, RBC-UKQCD</td>
</tr>
<tr>
<td>Standard Model</td>
<td>11659178.5</td>
<td>± 7.1</td>
<td></td>
</tr>
<tr>
<td>Experiment</td>
<td>11659208.9</td>
<td>± 6.3</td>
<td>E821, The $g − 2$ Collab. 2006</td>
</tr>
<tr>
<td>Difference (Exp-SM)</td>
<td>30.4</td>
<td>± 9.5</td>
<td></td>
</tr>
</tbody>
</table>

- More than 3 standard deviations due to mistake in the highly sophisticated perturbative calculation, inaccuracy of $e^+e^- \rightarrow$ hadrons experiments, under estimating sys/stat error in lattice calculation, or **new physics**?
1. Introduction

2. QED$_\infty$ in Muon $g - 2$
   
   Long-distance contribution to the HLbL in position space from the $\pi^0$ pole

3. QED$_\infty$ in hadron spectroscopy (X. Feng and L. Jin 2018)
   
   Pion mass splitting with the infinite volume reconstruction method

4. Conclusion
- We start the calculation by simulating both the QCD part and the QED part on the lattice within a finite volume using the $QED_L$ formalism. [T. Blum et al, 2014,2016,2017]

- Mainz’s group first demonstrated that it is possible to efficiently calculate the QED part semi-analytically in the infinite volume. [N. Asmussen et al, LATTICE2016]

- We can improve the infinite volume QED part to reduce the discretization error significantly with a subtracted QED kernel. [T. Blum et al, 2017]

At physical point, $\pi^0$ is very light, therefore:

- Large volume is needed.

- Large statistical error from the long distance region, especially for the disconnected diagrams. [L. Jin et al, 2016] [J. Bijnens and J. Relefors, 2016]
Norman’s initial idea for calculating the long-distance part of HLbL from $\pi^0$ pole.
(Norman’s talk at HLbL workshop in UCONN, 2018/03/13)

The hadronic 4-point function in the long-distance region can be calculated with two three-point functions, which can be directly calculated in a modest size lattice.
We can rotate the modest size lattice so its time direction is aligned with the pion propagating direction.

Only need to measure the $\pi^0 \rightarrow \gamma\gamma$ three-point function with zero momentum pion.

We only need the following three-point function for $\vec{p} = 0$ pion.

$$H_{\mu,\nu}(x - y) = \langle 0 | J_\mu(x) J_\nu(y) | \pi^0 \rangle$$
• UCONN graduate student Cheng Tu did the calculation.

• 32ID: $32^3 \times 64$, $a^{-1} = 1.015$ GeV, $M_\pi = 142$ MeV.

• $R_{\text{max}} = \max(|x-y|, |x-y'|, |y-y'|)$. Reverse partial sum plotted.

\[ a_\mu \times 10^{10} \]

\[ R_{\text{max}} (\text{fm}) \]
The three-point function: $\pi^0 \rightarrow \gamma \gamma$

$$H_{\mu,\nu}(x - y) = \langle 0 | J_\mu(x) J_\nu(y) | \pi^0 \rangle$$

- We calculated it using one point source propagator from $y$ and wall source propagators separated by enough distance from both $x$ and $y$ to create the pion.

- We averaged it within a configuration, saved it to disk, then performed further contractions.

- $\pi^0 \rightarrow \gamma \gamma$ decay width from a coordinate-space method.
  Poster: X. Feng (Tue 17:50).

- Yidi Zhao and Norman Christ’s very novel calculation $\pi^0 \rightarrow e^+ e^-$, which serves as the starting point for a more adventurous calculation $K_L \rightarrow \mu^+ \mu^-.$
  Talk: N. Christ (Tue 17:10).
  Talk: Y. Zhao (Tue 17:30).

- Likely more applications.
1. Introduction

2. $\text{QED}_\infty$ in Muon $g - 2$
   Long-distance contribution to the HLbL in position space from the $\pi^0$ pole

3. $\text{QED}_\infty$ in hadron spectroscopy (X. Feng and L. Jin 2018)
   Pion mass splitting with the infinite volume reconstruction method

4. Conclusion
QED correction to hadron masses

\[ \Delta M = \mathcal{I} = \frac{1}{2} \int d^4 x \, \mathcal{H}_{\mu,\nu}(x) S^\gamma_{\mu,\nu}(x) \]

\[ \mathcal{H}_{\mu,\nu}(x) = \frac{1}{2M} \langle N | T J_{\mu}(x) J_{\nu}(0) | N \rangle \]

\[ S^\gamma_{\mu,\nu}(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 \chi^2} \]

- We can evaluate the QED part, the photon propagator, in infinite volume.

- The hadronic function do not always fall exponentially in the long distance region.

\[ \Delta M = \mathcal{I} = \mathcal{I}^{(s)} + \mathcal{I}^{(l)} \]

\[ \mathcal{I}^{(s)} = \frac{1}{2} \int_{-t_s}^{t_s} dt \int d^3 x \, \mathcal{H}_{\mu,\nu}(x) S^\gamma_{\mu,\nu}(x) \]

\[ \mathcal{I}^{(l)} = \int_{t_s}^{\infty} dt \int d^3 x \, \mathcal{H}_{\mu,\nu}(x) S^\gamma_{\mu,\nu}(x) \]

- To eliminate all power-law suppressed finite volume effects, a different treatment for the long distance part is required. \((t_s \lesssim L)\)
The infinite volume reconstruction method

- For the short distance part, \( \mathcal{I}^{(s)} \) can be directly calculated on a finite volume lattice:

\[
\mathcal{I}^{(s)} \approx \mathcal{I}^{(s,L)} = \frac{1}{2} \int_{-t_s}^{t_s} dt \int_{L/2}^{L} d^3x \, \mathcal{H}_{\mu,\nu}(x) S^\gamma_{\mu,\nu}(x)
\]

- For the long distance part, we need to evaluate \( \mathcal{H}_{\mu,\nu}(x) \) indirectly.

Note that when \( t \) is large, the intermediate states between the two currents are mostly ground states (possibly with small momentum). Therefore:

\[
\mathcal{H}_{\mu,\nu}(x) \approx \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{2E_\beta} \frac{1}{2M} \langle N|J_\mu(0)|N(\vec{p})\rangle \langle N(\vec{p})|J_\nu(0)|N\rangle \right] \ e^{i\vec{p} \cdot \vec{x} - (E_\beta - M)t}
\]

- We only need to calculate the form factors: \( \langle N(\vec{p})|J_\nu(0)|N\rangle! \)

- Values for all \( \vec{p} \) are needed. Simply inverse Fourier transform the above relation!

\[
\int d^3x \mathcal{H}_{\mu,\nu}(t_s, \vec{x}) e^{-i\vec{p} \cdot \vec{x} + (E_\beta - M)t_s} \approx \frac{1}{2E_\beta} \frac{1}{2M} \langle N|J_\mu(0)|N(\vec{p})\rangle \langle N(\vec{p})|J_\nu(0)|N\rangle
\]
The final expression for QED correction to hadron mass is split into two parts:

$$\Delta M = \mathcal{I} \approx \mathcal{I}^{(s,L)} + \mathcal{I}^{(l,L)}$$

- For the short distance part:
  $$\mathcal{I}^{(s)} \approx \mathcal{I}^{(s,L)} = \frac{1}{2} \int_{-t_s}^{t_s} dt \int_{L/2}^{L/2} d^3x \mathcal{H}_{\mu,\nu}(x) S_{\mu,\nu}^\gamma(x)$$

- For the long distance part:
  $$\mathcal{I}^{(l)} \approx \mathcal{I}^{(l,L)} = \int_{-L/2}^{L/2} d^3x \mathcal{H}_{\mu,\nu}^L(t_s, \vec{x}) L_{\mu,\nu}(t_s, \vec{x})$$

- For Feynman gauge:
  $$S_{\mu,\nu}^\gamma(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 x^2} \quad L_{\mu,\nu}(t_s, \vec{x}) = \frac{\delta_{\mu,\nu}}{2\pi^2} \int_0^\infty dp \frac{\sin(p|\vec{x}|)}{2(p + E_p - M)|\vec{x}|} e^{-pt_s}$$

- Only exponentially suppressed finite volume errors.
  Power-law suppressed finite volume errors are removed to all orders.
Pion mass splitting: $M_{\pi^+} - M_{\pi^0}$

- Diagrams are very similar to the $\pi^- \rightarrow \pi^+ ee$ $0\nu2\beta$ decay calculation. (D. Murphy and W. Detmold, 2018.) Poster: X. Tuo (Tue 17:50).

- All UV divergence and other disconnected diagrams are cancelled. (RM123 2013)

- RM123 2013 (type 2 only): $M_{\pi^+}^2 - M_{\pi^0}^2 = 1.44(13)_{\text{stat}}(16)_{\text{chiral}} \times 10^3$ MeV


- Talk: J. Richings (Wed 11:50).
Pion mass splitting: $M_{\pi^+} - M_{\pi^0}$ (prelim)

Type 1 and Type 2

- $M_\pi = 142$ MeV.
- $a^{-1} = 1.015$ GeV,
- 24ID: $24^3 \times 64$,
- 32ID: $32^3 \times 64$,
Pion mass splitting: \( M_{\pi^+} - M_{\pi^0} \) (prelim)

\[
\Delta M^2(a, M_\pi) = 2M_\pi \Delta M_\pi = \Delta M^2(0, M_{\pi^+}^{\text{phys}}) + c_1 a^2 + c_2 (M_\pi^2 - (M_{\pi^+}^{\text{phys}})^2)
\]

\[
\Delta M^2_{\pi^0}(0, M_{\pi^+}^{\text{phys}}) = 1.275(15) \times 10^3 \text{ MeV}
\]

\[
\Delta M^2_{\pi^+}(0, M_{\pi^+}^{\text{phys}}) = 4.57(6) \text{ MeV}
\]

- Both type 1 and type 2 diagrams included.
- 10 times more accurate than previous.
The calculation of QED correction to QCD observables, and the appropriate treatment of the problems arising due to the finite size of the simulated lattice, is a *hot topic in the field*; nonetheless the main aspects debated at the moment concern actually *the infrared singularities* appearing in the matrix elements due to the presence of a finite box, which is not the subject of the present paper.

– Anonymous

- Infrared divergence shall *cancel analytically* as it always does in infinite volume perturbative calculations.
Outline

1. Introduction

2. QED$_\infty$ in Muon $g - 2$
   Long-distance contribution to the HLbL in position space from the $\pi^0$ pole

3. QED$_\infty$ in hadron spectroscopy (X. Feng and L. Jin 2018)
   Pion mass splitting with the infinite volume reconstruction method

4. Conclusion
In many cases, it is very beneficial to view the problem from a position space perspective.

For a class of problems, we can split the contribution into two different regions and adopt different treatment.

- Muon $g - 2$:
  * Long-distance $\pi^0$ pole contribution for HLbL.

- Hadron Spectroscopy:
  Applied in calculating $M_{\pi^+} - M_{\pi^0}$. Work in progress for leptonic decay.

- More applications?
Thank You!