Contents:

1. Introduction

Particles in electric and magnetic fields: **Lorentz-force**

$$\vec{F} = e\cdot\left(\vec{E} + \vec{\mathrm{v}}\times\vec{B}\right)$$

Energy increase is related to longitudinal electric fields:

$$\int \vec{F}\bullet d\vec{s} = e\cdot\left(\int \vec{E}\bullet d\vec{s} + \int\left(\vec{\mathrm{v}}\times\vec{B}\right)\bullet d\vec{s}\right) = e\cdot\int E_{\|}\,ds = e\cdot U$$

Beam deflection due to perpendicular fields, has to cancel centrifugal force:

$$\vec{F}_{\perp} = \gamma m_0 c\dot{\vec{\beta}}_{\perp} = e\cdot\left(\vec{E}_{\perp} + \vec{\beta}c\times\vec{B}\right) \overset{!}{=} \gamma m_0\frac{\mathrm{v}^2}{R}$$

Electric stiffness and **magnetic stiffness**:

$$\boxed{e\,R\,B_{\perp} \;=\; p \;=\; e\frac{1}{\mathrm{v}}R\,E_{\perp}}$$

Ultrarelativistic particles: $v\approx c \quad\Rightarrow\quad R\,B_{\perp} = 1\ \mathrm{Tm} \quad\Leftrightarrow\quad R\,E_{\perp}\approx 300\ \mathrm{MV/m}$

2. Magnets

2.1. General remarks on the calculation of magnetic fields

The calculation of magnetic fields is based on Maxwell's equation

$$\vec{\nabla} \times \vec{H} = \vec{j} \, .$$

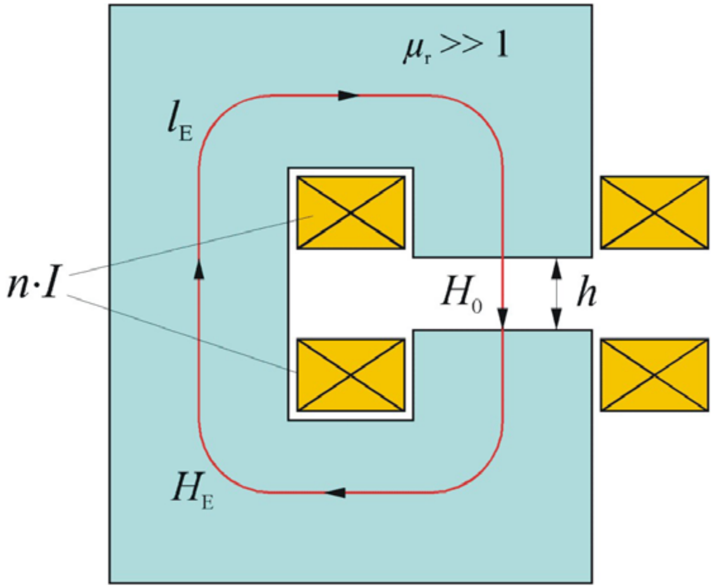
We are interested in the magnetic field close to the particles orbit where $\vec{j} = 0$, thus

yielding $$\vec{\nabla} \times \vec{H} = 0 \, .$$

$\vec{H}$ may therefore be expressed in terms of a scalar potential φ by

$$\vec{H} = -\vec{\nabla}\Phi \, , \qquad \text{giving} \qquad \Delta\Phi = 0 \, .$$

The magnetic field usually is generated by an electrical current I in current carrying coils surrounding magnet poles made of ferromagnetic material. A ferromagnetic return yoke surrounds the excitation coils providing an efficient return path for the magnetic flux.



The magnetic potential Φ is defined by its equipotential lines in the transverse plane and therefore by the surface of the ferromagnetic poles. Due to the relation

$$\vec{B} = \mu_0 \, \mu_r \, \vec{H} \,,$$

the magnetic field in the gap of a magnet may be calculated from a closed loop integral

$$n \cdot I = \oint \vec{H} \cdot d\,\vec{s} = \int\limits_{gap} \vec{H}_0 \cdot d\,\vec{s} + \int\limits_{yoke} H_E \,,$$

where n is the number of windings and I the current of the excitation coil of the magnet. At the surface of the magnetic poles, the continuity relation gives

$$|H_E| = \frac{1}{\mu_r}|H_0| \quad \Rightarrow \quad |H_0| \gg |H_E|.$$

Assuming a very large permeability μ_r, the second integral term may be neglected. We finally obtain for the magnetic field inside the gap

$$\boxed{\int_{gap} \vec{B} \cdot d\,\vec{s} = \mu_0 \cdot n \cdot I}$$

Concentrating on the major types of magnets (so called *upright magnets*) mainly used in particles accelerators and beam transport lines, it is sufficient to define the vertical component $G_z(x)$ of the magnetic field along the horizontal axis of the magnet (where we put $z = 0$) and to assume a constant field distribution along the longitudinal axis ($\partial\vec{B}/\partial s = 0$). This leads to the set-up

$$B_z(x,z) = G_z(x) + f(z)\,,$$

and with $\vec{B} = -\vec{\nabla}\Phi$, one has

$$\Phi(x,z) = -\int B_z \cdot dz = -G_z(x) \cdot z - \int f(z) \cdot dz\,,$$

and therefore

$$\Delta\Phi = -\frac{d^2\,G_z(x)}{d\,x^2}\,z - \frac{d\,f(z)}{d\,z} = 0$$

$$\Rightarrow \quad f(z) = -\int \frac{d^2\,G_z(x)}{d\,x^2}\,z \cdot dz = -\frac{1}{2}\frac{d^2\,G_z(x)}{d\,x^2}\,z^2$$

Finally, we obtain for the scalar potential of the magnetic field:

$$\Phi(x,z) = -G_z(x) \cdot z + \frac{1}{6}\frac{d^2\,G_z(x)}{d\,x^2} \cdot z^3$$

2.2. Particle Beam Guidance

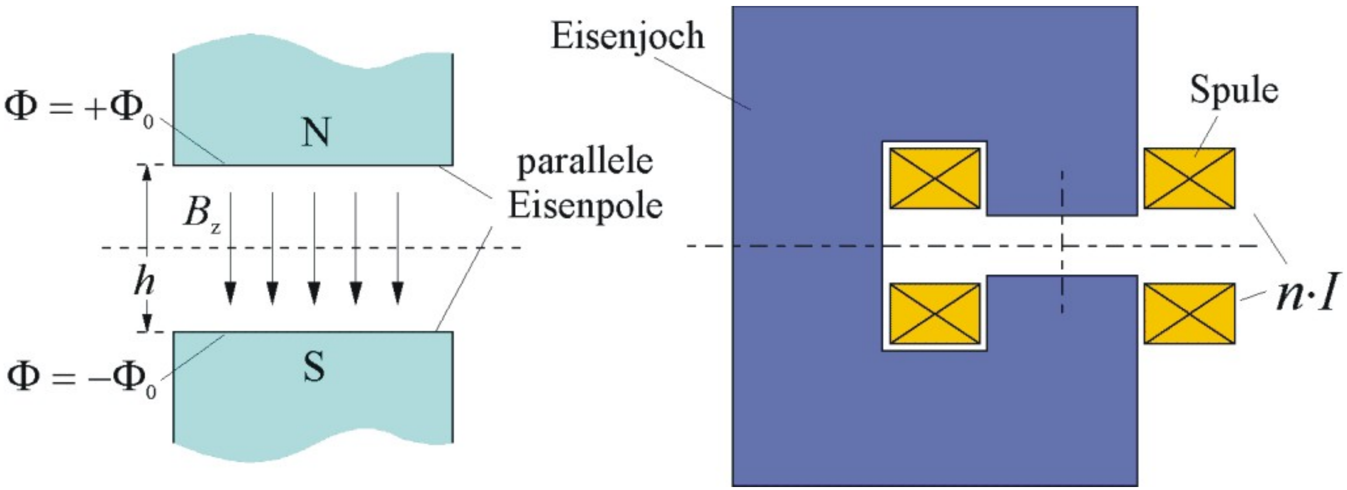
Guide fields are used to deflect particles along a predefined path. These fields, generated by **Dipole Magnets** have to be homogenous, leading to the set-up

$$G_z(x) = B_0 = \text{const.} \quad \Rightarrow \quad \frac{d^2 G_z(x)}{d\,x^2} = 0$$

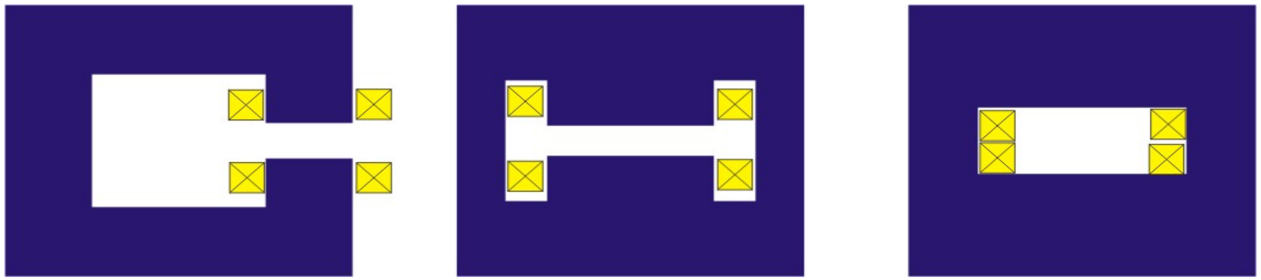
The corresponding magnetic potential follows to

$$\Phi(x,z) = -B_0 \cdot z\,,$$

defining the profile of the magnet poles to flat parallel poles at a distance *h*:



Typical types of dipole magnets are the C-Magnet, the H-Magnet and the Window-Frame Magnet:



C-Magnet H-Magnet Window-Frame Magnet

The magnetic field inside the gap is related to the current of the coils by

$$B_0 = \mu_0 \frac{n \cdot I}{h}$$

The strength of a dipole magnet is usually normalized to the particles momentum, giving the inverse bending radius $1/R$ or the curvature κ:

$$\boxed{\kappa = \frac{1}{R} = \frac{e}{p} B_0 = \frac{e\,\mu_0}{p} \frac{n \cdot I}{h}}$$

2.3. Particle Beam Focusing

Focusing fields are used to keep the particle beam together and to generate specifically desired beam properties at selected points along a beam transport line.

We need a magnetic field which increases linearly with increasing distance from the axis, leading to the set-up:

$$G_z(x) = g \cdot x \quad \text{with} \quad g = \frac{\partial B_z}{\partial x}$$

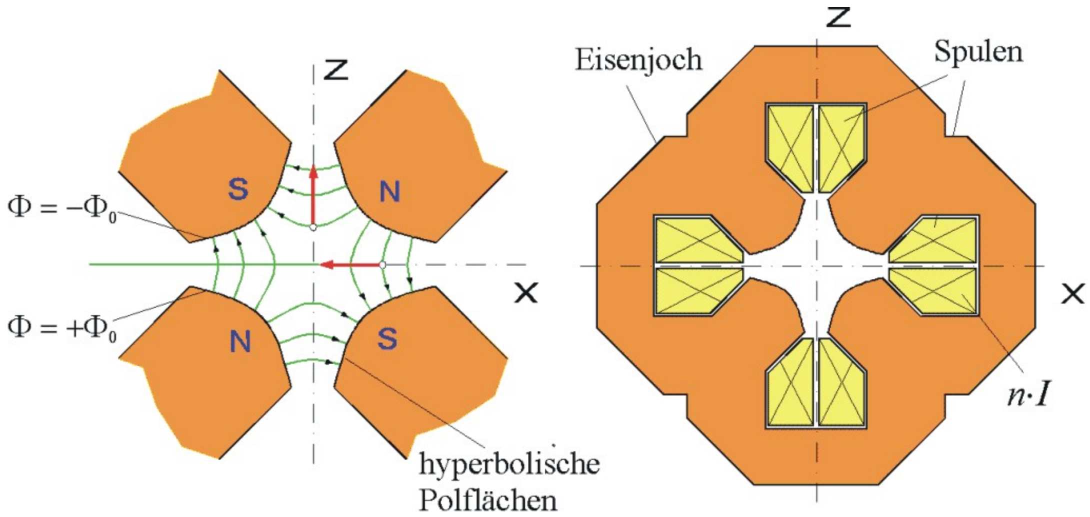
generated by **Quadrupole Magnets**. The corresponding potential follows to

$$\Phi(x,z) = -g \cdot x \cdot z \,,$$

defining the profile of the magnet poles to four hyperbolic poles

$$z(x) = \pm \frac{\Phi_0}{g \cdot x} = \pm \frac{a^2}{2x}$$

at a distance $a = \sqrt{2\Phi_0 / g}$ from the axis.



The magnetic field inside the gap is determined from the gradient of the magnetic potential, giving:

$$\boxed{B_x = -\frac{\partial \Phi}{\partial x} = g \cdot z \quad \text{and} \quad B_z = -\frac{\partial \Phi}{\partial z} = g \cdot x}$$

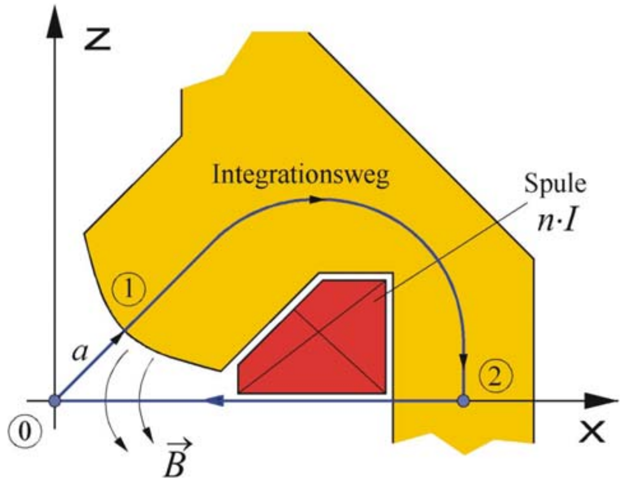
The “restoring” force acting on the particles is

$$\vec{F} = e \cdot \left(\vec{\mathrm{v}} \times \vec{B} \right) = e\,\mathrm{v}g \cdot \left(x\,\hat{e}_x - z\,\hat{e}_z \right)$$

A quadrupole magnet is therefore focusing only in one plane and defocusing in the other; depending on the sign of *g*.

The *g*-parameter may be related to the current of the coils by evaluating the closed

loop integral $$n \cdot I = \oint \vec{H} \cdot d\vec{s} = \int_0^1 \vec{H}_0 \cdot d\vec{s} + \int_1^2 \vec{H}_E \cdot d\vec{s} + \int_2^0 \vec{H}_0 \cdot d\vec{s} \approx \int_0^1 \vec{H}_0 \cdot d\vec{s} \,,$$



One obtains with $\vec{H} \cdot d\vec{s} = \frac{g}{\mu_0} r \cdot dr$:

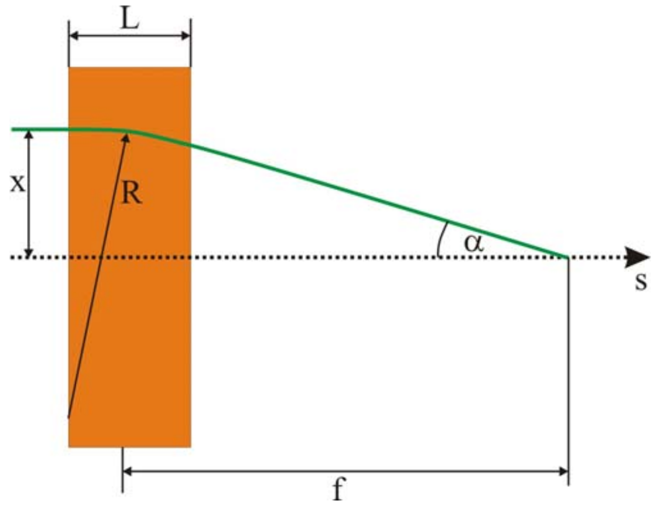
$$\boxed{g = \frac{2 \cdot \mu_0 \cdot n \cdot I}{a^2}}$$

Again we would like to normalize *g* to the particles momentum, giving the quadru-

pole strength *k*:

$$\boxed{k = \frac{e}{p} g = \frac{2e\mu_0}{p} \frac{n \cdot I}{a^2}}$$

The focal length of a thin quadrupole magnet of length L can be derived from the deflection angle α of the particles beam and its relation to the quadrupole strength k,



$$\tan\alpha = \frac{x}{f}$$

$$\tan\alpha = \frac{L}{R} = \frac{L}{p/(e B_z)} = \frac{e}{p} g\, x\, L = x\, k\, L$$

giving:

$$\boxed{\frac{1}{f} = k \cdot L}$$

Here we have assumed the length L to be short compared to the focal length f such that R does not change significantly within the quadrupole magnetic field.

2.4. Correction of Chromatic Errors

To correct focusing errors of the quadrupole magnets we need the nonlinear magnetic fields of **Sextupole Magnets**, in which the field increase is quadratic with increasing distance from the axis:

$$G_z(x) = \frac{1}{2} g' \cdot x^2 \quad \text{with} \quad g' = \frac{\partial^2 B_z}{\partial x^2} = \frac{d^2 G_z(x)}{d x^2}$$

This setting yields $f(z) \neq 0$!

We will therefore expect a coupling of particles motion in the horizontal and vertical plane due to the z-dependence of the vertical field.

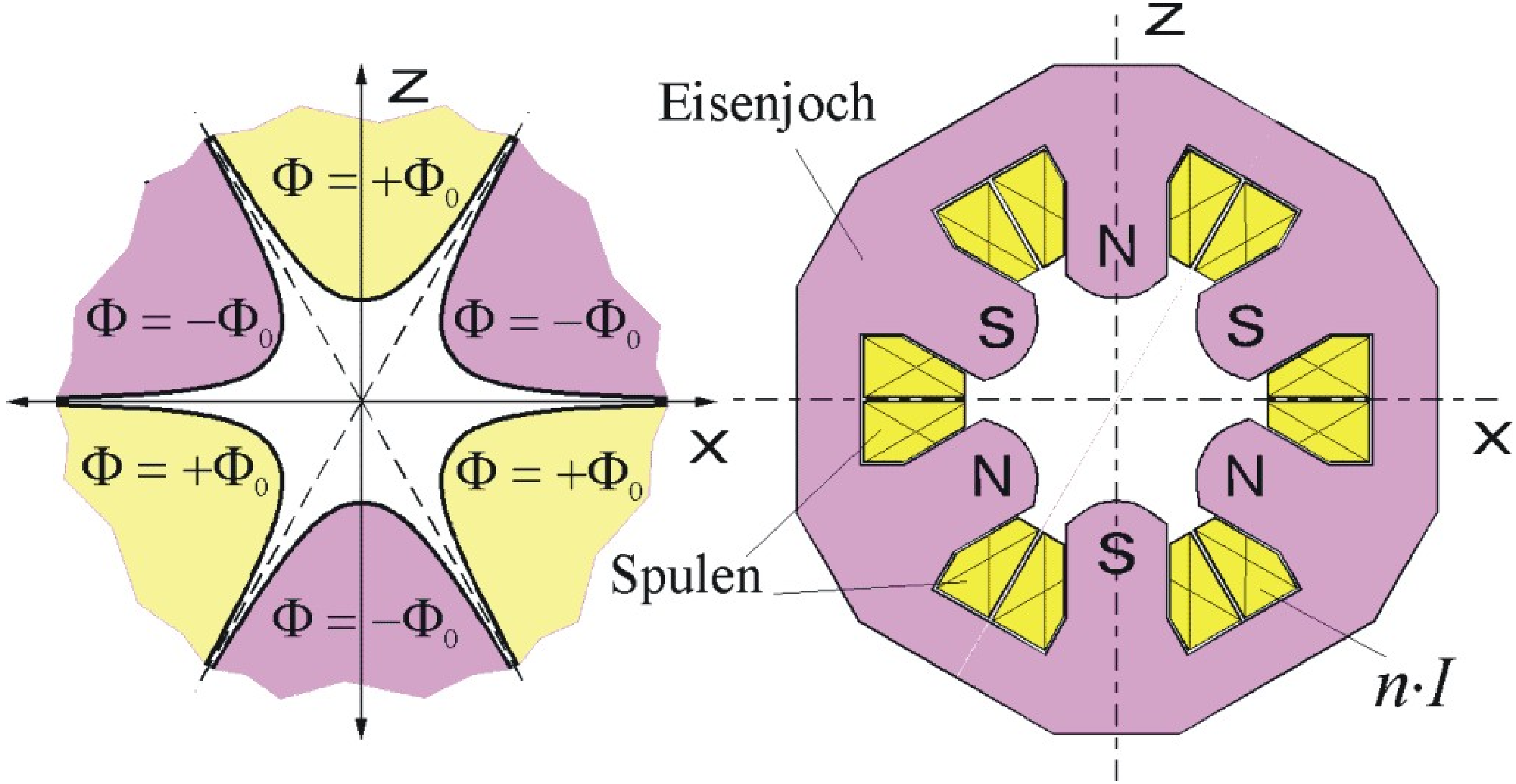
From the potential equation we obtain

$$\Phi(x, z) = \frac{1}{6} g' \cdot \left(z^3 - 3 \cdot x^2 \cdot z \right),$$

defining the profile of the magnet poles to

$$x(z) = \pm\sqrt{\frac{z^2}{3} \pm \frac{2\Phi_0}{g'z}} = \pm\sqrt{\frac{z^2}{3} \pm \frac{a^3}{3z}}$$

at a distance $a = \sqrt[3]{6\Phi_0/g'}$ from the axis.



Again the magnetic field is determined from the gradient of the magnetic potential, giving:

$$\boxed{B_x(x,z) = -\frac{\partial \Phi}{\partial x} = g' x z \quad \text{and} \quad B_z(x,z) = -\frac{\partial \Phi}{\partial z} = \frac{1}{2} g' \left(x^2 - z^2\right)}$$

The *g'*-parameter may be related to the current of the coils in the same manner as it was derived for the quadrupole magnets, yielding

$$g' = \frac{\partial^2 B_z}{\partial x^2} = 6\,\mu_0 \frac{n I}{a^3}$$

Normalizing *g'* to the particles momentum, we obtain the sextupole strength

$$\boxed{m = \frac{e}{p} g' = \frac{6 e \mu_0}{p} \frac{n I}{a^3}}.$$

2.5. Multipole Field Expansion

We will now derive a general form of the magnetic potential Φ using a cylinder coordinate system, in which the Laplace equation reads

$$\Delta\Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

A useful setup of Φ is a Taylor expansion with respect to the reference path (r=0), neglecting the z-dependence of the magnetic potential:

$$\Phi(r,\varphi) = -\sum_{n>0} \frac{1}{n!} c_n \, r^n \, e^{in\varphi} = \sum_{n>0} \Phi_n(r,\varphi)$$

Inserting this set-up into the Laplace equation, we get

$$\sum_n \frac{1}{n!} \frac{n(n-1)+n-n^2}{r^2} c_n \, r^n \, e^{in\varphi} = 0$$

Every multipole Φ_n is therefore a valid solution of the Laplace equation.

In cartesian coordinates, the multipoles of the magnetic potential read

$$\Phi_n(x,z) = -\frac{c_n}{n!}\left(x+iz\right)^n = -\sum_{k=0}^{n}\frac{c_n}{(n-k)!\cdot k!}x^{n-k}\left(iz\right)^k$$

Both the real and imaginary solutions are independent solutions:

$$\mathrm{Re}\left[\Phi_n(x,z)\right] = -\sum_{k=0}^{n/2} a_n \cdot \frac{(-1)^k}{(n-2k)!\cdot(2k)!}\cdot x^{n-2k}\cdot z^{2k}$$

$$\mathrm{Im}\left[\Phi_n(x,z)\right] = -\sum_{k=1}^{(n+1)/2} b_n \cdot \frac{(-1)^{k-1}}{(n-2k+1)!\cdot(2k-1)!}\cdot x^{n-2k+1}\cdot z^{2k-1}$$

Real and imaginary solutions differentiate between two classes of magnet orientation:

- the imaginary solution has mid plane symmetry $\mathrm{Im}\,\Phi(x,z) = -\mathrm{Im}\,\Phi(x,-z)$ and no horizontal field components in the mid plane → **upright magnets**
- the real solution has mid plane symmetry $\mathrm{Re}\,\Phi(x,z) = +\mathrm{Re}\,\Phi(x,-z)$ and not vanishing vertical field components in the mid plane → **rotated magnets**

The magnetic field components for the n[th] order multipoles are derived from

upright magnets:	**rotated magnets:**
$B_{nx} = -\frac{\partial}{\partial x}\operatorname{Im}\Phi, \quad B_{nz} = -\frac{\partial}{\partial z}\operatorname{Im}\Phi$	$B_{nx} = -\frac{\partial}{\partial x}\operatorname{Re}\Phi, \quad B_{nz} = -\frac{\partial}{\partial z}\operatorname{Re}\Phi$

A particle traveling in the horizontal mid plane through an upright magnet will remain in the horizontal plane!

We finally derive for the potential Φ and the magnetic field B of the n[th] multipole by setting $a_n = -e/p \cdot \underline{S}_n$, $b_n = e/p \cdot S_n$ with the multipole strengths $S_n = -\kappa, k, m, r$:

Dipole	$-\frac{e}{p}\Phi_1 = \kappa_z\, x - \kappa_x\, z$
Quadrupole	$-\frac{e}{p}\Phi_2 = -\frac{1}{2}\underline{k}\left(x^2 - z^2\right) + k\, x\, z$
Sextupole	$-\frac{e}{p}\Phi_3 = -\frac{1}{6}\underline{m}\left(x^3 - 3\,x\,z^2\right) + \frac{1}{6}m\left(3\,x^2\,z - z^3\right)$
Octupole	$-\frac{e}{p}\Phi_4 = -\frac{1}{24}\underline{r}\left(x^4 - 6\,x^2\,z^2 + z^4\right) + \frac{1}{6}r\left(x^3\,z - x\,z^3\right)$

Upright Magnets:

Dipole	$\frac{e}{p}\vec{B}_1 = -\kappa_x \hat{e}_z$
Quadrupole	$\frac{e}{p}\vec{B}_2 = k\,z\,\hat{e}_x + k\,x\,\hat{e}_z$
Sextupole	$\frac{e}{p}\vec{B}_3 = m\,x\,z\,\hat{e}_x + \frac{1}{2}m\left(x^2 - z^2\right)\hat{e}_z$
Octupole	$\frac{e}{p}\vec{B}_4 = \frac{1}{6}r\left(3x^2 z - z^3\right)\hat{e}_x + \frac{1}{6}r\left(x^3 - 3xz^2\right)\hat{e}_z$

Rotated (Skew) Magnets:

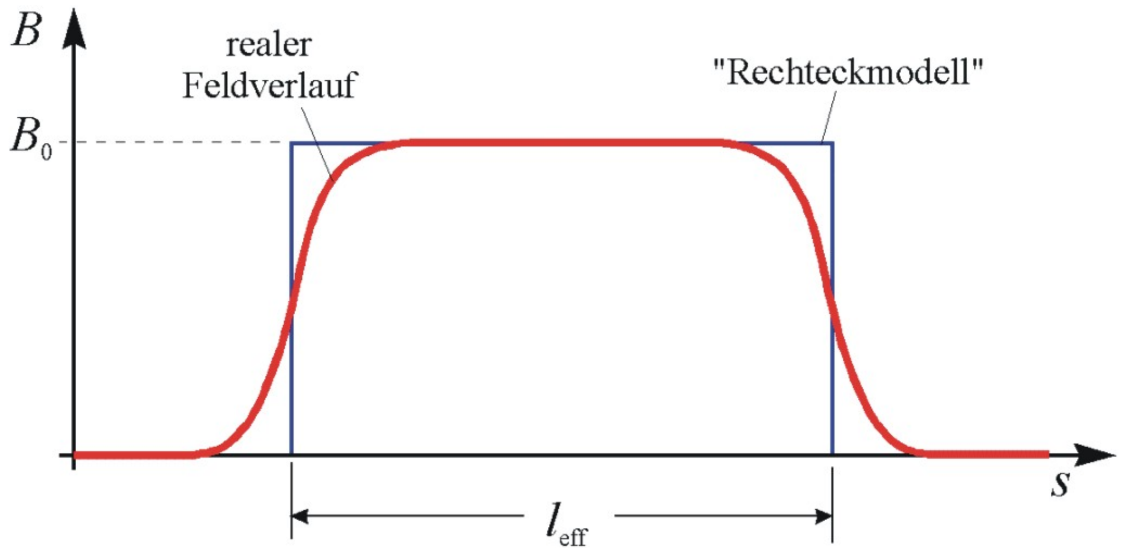
Dipole (90^0)	$\frac{e}{p}\vec{B}_1 = \kappa_z \hat{e}_x$
Quadrupole (45^0)	$\frac{e}{p}\vec{B}_2 = -\underline{k}\,x\,\hat{e}_x + \underline{k}\,z\,\hat{e}_z$
Sextupole (30^0)	$\frac{e}{p}\vec{B}_3 = -\frac{1}{2}\underline{m}\left(x^2 - z^2\right)\hat{e}_x + \underline{m}\,x\,z\,\hat{e}_z$
Octupole ($22{,}5^0$)	$\frac{e}{p}\vec{B}_4 = -\frac{1}{6}r\left(x^3 - 3xz^2\right)\hat{e}_x + \frac{1}{6}r\left(3x^2 z - z^3\right)\hat{e}_z$

2.6. Effective Field length

The assumption of a constant field distribution along the longitudinal axis ($\partial\vec{B}/\partial s = 0$) is not valid in general due to the fringing fields at the end of the magnets.

In order to simplify the calculation of the optics of particle accelerators, an effective field length l_{eff} of each magnet is usually defined, calculated from the path-integral

$$\boxed{\int_{-\infty}^{\infty} \vec{B} \cdot d\,\vec{s} = \vec{B}_0 \cdot l_{eff}}$$

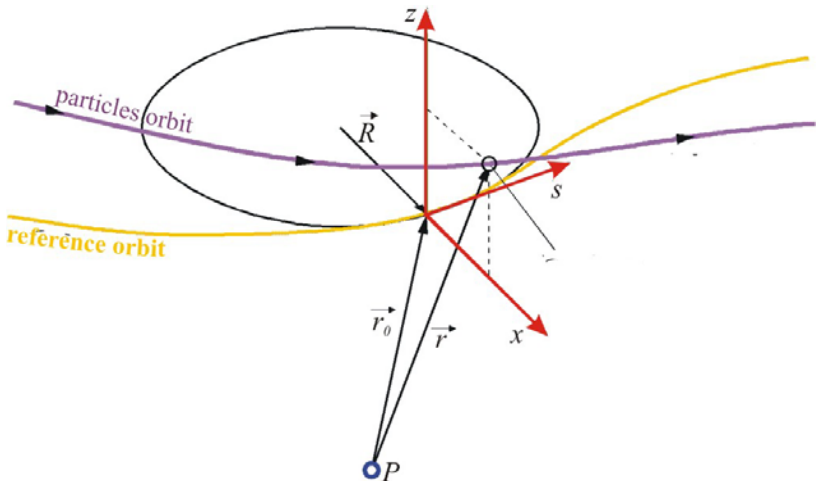


and approximating the real longitudinal field by a rectangular shaped profile.

3. Linear Beam Optics

3.1. Equations of Motion in a moving Reference System

In order to have a meaningful formulation of the particles trajectories in the vicinity of a once defined ideal path, we will use a **moving orthogonal, right-handed coordinate system (*x, s, z*)** that follows an ideal particle traveling along its ideal path:



We will concentrate on ideal orbits, laying within the horizontal plane, therefore

$$\boxed{\vec{r} = (R + x) \cdot \hat{e}_x + z \cdot \hat{e}_z}$$

Using this reference system moving with $\dot{s} = R \cdot \dot{\varphi}$, we obtain the following time derivatives of the coordinate vectors

$$\left.\begin{aligned} \dot{\hat{e}}_x &= +\dot{\varphi}\,\hat{e}_s = +\frac{\dot{s}}{R}\hat{e}_s \\ \dot{\hat{e}}_s &= -\dot{\varphi}\,\hat{e}_x = -\frac{\dot{s}}{R}\hat{e}_x \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} \ddot{\hat{e}}_x &= -\dot{\varphi}^2\,\hat{e}_x = -\left(\frac{\dot{s}}{R}\right)^2 \hat{e}_x \\ \ddot{\hat{e}}_s &= -\dot{\varphi}^2\,\hat{e}_s = -\left(\frac{\dot{s}}{R}\right)^2 \hat{e}_s \end{aligned}$$

$$\dot{\hat{e}}_z = 0$$

and by using $\dot{x} = \frac{dx}{ds} \cdot \frac{ds}{dt} = x' \cdot \dot{s}$, $\dot{z} = \frac{dz}{ds} \cdot \frac{ds}{dt} = z' \cdot \dot{s}$, we obtain

$$\dot{\vec{r}} = x'\,\dot{s}\,\hat{e}_x + \left(1 + \frac{x}{R}\right)\dot{s}\,\hat{e}_s + z'\,\dot{s}\,\hat{e}_z$$

$$\ddot{\vec{r}} = \left\{ x''\dot{s}^2 - \left(1 + \frac{x}{R}\right)\frac{\dot{s}^2}{R} + \underbrace{x'\ddot{s}}_{\approx 0} \right\}\hat{e}_x + \left\{ 2x'\frac{\dot{s}^2}{R} + \underbrace{\left(1 + \frac{x}{R}\right)\ddot{s}}_{\approx 0} \right\}\hat{e}_s + \left\{ z''\dot{s}^2 + \underbrace{z'\ddot{s}}_{\approx 0} \right\}\hat{e}_z$$

where we neglect the second derivatives $\ddot{s}$ due to very slow changes of the longitudinal component of particles velocity.

We will further concentrate on linear magnetic fields, generated by **upright** dipole and quadrupole magnets:

$$\frac{e}{p_0}\vec{B} = k\,z\,\hat{e}_x + \left\{-\frac{1}{R} + k\,x\right\}\hat{e}_z\,,$$

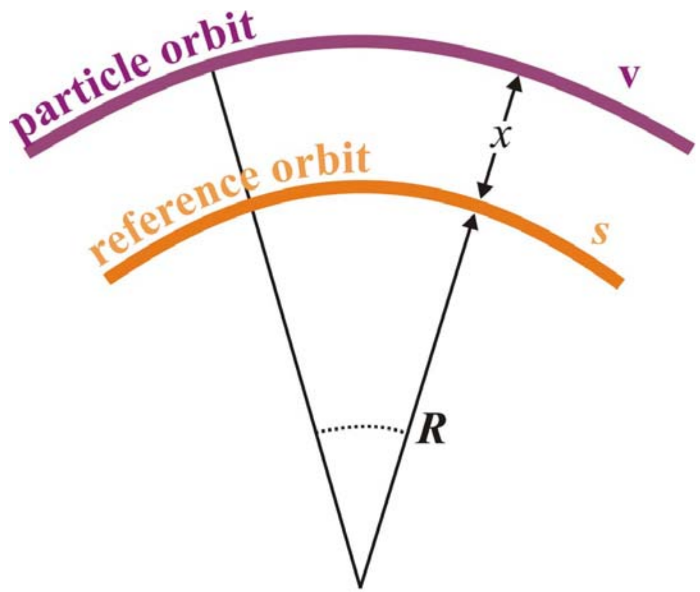
and from simple geometrical considerations, we may link v to $\dot{s}$:

$$\dot{s} = R\,\dot{\varphi}, \quad \mathrm{v} = (R + x)\,\dot{\varphi}$$

$$\Rightarrow \quad \mathrm{v} = \frac{R + x}{R}\dot{s} = \left(1 + \frac{x}{R}\right)\dot{s}$$

and for $p = m \cdot \mathrm{v} \approx p_0$ we have

$$\frac{1}{p} \approx \frac{1}{p_0}\left(1 - \frac{\Delta p}{p_0}\right)$$



The particles are deflected due to the Lorentz force $m \cdot \ddot{\vec{r}} = e \cdot \left(\dot{\vec{r}} \times \vec{B} \right)$, thus

$$\begin{pmatrix} x''\dot{s}^2 - \left(1+\frac{x}{R}\right)\frac{\dot{s}^2}{R} \\ 2x'\dot{s}^2/R \\ z''\dot{s}^2 \end{pmatrix} = \frac{e}{m} \cdot \begin{pmatrix} \left(1+\frac{x}{R}\right)\dot{s}B_z \\ \dot{s}\left(z'B_x - x'B_z\right) \\ -\left(1+\frac{x}{R}\right)\dot{s}B_x \end{pmatrix}$$

Usually, the effect of particle deflection on the longitudinally velocity may be neglected due to small deflection angles. We will concentrate on the transverse planes. With the corresponding multipole strengths and the momentum expansion we get

$$x'' - \left(1+\frac{x}{R}\right)\frac{1}{R} = \frac{e}{\dot{s}}\frac{\mathrm{v}}{p}\left(1+\frac{x}{R}\right)\underbrace{\frac{p_0}{e}\left(kx-\frac{1}{R}\right)}_{=B_z} = \left(1+\frac{x}{R}\right)^2\left(kx-\frac{1}{R}\right)\left(1-\frac{\Delta p}{p_0}\right)$$

$$z'' = -\frac{e}{\dot{s}}\frac{\mathrm{v}}{p}\left(1+\frac{x}{R}\right)\underbrace{\frac{p_0}{e}kz}_{=B_x} = -\left(1+\frac{x}{R}\right)^2 kz\left(1-\frac{\Delta p}{p_0}\right)$$

Neglecting all nonlinear terms in $x, z, and\ \Delta p/p_0$, we obtain the linear equations of motion of a particle traversing through a system of magnets, the Hill equations:

$$x''(s) + \left(\frac{1}{R^2(s)} - k(s)\right) \cdot x(s) = \frac{1}{R(s)} \frac{\Delta p}{p}$$

$$z''(s) + k(s) \cdot z(s) = 0$$

Note that these equations were derived for upright magnets only!

3.2. Matrix Formalism

We will characterize a particles state by a vector built from its relative coordinates:

$$\vec{X} = \begin{pmatrix} x \\ x' \\ z \\ z' \\ s \\ \delta \end{pmatrix} = \begin{pmatrix} \text{radial displacement} \\ \text{radial angular displacement} \\ \text{axial displacement} \\ \text{axial angular displacement} \\ \text{longitudinal displacement} \\ \text{relative momentum deviation} \end{pmatrix}$$

Matrix formalism to describe particles trajectories: $\vec{X} = \mathbf{M} \cdot \vec{X}_0$. Upright magnets:

$$\mathbf{M} = \begin{pmatrix} r_{11} & r_{12} & 0 & 0 & 0 & r_{16} \\ r_{21} & r_{22} & 0 & 0 & 0 & r_{26} \\ 0 & 0 & r_{33} & r_{34} & 0 & 0 \\ 0 & 0 & r_{43} & r_{44} & 0 & 0 \\ r_{51} & r_{52} & 0 & 0 & 1 & r_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \langle x|x\rangle & \langle x|x'\rangle & 0 & 0 & 0 & \langle x|\delta\rangle \\ \langle x'|x\rangle & \langle x'|x'\rangle & 0 & 0 & 0 & \langle x'|\delta\rangle \\ 0 & 0 & \langle z|z\rangle & \langle z|z'\rangle & 0 & 0 \\ 0 & 0 & \langle z'|z\rangle & \langle z'|z'\rangle & 0 & 0 \\ \langle s|x\rangle & \langle s|x'\rangle & 0 & 0 & \langle s|s\rangle & \langle s|\delta\rangle \\ 0 & 0 & 0 & 0 & 0 & \langle \delta|\delta\rangle \end{pmatrix}$$

3.2.1. Drift Space

Due to $1/R(s) = k(s) = 0$, the equations of motion transform to $x''(s) = z''(s) = 0$.

This gives $x'(s) = x_0' = \text{const.},\ z'(s) = z_0' = \text{const.}$

For the longitudinal coordinate, we have for a drift of length L

$$s = l - l_0 = \left(\mathrm{v} - \mathrm{v}_0\right) \cdot t = \left(\mathrm{v} - \mathrm{v}_0\right) \frac{L}{\mathrm{v}_0} = \frac{\Delta \mathrm{v}}{\mathrm{v}_0} L = \frac{\Delta p}{\beta \cdot \left(\partial p / \partial \beta\right)} L = \frac{1}{\gamma^2} \frac{\Delta p}{p} L = \frac{\delta L}{\gamma^2}$$

Combining these two relations, we obtain the transformation for a drift space:

$$\mathbf{M}_{drift} = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3.2.2. Dipole Magnets

Assuming a constant bending radius within the magnet, we have $k = 0$. Therefore, $z'(s) = z_0' =$ const. Thus in the vertical plane a dipole acts like a drift space if we neglect fringe field effects.

The homogenous equation of motion for the horizontal plane ($\Delta p/p = 0$) is solved by

$$x_h(s) = a \cdot \cos\left(s/R\right) + b \cdot \sin\left(s/R\right)$$

For a given (and therefore constant) $\delta = \Delta p/p$, a particular solution of the inhomogeneous equation of motion is $x_D(s) = R \cdot \delta$. This gives the general solution

$$\boxed{x(s) = x_h(s) + x_D(s) = a \cdot \cos\left(s/R\right) + b \cdot \sin\left(s/R\right) + R \cdot \delta}$$

The integration constants *a*, *b* are derived from the boundary conditions at $s = 0$

$$x(s=0) = a + R \cdot \delta = x_0, \quad x'(s=0) = \frac{b}{R} = x_0',$$

and by defining the bending angle $\varphi = L/R$ of the dipole magnet, we obtain :

$$x(L) = x_0 \cdot \cos\varphi + R \cdot x_0{}' \cdot \sin\varphi + R\left(1 - \cos\varphi\right) \cdot \delta$$
$$z(L) = z_0 + R \cdot \varphi \cdot z_0{}'$$

For the longitudinal coordinate, we have for a drift length $L = \varphi \cdot R$:

$$s \approx \frac{\Delta \mathrm{v}}{\mathrm{v}_0} L - \Delta L = \frac{L}{\gamma^2}\delta - \left(\int ds_D - \int ds\right)$$

The infinitesimal path length element along a trajectory s_D in our curvilinear coordinate system is

$$ds_D^2 = dx^2 + dz^2 + \left(x + R\right)^2 d\varphi^2$$

and with $ds = R \cdot d\varphi$ we obtain for the path length difference ΔL in linear approximation:

$$\Delta L = \int \left\{ \sqrt{x'^2 + z'^2 + \left(1 + x/R\right)^2} - 1 \right\} \cdot ds \approx \int x \cdot \frac{ds}{R}$$

With the above derived general solution $x(s)$, this gives

$$\Delta L = \frac{x_0}{R} \cdot \int \cos\frac{s}{R} \cdot ds + x_0' \cdot \int \sin\frac{s}{R} \cdot ds + \delta \cdot \int \left(1 - \cos\frac{s}{R}\right) \cdot ds$$

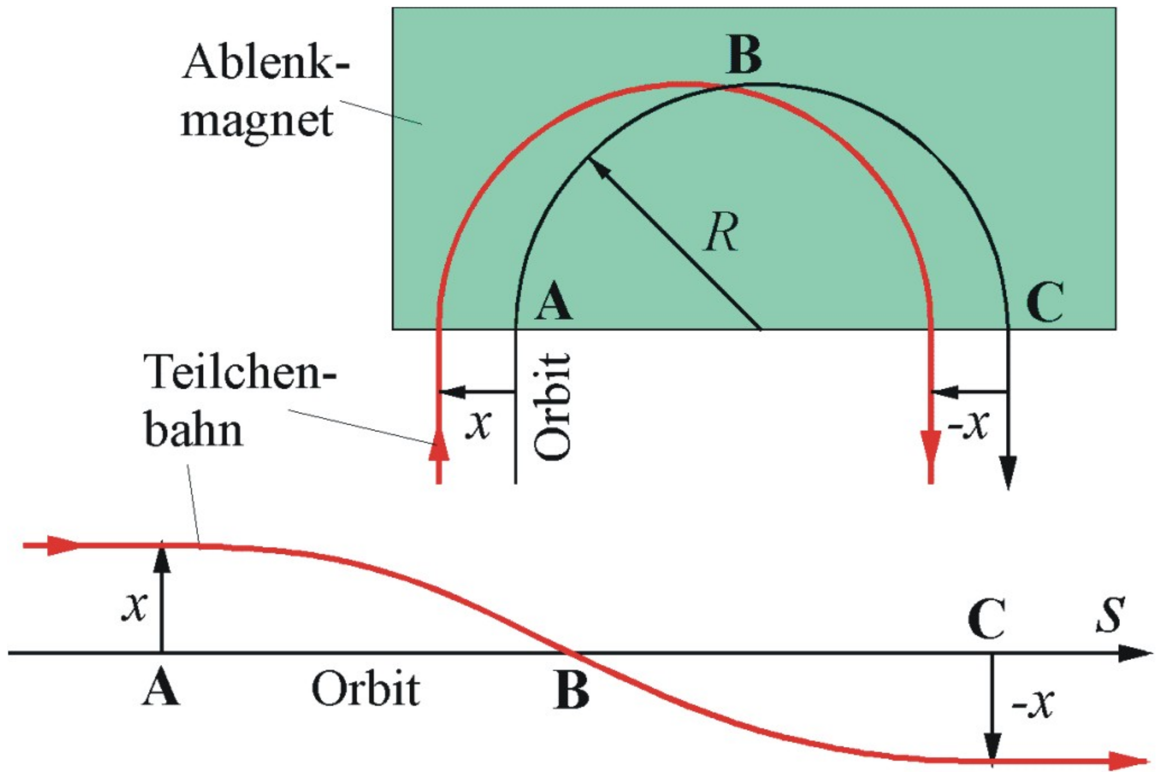
$$\Rightarrow \qquad \Delta L = x_0 \cdot \sin\varphi + x_0' \cdot R \cdot (1 - \cos\varphi) + \delta \cdot (R\varphi - R\sin\varphi)$$

Combining all these relations, we obtain for the transformation for a conventional sector dipole magnet:

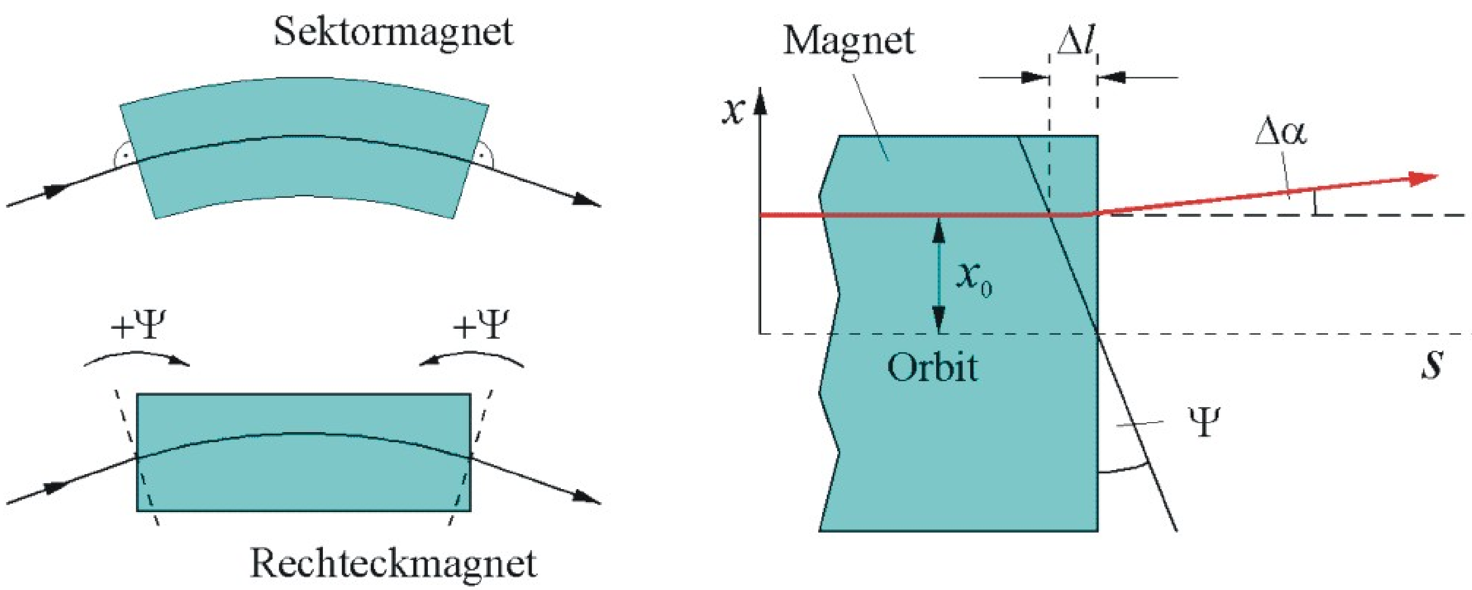
$$\mathbf{M}_{dipole} = \begin{pmatrix} \boxed{\begin{matrix} \cos\varphi & R\sin\varphi \\ -1/R \cdot \sin\varphi & \cos\varphi \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 0 & R(1-\cos\varphi) \\ 0 & \sin\varphi \end{matrix}} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & R\varphi \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \boxed{\begin{matrix} -\sin\varphi & -R(1-\cos\varphi) \\ 0 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & R\varphi/\gamma^2 - R(\varphi - \sin\varphi) \\ 0 & 1 \end{matrix}} \end{pmatrix}$$

A sector magnet is therefore focusing in the horizontal plane.

This effect is purely geometric in nature:



Due to technical constraints, particle accelerators often use rectangular magnets which have parallel end faces. Commonly these magnets are installed symmetrically about the intended particle trajectory:



For a deflection angle φ we have an entrance and exit angle $\psi = \varphi/2$. The orbit of a particle traveling with a radial displacement x_0 at the position of the magnet pole of a rectangular dipole is shorter than the reference orbit (with $x_0 = 0$) by

$$\Delta l = x_0 \cdot \tan\psi$$

This leads to a decrease of the total deflection angle by

$$\Delta\varphi = \frac{\Delta l}{R} = \frac{x_0}{R} \cdot \tan\psi$$

Therefore the position of a particle while leaving a rectangular magnet remains unaltered, whereas the radial angular displacement is increased by $\Delta\varphi$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{\tan\psi}{R} & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_0{}' \end{pmatrix}$$

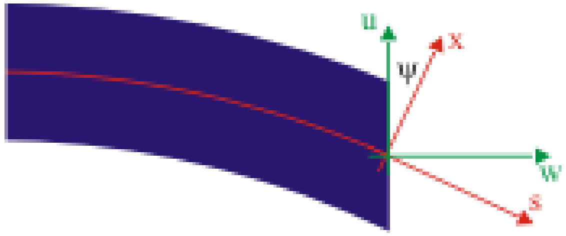
A particle traveling with a vertical displacement z_0 at the position of the magnet pole will be vertically deflected due to the horizontal magnetic field B_x in the fringe field region of length Δs by

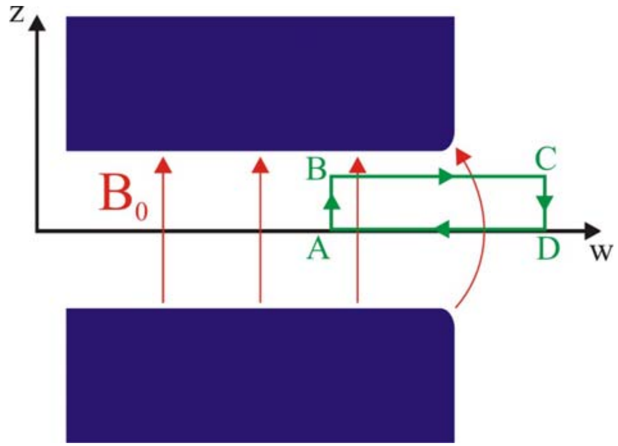
$$\Delta z' = \tan\alpha = \frac{\Delta s}{r} = \frac{\Delta s}{p/(e B_x)} = \frac{e}{p} B_x \Delta s$$

and with $p = e R B_0$ we obtain

$$\Delta z' = \frac{1}{R B_0} \int B_x \cdot ds$$

The integral may be easily calculated in a reference system rotated by the exit angle $\psi = \varphi/2$:





$$B_x = \underbrace{B_u \cos\psi}_{\approx 0} + B_w \sin\psi \approx B_w \sin\psi$$

and with $ds = dw/\cos\psi$ we obtain

$$\int B_x \, ds \approx \tan\psi \cdot \int B_w \, dw$$

Integration over a closed loop ABCD in the fringe field region leads to

$$0 = \oint \vec{B} \cdot d\vec{s} = \underbrace{\int_A^B B_z\, dz}_{=z_0 B_0} + \int_B^C B_w\, dw + \underbrace{\int_C^D B_z\, dz}_{=0} + \underbrace{\int_D^A B_w\, dw}_{\approx 0} \approx z_0 B_0 + \int_B^C B_w\, dw$$

and we obtain a total vertical deflection angle

$$\Delta z' = \frac{1}{R B_0} \cdot \tan\psi \cdot \int_B^C B_w\, dw \approx -\frac{z_0}{R} \tan\psi$$

Again the position of a particle while leaving a rectangular magnet remains unaltered, whereas the vertical angular displacement is changed by $\Delta z'$:

$$\begin{pmatrix} z \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{\tan\psi}{R} & 1 \end{pmatrix} \cdot \begin{pmatrix} z_0 \\ z_0' \end{pmatrix}$$

The focusing / defocusing effect of the fringe fields (edge focusing) depends on the entrance (exit) angle ψ and may again be described by a linear transformation matrix

$$\mathbf{M}_\psi = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ \frac{\tan\psi}{R} & 1 \end{matrix}} & \cdots & 0 \\ \vdots & \boxed{\begin{matrix} 1 & 0 \\ -\frac{\tan\psi}{R} & 1 \end{matrix}} & \vdots \\ 0 & \cdots & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \end{pmatrix}$$

We finally obtain for a rectangular dipole magnet with $\psi = \varphi/2$

$$\mathbf{M}_{rect} = \mathbf{M}_\psi \cdot \mathbf{M}_{dipole} \cdot \mathbf{M}_\psi$$

and with the relations

$$\sin\varphi = 2 \cdot \sin\psi \cdot \cos\psi \,,$$

$$\cos\varphi = \cos^2\psi - \sin^2\psi$$

this reads

$$\mathbf{M}_{rect} = \begin{pmatrix} \boxed{\begin{matrix} 1 & R\sin\varphi \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 0 & R(1-\cos\varphi) \\ 0 & \sin\varphi \end{matrix}} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1-\dfrac{R\varphi}{f} & R\varphi \\ \dfrac{R\varphi}{f^2}-\dfrac{2}{f} & 1-\dfrac{R\varphi}{f} \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \boxed{\begin{matrix} -\sin\varphi & -R(1-\cos\varphi) \\ 0 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & R\varphi/\gamma^2 - R(\varphi-\sin\varphi) \\ 0 & 1 \end{matrix}} \end{pmatrix}$$

where we have defined the focal length in the vertical plane

$$\boxed{\frac{1}{f} \approx \frac{1}{R}\tan\psi}$$

A rectangular dipole magnet is therefore focusing in the vertical plane.
It acts like a drift space in the horizontal plane!

3.2.3. Quadrupole Magnets

Assuming a pure quadrupole magnet we set the bending term $1/R = 0$. The solution of the equation of motion depends on the sign of the quadrupole strength *k*. For $k < 0$ we get the solution of a quadrupole magnet, which is horizontal focusing and vertical defocusing:

$$x(s) = a \cdot \cos\left(\sqrt{k} \cdot s\right) + b \cdot \sin\left(\sqrt{k} \cdot s\right)$$

$$z(s) = c \cdot \cosh\left(\sqrt{k} \cdot s\right) + d \cdot \sinh\left(\sqrt{k} \cdot s\right)$$

The integration constants *a, b, c, d* are derived from the boundary conditions at $s = 0$:

$$x(s=0) = a = x_0, \qquad x'(s=0) = b = x_0'$$

$$z(s=0) = c = z_0, \qquad z'(s=0) = d = z_0'$$

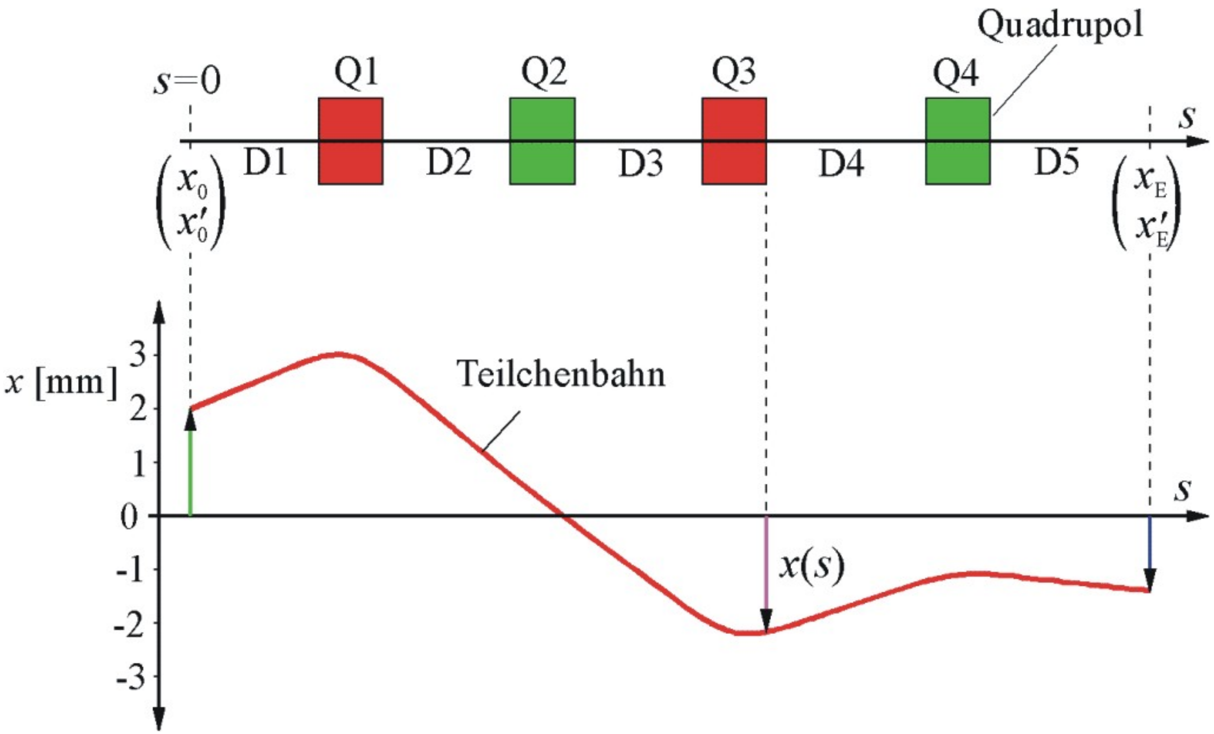
Substitution and building the first derivative, we obtain the transformation of a **hori-zontal focusing (FQ) and a horizontal defocusing (DQ) quadrupole**: where we put $\Omega = \sqrt{k} \cdot L$ with the quadrupole length *L*.

$$\mathbf{M}_{FQ} = \begin{pmatrix} \boxed{\begin{matrix} \cos\Omega & \frac{1}{\sqrt{k}}\sin\Omega \\ -\sqrt{k}\sin\Omega & \cos\Omega \end{matrix}} & \cdots & 0 \\ \vdots & \boxed{\begin{matrix} \cosh\Omega & \frac{1}{\sqrt{k}}\sinh\Omega \\ \sqrt{k}\sinh\Omega & \cosh\Omega \end{matrix}} & \vdots \\ 0 & \cdots & \boxed{\begin{matrix} 1 & L/\gamma^2 \\ 0 & 1 \end{matrix}} \end{pmatrix} \quad (k<0)$$

$$\mathbf{M}_{DQ} = \begin{pmatrix} \boxed{\begin{matrix} \cosh\Omega & \frac{1}{\sqrt{k}}\sinh\Omega \\ \sqrt{k}\sinh\Omega & \cosh\Omega \end{matrix}} & \cdots & 0 \\ \vdots & \boxed{\begin{matrix} \cos\Omega & \frac{1}{\sqrt{k}}\sin\Omega \\ -\sqrt{k}\sin\Omega & \cos\Omega \end{matrix}} & \vdots \\ 0 & \cdots & \boxed{\begin{matrix} 1 & L/\gamma^2 \\ 0 & 1 \end{matrix}} \end{pmatrix} \quad (k>0)$$

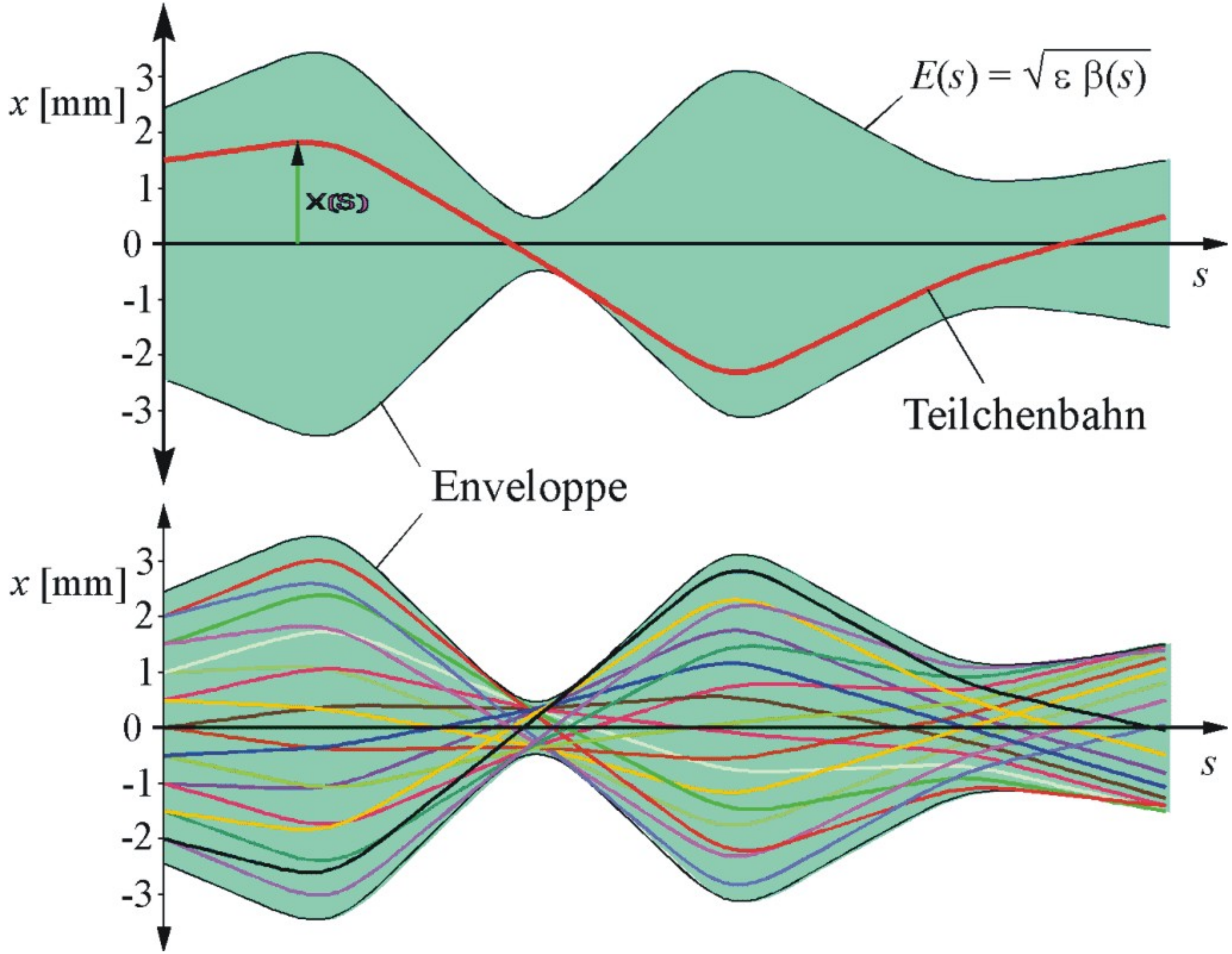
3.2.4. Particle Orbits in a System of Magnets

With the derived matrixes particle trajectories may be calculated for any given arbitrary beam transport line by cutting this beam line into smaller uniform pieces so that *k*=const. and *R*=const. in each of these pieces:



$$\boxed{\vec{X}_E = \mathbf{M}_{D5} \cdot \mathbf{M}_{Q4} \cdot \mathbf{M}_{D4} \cdot \mathbf{M}_{Q3} \cdot \mathbf{M}_{D3} \cdot \mathbf{M}_{Q2} \cdot \mathbf{M}_{D2} \cdot \mathbf{M}_{Q1} \cdot \mathbf{M}_{D1} \cdot \vec{X}_0}$$

but:



x [mm]
3
2
1
0
-1
-2
-3
x(s)
$E(s) = \sqrt{\varepsilon\,\beta(s)}$
s
Teilchenbahn
Enveloppe
x [mm]
3
2
1
0
-1
-2
-3
s

3.3. Particle Beams and Phase Space

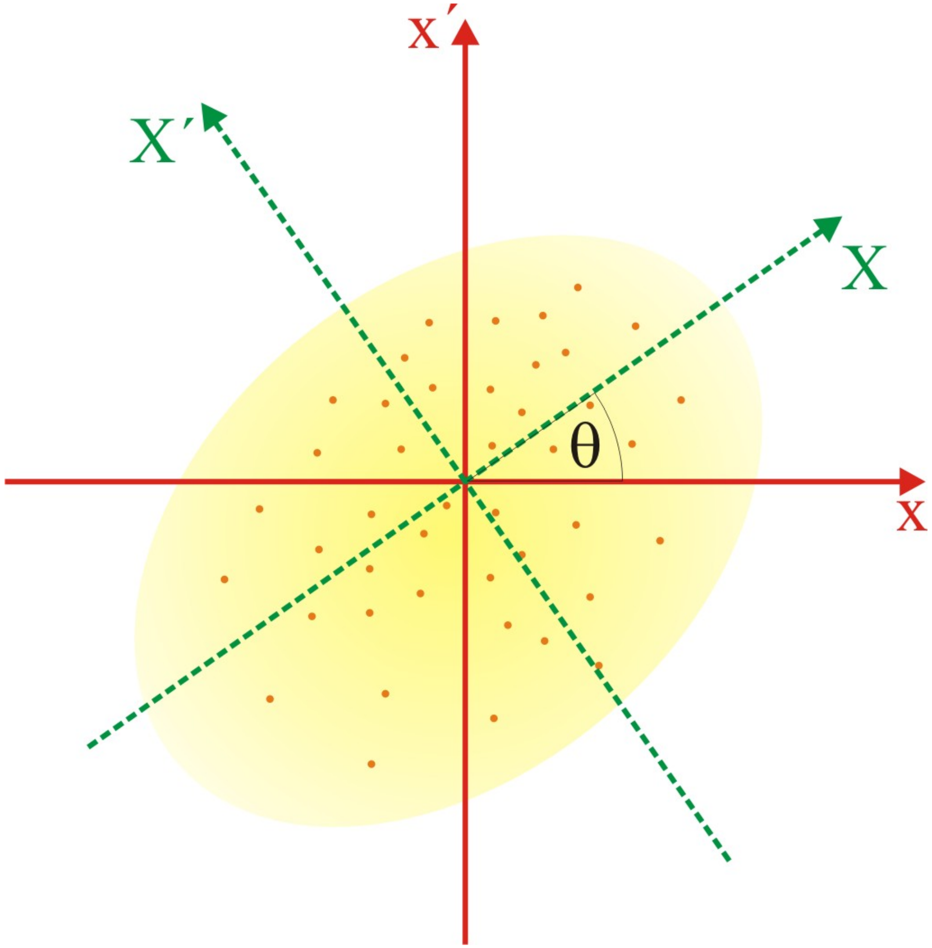
3.3.1. Beam Emittance

We consider a beam as a statistical set of points in the phase space. In linear beam optics, the horizontal and vertical planes are decoupled and we will therefore concentrate on a two-dimensional phase space (x, x'). A distribution of points can be translated and rotated without changing its spread. We choose the origin of the two coordinate axes $\hat{e}_x$ and $\hat{e}_{x'}$ at the barycentre of the points, such that the averages of their coordinates vanish:

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i = 0$$

$$\overline{x'} = \frac{1}{N}\sum_{i=1}^{N} x_i{}' = 0$$

The variances $\sigma_X{}^2$ and $\sigma_{X'}{}^2$ are determined in a reference system (X, X') which is rotated by a polar angle θ with respect to (x, x') in order to minimize (maximize) the sum of their squared distances to these axes:



$$\boxed{\begin{aligned} X_i &= x_i \cdot \cos\theta + x_i' \cdot \sin\theta \\ X_i' &= -x_i \cdot \sin\theta + x_i' \cdot \cos\theta \end{aligned}}$$

The mean square distances

$$d_{X'}^2 = \sigma_X{}^2 = \frac{1}{N}\sum_{i=1}^{N} X_i{}^2 = \overline{x^2}\cos^2\theta + \overline{x'^2}\sin^2\theta + \overline{x\,x'}\sin 2\theta$$

$$d_X^2 = \sigma_{X'}{}^2 = \frac{1}{N}\sum_{i=1}^{N} X_i{}'^2 = \overline{x^2}\sin^2\theta + \overline{x'^2}\cos^2\theta - \overline{x\,x'}\sin 2\theta$$

are minimized with respect to the angle θ ($\frac{\partial \sigma_X{}^2}{\partial\theta} = \frac{\partial \sigma_{X'}{}^2}{\partial\theta} = 0$) when

$$\tan 2\theta = \frac{2\overline{x\,x'}}{\overline{x^2} - \overline{x'^2}}$$

giving (from $\sigma_X{}^2 + \sigma_{X'}{}^2 = \overline{x^2} + \overline{x'^2}$ and $\sigma_X{}^2 - \sigma_{X'}{}^2 = 2\overline{x\,x'}/\sin 2\theta$)

$$\sigma_X{}^2 = \frac{1}{2}\left(\overline{x^2} + \overline{x'^2} + \frac{2\overline{x\,x'}}{\sin 2\theta}\right)$$

$$\sigma_{X'}{}^2 = \frac{1}{2}\left(\overline{x^2} + \overline{x'^2} - \frac{2\overline{x\,x'}}{\sin 2\theta}\right)$$

We will define the spread of the distribution, which is called the **emittance ε**, by

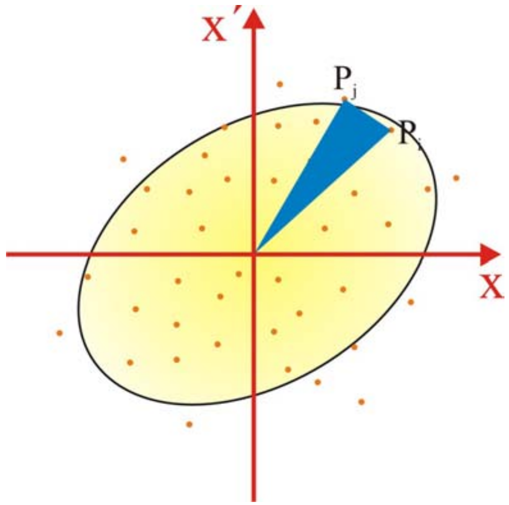
$$\boxed{\varepsilon = \sigma_X \cdot \sigma_X{}' = \sqrt{\overline{x^2} \cdot \overline{x'^2} - \overline{x\,x'}^2}}$$

It is important to note that this definition usually is used for electron beams, the emittance of proton beams is typically defined as 4ε !

The emittance can be considered as a statistical mean area:

$$\varepsilon = \frac{1}{N}\sqrt{\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(x_i x_j{}' - x_j x_i{}'\right)^2} = \frac{1}{N}\sqrt{2\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}{}^2}$$

where A_{ij} is the area of the triangle $0P_iP_j$ and ε is a measure of the spread of the points around their barycentre.



The area of the envelope-ellipse is just π times the emittance ε

$$A = \pi\, a\, b = \pi\, \sigma_X\, \sigma_{X'} = \pi\, \varepsilon$$

and its equation with respect to the axes X and X' is

$$\boxed{\frac{X^2}{{\sigma_X}^2} + \frac{X'^2}{{\sigma_{X'}}^2} = \frac{X^2}{a^2} + \frac{X'^2}{b^2} = 1}$$

where a and b are the two semi-axes of the envelope-ellipse.

3.3.2. Twiss Parameters

The emittance does not give all the information that is contained in the second-order moments. We will therefore define an ellipse with parameters involving the second-order moments of the particle spread in phase space. By an inverse rotation of angle $-\theta$ in phase space we obtain

$$\varepsilon^2 = x^2 \cdot \sigma_{x'}^2 - 2\, x\, x' \cdot \overline{x\, x'} + x'^2 \cdot \sigma_x^2 = x^2 \cdot \sigma_{x'}^2 - 2\, x\, x' \cdot r\, \sigma_x\, \sigma_{x'} + x'^2 \cdot \sigma_x^2$$

where we have defined the correlation coefficient

$$r = \frac{\overline{x\,x'}}{\sqrt{\overline{x^2} \cdot \overline{x'^2}}}$$

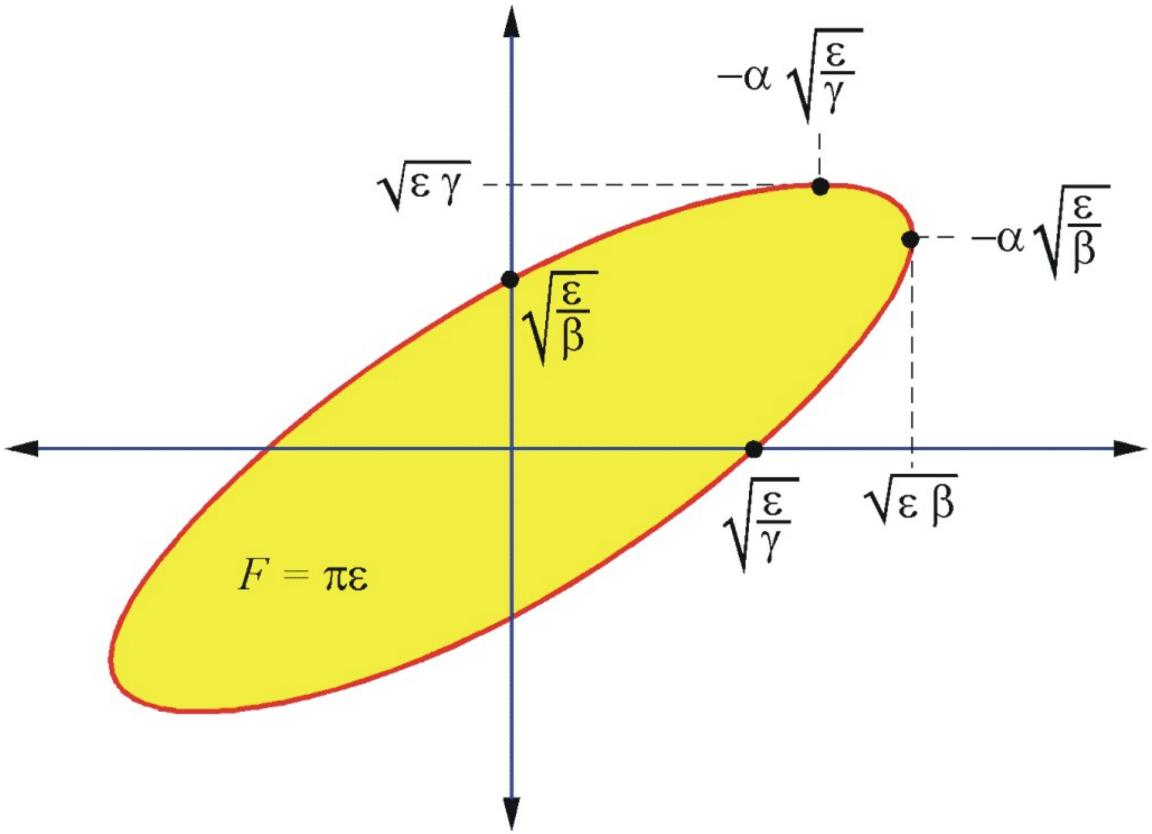
We may define the so called Twiss-parameters α, β, and γ such that

$$\boxed{\begin{aligned} \sigma_x &= \sqrt{\overline{x^2}} = \sqrt{\beta\,\varepsilon} \\ \sigma_{x'} &= \sqrt{\overline{x'^2}} = \sqrt{\gamma\,\varepsilon} \\ r\,\sigma_x\,\sigma_{x'} &= \overline{x\,x'} = -\alpha\,\varepsilon \end{aligned}}$$

and the equation of the envelope-ellipse reads in the "conventional" form:

$$\boxed{\gamma\,x^2 + 2\alpha\,x\,x' + \beta\,x'^2 = \varepsilon}$$

The meaning of the Twiss-parameters can be read off from the graphical representation of the envelope-ellipse:



- $\sqrt{\beta}$ represents the r.m.s. beam-envelope per unit emittance,
- $\sqrt{\gamma}$ represents the r.m.s. beam divergence per unit emittance,
- α is proportional to the correlation between x and x'.

3.3.3. Betatron Functions

In the following, we will first concentrate on the situation where $\Delta p / p = 0$. With $K(s) = 1/R^2(s) - k(s)$ the equation of motion reads

$$x''(s) + K(s) \cdot x(s) = 0$$

It describes a transverse oscillation with position dependent amplitude and phase, which is called betatron oscillation. We try to solve this equation with the set-up

$$x(s) = \sqrt{\varepsilon} \cdot u(s) \cdot \cos\left(\phi(s) + \varphi_0\right)$$

and obtain:

$$\left[u'' - u \cdot \phi'^2 + K \cdot u\right] \cdot \cos\left(\phi + \varphi_0\right) - \left[2 \cdot u' \cdot \phi' + u \cdot \phi''\right] \sin\left(\phi + \varphi_0\right) = 0$$

This relation is valid for any given phase $\phi(s)$ at any given position s, therefore

$$u'' - u \cdot \phi'^2 + K \cdot u = 0$$
$$2 \cdot u' \cdot \phi' + u \cdot \phi'' = 0$$

By integration of the second equation we obtain $\phi(s) = \int_0^s \frac{d\tilde{s}}{u^2(\tilde{s})}$

and by using this relation $u'' - \frac{1}{u^3} + K \cdot u = 0$.

With the definition of the betatron-function $\beta(s) := u^2(s)$ we derive for the amplitude and phase of the oscillation:

$$\boxed{\begin{aligned} x(s) &= \sqrt{\varepsilon} \cdot \sqrt{\beta(s)} \cdot \cos\left(\phi(s) + \varphi_0\right) \\ \phi(s) &= \int_0^s \frac{d\tilde{s}}{\beta(\tilde{s})} \end{aligned}}$$

Building the first derivative and defining $\alpha(s) := -\frac{\beta'(s)}{2}$, we obtain

$$\boxed{x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \left\{ \alpha(s) \cdot \cos\left(\phi(s) + \varphi_0\right) + \sin\left(\phi(s) + \varphi_0\right) \right\}}$$

The equation for x can be transformed to

$$\cos^2(\phi+\varphi_0)=\frac{x^2}{\varepsilon\cdot\beta},$$

which can be used in combination with the equation for x′ to obtain

$$\sin^2(\phi+\varphi_0)=\left(\sqrt{\frac{\beta}{\varepsilon}}\cdot x'+\frac{\alpha}{\sqrt{\varepsilon\beta}}\cdot x\right)^2$$

Using $\cos^2+\sin^2=1$ we derive

$$\frac{x^2}{\beta(s)}+\left(\frac{\alpha(s)}{\sqrt{\beta(s)}}\cdot x+\sqrt{\beta(s)}\cdot x'\right)^2=\varepsilon$$

which can be transformed by defining $\gamma(s):=\dfrac{1+\alpha^2(s)}{\beta(s)}$ to:

$$\gamma x^2+2\alpha\, x\, x'+\beta x'^2 = \varepsilon, \qquad \text{where} \quad \alpha=-\frac{\beta'}{2} \quad \text{and} \quad \gamma=\frac{1+\alpha^2}{\beta}$$

3.3.4. Transformation in Phase Space

According to Liouville's theorem, all particles enclosed by an envelope-ellipse will stay within that ellipse. The transformation of the horizontal and vertical ellipse parameters along the beam line may be derived from the transport matrixes in the horizontal and vertical plane. Starting at $s=0$, we have

$$\gamma_0 x_0^{\,2} + 2\alpha_0 x_0 x_0' + \beta_0 x_0'^{\,2} = \varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

Any particle trajectory starting at $s=0$ transforms to $s\neq 0$ by

$$\vec{X} = \mathbf{M} \cdot \vec{X}_0$$

Defining the Beta-matrix **B**

$$\mathbf{B} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}, \qquad \left|\mathbf{B}\right| = \beta\gamma - \alpha^2 = 1$$

the equation of the envelope-ellipse can be transformed to:

$$\varepsilon = {}^T\vec{X}_0 \cdot \mathbf{B}_0^{-1} \cdot \vec{X}_0 = {}^T\vec{X}_1 \cdot \mathbf{B}_1^{-1} \cdot \vec{X}_1$$

where the inverse of the Beta-matrix is

$$\mathbf{B}^{-1} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$$

and displacement-vector $\vec{X}$ transforms according to

$$\vec{X}_1 = \mathbf{M} \cdot \vec{X}_0, \qquad {}^T\vec{X}_1 = {}^T\!\left(\mathbf{M} \cdot \vec{X}_0\right) = {}^T\vec{X}_0 \cdot {}^T\mathbf{M}$$

By inserting $\mathbf{1} = \mathbf{M}^{-1} \cdot \mathbf{M}$, we obtain:

$$\begin{aligned} \varepsilon &= {}^T\vec{X}_0 \cdot {}^T\mathbf{M} \cdot {}^T\mathbf{M}^{-1} \cdot \mathbf{B}_0{}^{-1} \cdot \mathbf{M}^{-1} \cdot \mathbf{M} \cdot \vec{X}_0 \\ &= {}^T\!\left(\mathbf{M} \cdot \vec{X}_0\right) \cdot \left({}^T\mathbf{M}^{-1} \cdot \mathbf{B}_0{}^{-1} \cdot \mathbf{M}^{-1}\right) \cdot \left(\mathbf{M} \cdot \vec{X}_0\right) \\ &= {}^T\vec{X}_1 \cdot \left(\mathbf{M} \cdot \mathbf{B}_0 \cdot {}^T\mathbf{M}\right)^{-1} \cdot \vec{X}_1 \end{aligned}$$

and we can read off the transformation of the Beta-matrix:

$$\boxed{\mathbf{B}_1 = \mathbf{M} \cdot \mathbf{B}_0 \cdot {}^T\mathbf{M}}$$

This can e.g. be used to derive the beta-function around a symmetry-point of a transfer-line where $\alpha = 0$ in a simple way:

$$\mathbf{B}_1(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \beta_{sym} & 0 \\ 0 & 1/\beta_{sym} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_{sym} + \dfrac{s^2}{\beta_{sym}} & \dfrac{s}{\beta_{sym}} \\ \dfrac{s}{\beta_{sym}} & \dfrac{1}{\beta_{sym}} \end{pmatrix}$$

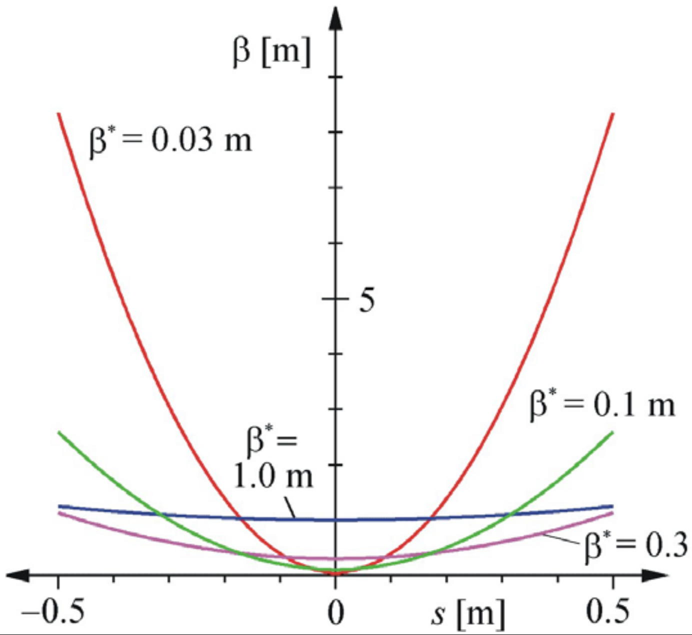
This gives the relations for the Twiss-parameters around a symmetry-point:

$$\beta(s) = \beta_{sym} + \frac{s^2}{\beta_{sym}}$$

$$\alpha(s) = -\frac{s}{\beta_{sym}}$$

The corresponding beam size scales with

$\sigma_x = \sqrt{\varepsilon \cdot \beta(s)}$!



The transformation matrix **M** can be derived also from the Twiss-parameters. With

$$x(s) = \sqrt{\varepsilon}\cdot\sqrt{\beta}\cdot\left\{\cos\phi\cdot\cos\varphi_0 - \sin\phi\cdot\sin\varphi_0\right\}$$

$$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta}}\cdot\left\{\alpha\cdot\left[\cos\phi\cdot\cos\varphi_0 - \sin\phi\cdot\sin\varphi_0\right] - \sin\phi\cdot\cos\varphi_0 + \cos\phi\cdot\sin\varphi_0\right\}$$

and the starting conditions $x(0) = x_0,\ x'(0) = x_0',\ \phi(0) = 0$, which transform to

$$\cos\varphi_0 = x_0 \Big/ \sqrt{\varepsilon\,\beta_0}$$

$$\sin\varphi_0 = -\,1\Big/\sqrt{\varepsilon}\left(x_0'\sqrt{\beta_0} + \alpha_0\, x_0 \Big/ \sqrt{\beta_0}\right)$$

we obtain:

$$\boxed{\mathbf{M}(s) = \begin{pmatrix} \dfrac{\sqrt{\beta}}{\sqrt{\beta_0}}\left(\cos\phi + \alpha_0\sin\phi\right) & \sqrt{\beta\,\beta_0}\,\sin\phi \\[2ex] \dfrac{\alpha_0 - \alpha}{\sqrt{\beta\,\beta_0}}\cos\phi - \dfrac{1+\alpha\,\alpha_0}{\sqrt{\beta\,\beta_0}}\sin\phi & \dfrac{\sqrt{\beta}}{\sqrt{\beta_0}}\left(\cos\phi - \alpha\sin\phi\right) \end{pmatrix}}$$

3.3.6. Dispersion Functions

Including a relative momentum deviation $\frac{\Delta p}{p} = \delta$, we may use the principal solutions $C(s), S(s)$ to find a particular solution $D(s)$ of the equation of motion where we have set $\delta = 1$:

$$D''(s) + K(s) \cdot D(s) = \frac{1}{R(s)}$$

$$D(s) = \int_0^s \frac{1}{R(\tilde{s})} \cdot \underbrace{\left[S(s) \cdot C(\tilde{s}) - C(s) \cdot S(\tilde{s}) \right]}_{G(s,\tilde{s})} \cdot d\,\tilde{s}$$

Forming the second derivative and using the properties of the Wronskian, we finally obtain

$$D''(s) = \frac{1}{R(s)} + \underbrace{S''(s)}_{=-K \cdot S} \cdot \int_0^s \frac{1}{R(\tilde{s})} \cdot C(\tilde{s}) \cdot d\,\tilde{s} - \underbrace{C''(s)}_{=-K \cdot C} \cdot \int_0^s \frac{1}{R(\tilde{s})} \cdot S(\tilde{s}) \cdot d\,\tilde{s}$$

and verify the validity of the setup using the Green's function $G(s,\tilde{s})$ defined above. We may then generally calculate the dispersion function $D(s)$ from the principal solutions by solving

$$\boxed{D(s) = S(s)\cdot\int_0^s \frac{1}{R(\tilde{s})}\cdot C(\tilde{s})\cdot d\,\tilde{s} - C(s)\cdot\int_0^s \frac{1}{R(\tilde{s})}\cdot S(\tilde{s})\cdot d\,\tilde{s}}$$

This relation may also be used in order to derive the parameter $r_{16} = \langle x|\delta\rangle$ in the transport matrix of dipole magnets. The parameter $r_{26} = \langle x'|\delta\rangle$ is then calculated by forming the derivative $D'(s)$.

3.3.7. Path Length and Momentum Compaction

The path length of a particle with horizontal orbit displacement x_D is influenced by the curved sections of the beam line. The total path length is therefore given by

$$L = \int_{s_0}^{s} \left[\frac{R(\tilde{s}) + x_D(\tilde{s})}{R(\tilde{s})} \right] d\tilde{s} = L_0 + \int_{s_0}^{s} \frac{x_D(\tilde{s})}{R(\tilde{s})} d\tilde{s}$$

With a given relative momentum deviation $\delta = \Delta p / p$, we have $x_D(s) = D(s) \cdot \delta$ and obtain the deviation $\Delta L = L - L_0$ from the ideal path length

$$\Delta L = \delta \int_{s_0}^{s} \frac{D(\tilde{s})}{R(\tilde{s})} d\tilde{s}$$

This variation is determined by the so called **momentum compaction factor α_c**, defined by

$$\boxed{\alpha_c = \frac{\Delta L / L_0}{\Delta p / p} = \frac{1}{L_0} \cdot \int_{s_0}^{s} \frac{D(\tilde{s})}{R(\tilde{s})} d\tilde{s}}$$

The travel time is given by $\tau = L/(\beta c)$, and its relative variation is obtained from the logarithmic differentiation

$$\Delta \ln \tau = \frac{\Delta \tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta \beta}{\beta} = \left(\alpha_c - \frac{1}{\gamma^2} \right) \cdot \delta = -\eta \cdot \delta$$

where we have defined the **slip factor** η by

$$\boxed{\eta = \frac{1}{\gamma^2} - \alpha_c}$$

The momentum compaction factor therefore characterizes a critical energy

$$\boxed{\gamma_{tr} = \frac{1}{\sqrt{\alpha_c}}},$$

which is called the **transition energy**, where the slip factor vanishes.

4. Circular Accelerators

4.1. Periodic Focusing Systems

After the discussion of linear beam optics of beam transfer lines which in principle can be made of an irregular array of magnets, we will now discuss a repetitive sequence of a special magnet arrangement, which is called a periodic lattice.

4.1.1. Periodic Betatron Functions

Periodic solutions of a periodic lattice of period-length L will be

$$\begin{aligned} \beta(s_0 + L) &= \beta(s_0) = \beta_0 \\ \alpha(s_0 + L) &= \alpha(s_0) = \alpha_0 \\ D(s_0 + L) &= D(s_0) = D_0 \\ D'(s_0 + L) &= D'(s_0) = D_0' \end{aligned}$$

Using the transformation of the Beta-matrix and the 2x2 transport matrix $\mathbf{M}_2$ (for the horizontal or vertical plane), these relations transform to

$$\mathbf{B}_0 = \mathbf{M}_2 \cdot \mathbf{B}_0 \cdot {}^{\mathrm{T}}\mathbf{M}_2$$

which yields to

$$\beta_0 = \frac{2 r_{12}}{\sqrt{2 - r_{11}^2 - 2 r_{12} r_{21} - r_{22}^2}}$$

$$\alpha_0 = \frac{r_{11} - r_{22}}{2 r_{12}} \beta_0$$

The dispersion function is obtained from the transformation using the 3x3 transport matrix $\mathbf{M}_3$ (in case of upright magnets for the horizontal plane only):

$$\begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix} = \mathbf{M}_3 \cdot \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

yielding:

$$D_0' = \frac{r_{21} r_{13} + r_{23}(1 - r_{11})}{2 - r_{11} - r_{22}}$$

$$D_0 = \frac{r_{12} D_0' + r_{13}}{1 - r_{11}}$$

If the periodic lattice includes a symmetry point, we will have $\alpha_0 = 0$ and $D_0' = 0$ for this point. We then get the simple solutions

$$\beta_0^{\text{sym}} = \frac{r_{12}}{\sqrt{1-r_{11}^2}}, \qquad D_0^{\text{sym}} = \frac{r_{13}}{1-r_{11}}$$

4.1.2. Stability Criterion

If $\mathbf{M}(L)$ is the transformation matrix for one periodic cell we will have for N cells:

$$\mathbf{M}(N \cdot L) = \left[\mathbf{M}(L)\right]^N$$

We derive the eigenvalues from $\quad |\mathbf{M} - \lambda \cdot \mathbf{I}| = \lambda^2 - \mathrm{Tr}\{\mathbf{M}\} \cdot \lambda + 1 = 0$

With the substitution $\mathrm{Tr}\{\mathbf{M}\} = 2 \cdot \cos\phi$ we obtain $\quad \lambda_{1,2} = \cos\phi \pm i \sin\phi = e^{\pm i\phi}$

We require that the eigenvalues remain finite yielding a real betatron phase ϕ . This gives the general stability condition:

$$|\mathrm{Tr}\{\mathbf{M}\}| = |r_{11} + r_{22}| \leq 2$$

For a full lattice period, we have $\beta = \beta_0$, $\alpha = \alpha_0$ and therewith

$$\mathbf{M} = \begin{pmatrix} \cos\phi + \alpha_0 \sin\phi & \beta_0 \sin\phi \\ -\gamma_0 \sin\phi & \cos\phi - \alpha_0 \sin\phi \end{pmatrix}$$

Defining the Twiss-matrix

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \qquad \mathbf{J}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$$

the transformation matrix for one period can be expressed by

$$\mathbf{M} = \mathbf{I} \cdot \cos\phi + \mathbf{J} \cdot \sin\phi$$

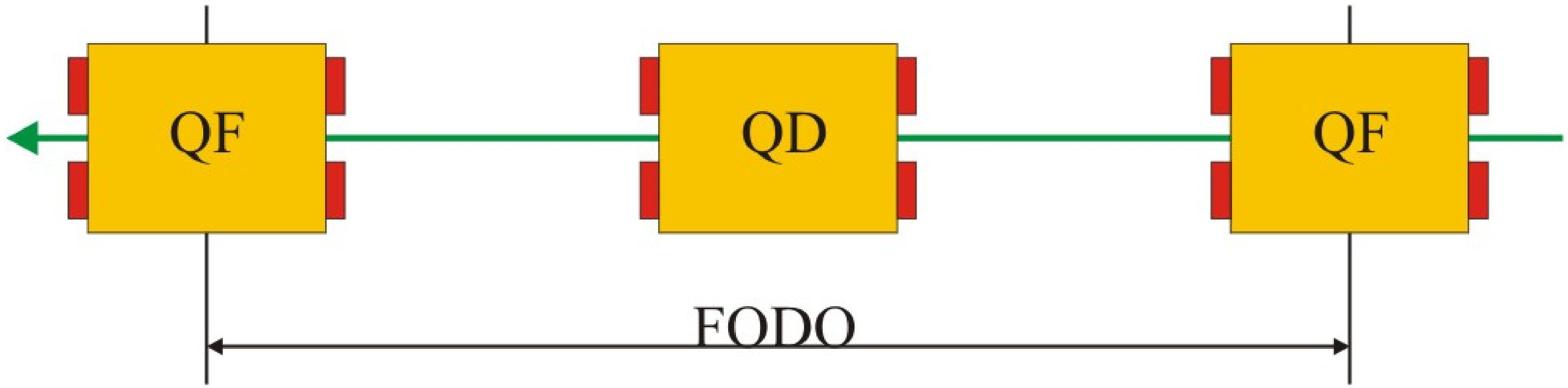
Similar to Moivre's formula we get for N equal periods

$$\mathbf{M}^N = \left(\mathbf{I} \cdot \cos\phi + \mathbf{J} \cdot \sin\phi\right)^N = \mathbf{I} \cdot \cos(N\phi) + \mathbf{J} \cdot \sin(N\phi)$$

and

$$\left|\mathrm{Tr}\left\{\mathbf{M}^N\right\}\right| = \left|2 \cdot \cos(N\phi)\right| \leq 2$$

4.1.3. General FODO Lattice



The FODO geometry can be expressed symbolically by the sequence

$$\underbrace{\tfrac{1}{2}\mathrm{QF},\, \mathrm{D},\, \tfrac{1}{2}\mathrm{QD}}_{=\mathbf{M}_{-1/2}},\ \underbrace{\tfrac{1}{2}\mathrm{QD},\, \mathrm{D},\, \tfrac{1}{2}\mathrm{QF}}_{=\mathbf{M}_{1/2}}$$

It is sufficient to use the thin lens approximation ($l_Q \ll f$). We will set the focal lengths to $f_2 = 2 f_D$, $f_1 = 2 f_F$, the drift length to *L*. Defining

$$1/f^* = 1/f_1 + 1/f_2 - L/(f_1 \cdot f_2)$$

the transformation matrix of half a FODO cell is

$$\mathbf{M}_{1/2} = \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} = \begin{pmatrix} 1-\dfrac{L}{f_1} & L \\ -\dfrac{1}{f^*} & 1-\dfrac{L}{f_2} \end{pmatrix}$$

Multiplication with the reverse matrix gives

$$\mathbf{M}_{\mathrm{FODO}} = \begin{pmatrix} 1-2\dfrac{L}{f^*} & 2L\cdot\left(1-\dfrac{L}{f_2}\right) \\ -\dfrac{2}{f^*}\cdot\left(1-\dfrac{L}{f_1}\right) & 1-2\dfrac{L}{f^*} \end{pmatrix} \quad \text{and} \quad \left|\mathrm{Tr}\{\mathbf{M}\}\right| = \left|2-\frac{4L}{f^*}\right| < 2$$

This is equivalent to $0 < \dfrac{L}{f^*} < 1$, and defining $u = \dfrac{L}{f_1}$, $v = \dfrac{L}{f_2}$ we get

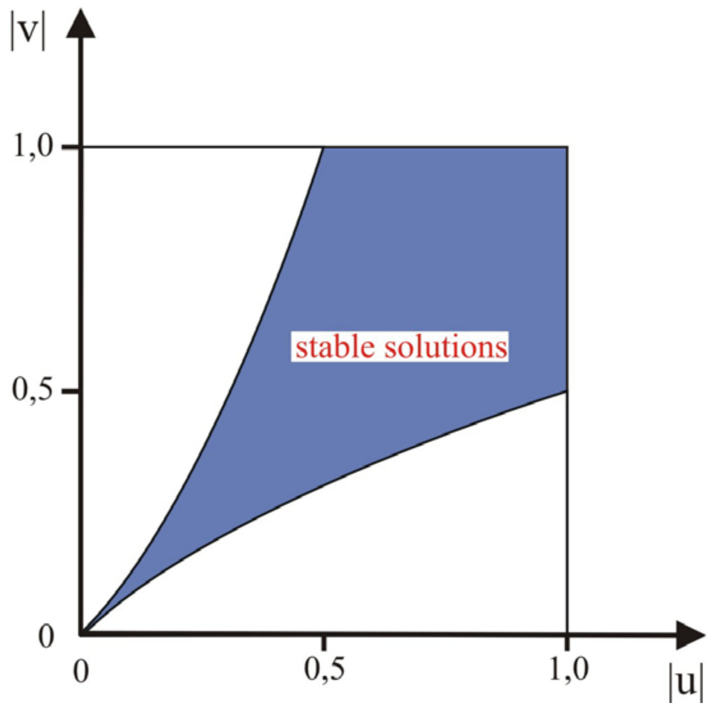
$$\boxed{0 < u + v - u\cdot v < 1}$$

from which we derive the boundaries of the stability region

$$|u_1| = 1, \quad |v_2| = \frac{|u|}{1-|u|}$$

$$|v_1| = 1, \quad |v_3| = \frac{|u|}{1+|u|}$$

which gives the famous necktie-diagram for thin lens approximation:



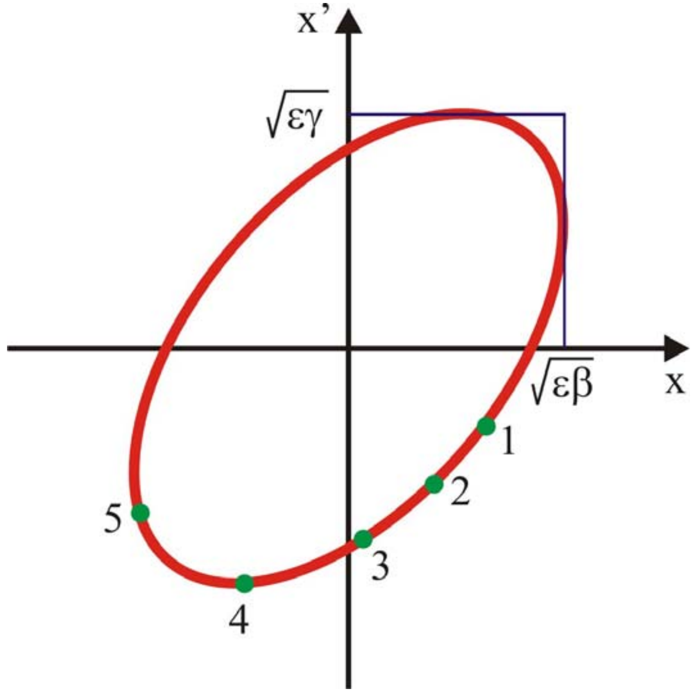
4.2. Transverse Beam Dynamics

4.2.1. Betatron Tune

The betatron tune Q is defined as the number of oscillations per revolution:

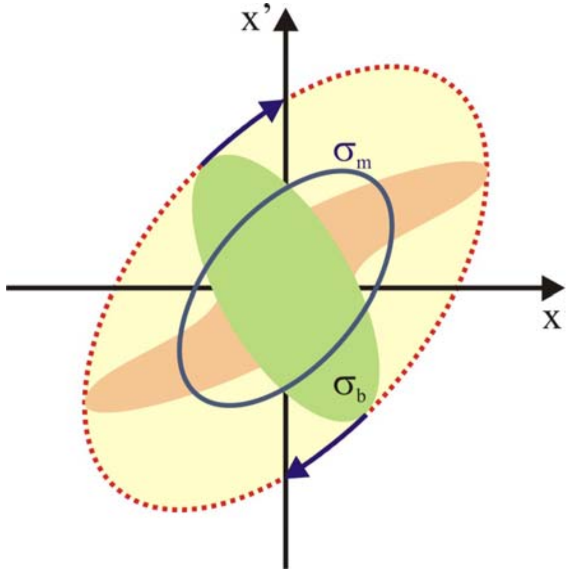
$$Q_{x,z} = \frac{1}{2\pi} \cdot \oint \frac{d\,s}{\beta_{x,z}(s)}$$

If one regards the phase space at an arbitrarily chosen point, a single particle moves on its phase space ellipse ,where the points represents the parameters after 1,2, … 5 revolutions.



4.2.2. Filamentation

If the envelope ellipse σ_b of the beam is not matched to the ellipse σ_m of the periodic lattice, it will start to rotate with a phase advance per revolution of $2\pi Q$



Due to effects of higher order the quadrupole strengths and therefore the phase advance depends on the amplitude (horizontal and vertical displacements). In case of mismatch, the beam phase space distribution starts to filament. After a large number of revolutions, the distribution may be surrounded by a large ellipse.

4.2.3. Normalized Coordinates

It is useful to transform the oscillatory solution with varying amplitude and frequency $x(s) = \sqrt{\varepsilon\,\beta(s)} \cdot \cos\left(\phi(s) + \varphi_0\right)$ to a solution which looks exactly like that of a harmonic oscillator. We introduce Floquet's coordinates through the transformation:

$$\boxed{\begin{aligned} \psi(s) &= \frac{\phi(s)}{Q} \\ \eta(\psi) &= \frac{x(s)}{\sqrt{\beta(s)}} \end{aligned}}$$

The angle ψ advances by 2π every revolution. It coincides with θ at each $\beta^{\max}$ and $\beta^{\min}$ location and does not depart very much from θ in between.

Using these normalized coordinates, the equation of motion can be transformed to

$$\frac{d^2\eta}{d\psi^2} + Q^2\eta = 0$$

The solution transforms to

$$\eta(\psi) = \eta_0 \cdot \cos(Q\psi + \lambda)$$

and the phase space ellipse becomes an invariant cycle of radius η_0.

The use of normalized coordinates is convenient in the discussions of perturbations and aberrations:

4.2.4. Closed orbit distortions

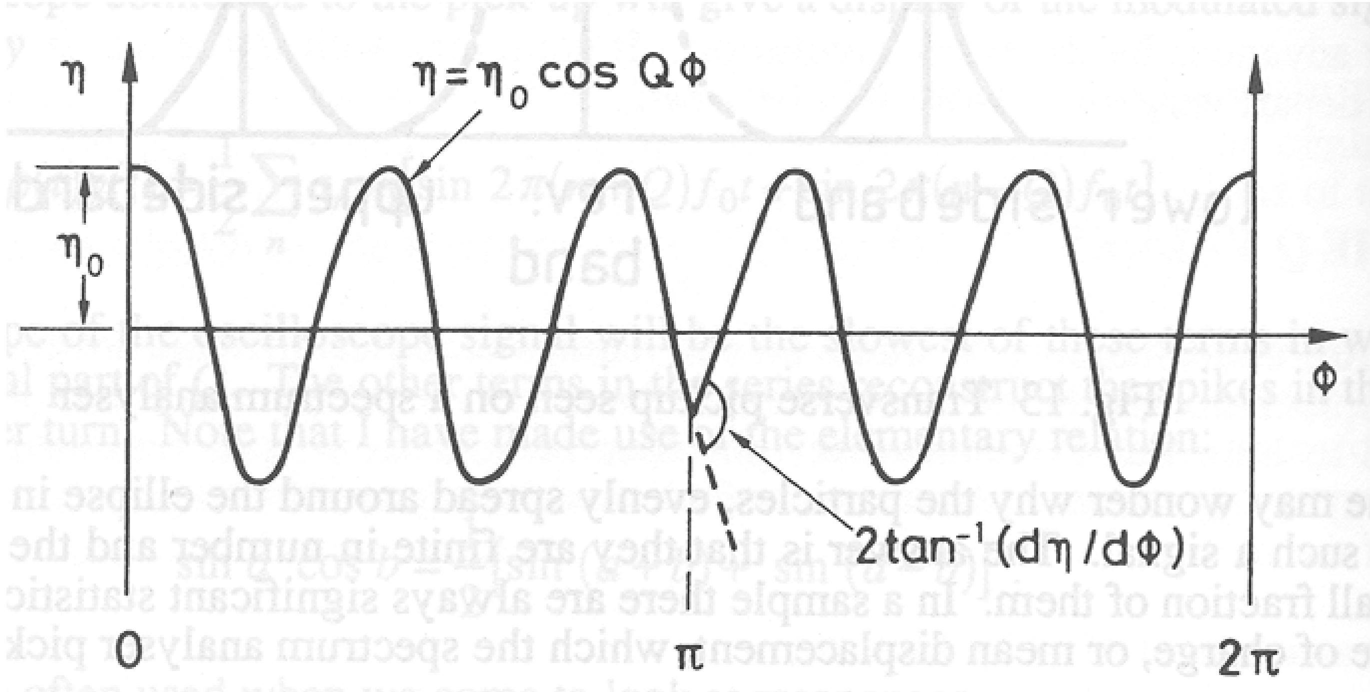
Let us assume a dipole field error produced by a short dipole which makes a constant angular kick in divergence

$$\delta x' = \frac{\delta(B\,l)}{B\,R}$$

which perturbs the orbit trajectory which elsewhere obeys

$$\frac{d^2\eta}{d\psi^2} + Q^2\eta = 0, \quad \text{with} \quad \eta = \eta_0 \cos(Q\psi + \lambda)$$

We choose $\psi = 0$ to be diametrically opposite to the kick. Then by symmetry $\lambda = 0$ and the disturbed orbit oscillates around the ideal path



Differentiation gives $\dfrac{d\eta}{d\psi} = -\eta_0 Q \cdot \sin\left(Q\psi\right) = -\eta_0 Q \cdot \sin\left(\pi Q\right)$ at $\psi = \pi$.

With $\dfrac{d\psi}{ds} = \dfrac{1}{Q\beta_0}$ and $\dfrac{dx}{ds} = \sqrt{\beta_0} \cdot \dfrac{d\eta}{ds}$, we may relate the orbit displacement η_0 to the real kick by $$\frac{\delta x'}{2} = -\frac{dx}{ds} = -\sqrt{\beta_0}\cdot\frac{d\eta}{d\psi}\cdot\frac{d\psi}{ds} = \frac{\eta_0}{\sqrt{\beta_0}}\cdot\sin\left(\pi Q\right)$$

giving $$\eta_0 = \frac{\sqrt{\beta_0}}{2\sin\left(\pi Q\right)}\delta x'$$

Returning to physical coordinates, we obtain the closed orbit displacement $x_c(s)$ at position s for a field error at s_0 with $x = \sqrt{\beta}\cdot\eta$ and $\phi(s)-\phi(s_0)+Q\pi = Q\cdot\psi$:

$$x_c(s) = \sqrt{\beta}\,\eta_0\cos\left(Q\psi\right) = \underbrace{\left[\frac{\sqrt{\beta(s)\beta(s_0)}}{2\sin\left(\pi Q\right)}\frac{\delta\left(Bl\right)}{B\,R}\right]}_{=\text{ amplitude at position } s}\cdot\cos\left(\phi(s)-\phi(s_0)+Q\pi\right)$$

The effect of a random distribution of dipole errors can be estimated from the r.m.s. average, weighted according to the β_0 values of the kicks δx_i :

$$\boxed{x_c(s) = \frac{\sqrt{\beta(s)}}{2\sin\left(\pi Q\right)}\cdot\oint_s\sqrt{\beta(s_0)}\cdot\frac{\delta Bl}{B\,R}\cdot\cos\left(\phi(s)-\phi(s_0)+Q\pi\right)\cdot ds_0}$$

Using matrix algebra, the displacement of the closed orbit at the position of the field error can be calculated from the displacement vector just before and just after the kick element:

$$\begin{pmatrix} x_0 \\ x_0' - \delta x' \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} \cos\phi + \alpha_0 \sin\phi & \beta_0 \sin\phi \\ -\gamma_0 \sin\phi & \cos\phi - \alpha_0 \sin\phi \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

with $\phi = 2\pi Q$, giving

$$x_0 = \frac{\beta_0\, \delta x'}{2\sin(\pi Q)} \cos(\pi Q)$$

$$x_0' = \frac{\delta x'}{2\sin(\pi Q)} \left[\sin(\pi Q) - \alpha_0 \cos(\pi Q) \right]$$

The closed orbit displacement $x_c(s)$ is calculated from $\vec{x}_c(s) = \mathbf{M}(s_0, s) \cdot \vec{x}_0$

$$\begin{pmatrix} x_c(s) \\ x_c'(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\dfrac{\beta(s)}{\beta_0}} \left(\cos\phi + \alpha_0 \sin\phi\right) & \sqrt{\beta(s)\beta_0}\, \sin\phi \\ -\dfrac{1 + \alpha(s)\alpha_0}{\sqrt{\beta(s)\beta_0}} \sin\phi + \dfrac{1 - \alpha(s)\alpha_0}{\sqrt{\beta(s)\beta_0}} \cos\phi & \sqrt{\dfrac{\beta(s)}{\beta_0}} \left(\cos\phi - \alpha_0 \sin\phi\right) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

4.2.5. Gradient Errors

Consider a small gradient error which affects a quadrupole at position *s* in the lattice of a circular accelerator. Translated to matrix algebra, we have to multiply a perturbation matrix

$$\boldsymbol{\delta}\mathbf{Q}(s) = \begin{pmatrix} 1 & 0 \\ -\delta k(s)\cdot ds & 1 \end{pmatrix}$$

with the unperturbed matrix for one circle staring at s (where $\alpha(s)=\alpha_0$, …)

$$\mathbf{M}_0 = \begin{pmatrix} \cos\phi_0 + \alpha_0 \sin\phi_0 & \beta_0 \sin\phi_0 \\ -\gamma_0 \sin\phi_0 & \cos\phi_0 - \alpha_0 \sin\phi_0 \end{pmatrix}$$

giving:

$$\begin{aligned} \tilde{\mathbf{M}}(s) &= \boldsymbol{\delta}\mathbf{Q}(s)\cdot\mathbf{M}_0 \\ &= \begin{pmatrix} \cos\phi_0 + \alpha_0 \sin\phi_0 & \beta_0 \sin\phi_0 \\ -\delta k\, ds\left(\cos\phi_0 + \alpha_0 \sin\phi_0\right) - \gamma_0 \sin\phi_0 & -\delta k\, ds\, \beta_0 \sin\phi_0 + \cos\phi_0 - \alpha_0 \sin\phi_0 \end{pmatrix} \end{aligned}$$

From $1/2 \cdot \mathrm{Tr}\left\{\tilde{\mathbf{M}}\right\} = \cos\phi$ we can calculate the change in $\cos\phi$:

$$\Delta\left(\cos\phi\right) = -\Delta\phi \cdot \sin\phi_0 = -\frac{1}{2}\sin\phi_0\,\beta_0\,\delta k\,ds$$

$$2\pi\Delta Q = \Delta\phi = \frac{1}{2}\beta(s)\,\delta k(s)\,ds$$

Integrating over the length of the quadrupol perturbation, one obtains

$$\boxed{\Delta Q = \frac{1}{4\pi}\int \beta(s)\,\delta k(s)\,ds}$$

A gradient error will not influence the closed orbit but the betatron function of the lattice. In order to calculate the betatron amplitude modulation, we have to determine the single turn transport matrix starting at a given observer position s, introducing a small gradient perturbation at position s_0:

$$\tilde{\mathbf{M}}_s = \mathbf{M}(s,s_0)\cdot\boldsymbol{\delta}\mathbf{Q}(s_0)\cdot\mathbf{M}(s_0,s) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ -\delta k\,ds_0 & 1 \end{pmatrix}\cdot\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

It is only necessary to evaluate the element $\tilde{r}_{12}$ which is

$$\tilde{r}_{12} \;=\; b_{11}a_{12} + b_{12}\left(-\delta k\, ds \cdot a_{12} + a_{22}\right) \;=\; r_{12} - \delta k\, ds_0 \cdot a_{12}b_{12}$$

where r_{12} from the unperturbed matrix found by putting $\delta k\, ds_0 = 0$. Thus the variation in the r_{12} term due to the perturbation is

$$\begin{aligned}\Delta\left[\beta(s)\sin\left(2\pi Q_0\right)\right] &= -\delta k\, ds_0\, \beta(s)\,\beta(s_0)\cdot\sin\left(\phi(s)-\phi(s_0)\right)\cdot\sin\left(\phi(s_0)-\phi(s)\right)\\ &= -\delta k\, ds_0\, \beta(s)\,\beta(s_0)\cdot\sin\left(\phi(s)-\phi(s_0)\right)\cdot\sin\left[2\pi Q_0-\left(\phi(s)-\phi(s_0)\right)\right]\end{aligned}$$

Using $\sin\alpha\cdot\sin\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right)-\cos\left(\alpha+\beta\right)\right]$ the left-hand and right-hand sides can be expanded to give

$$\Delta\beta(s)\sin\left(2\pi Q_0\right) + \beta(s)\cdot\underbrace{2\pi\Delta Q}\cdot\cos\left(2\pi Q_0\right) =$$

$$\overbrace{\frac{1}{2}\delta k\, ds_0\, \beta(s_0)}\,\beta(s)\left\{\cos\left(2\pi Q_0\right)-\cos\left[2\left(\phi(s)-\phi(s_0)-\pi Q_0\right)\right]\right\}$$

This leaves the final expression for the betatron amplitude modulation (the so called **beta-beating**):

$$\boxed{\Delta\beta(s) = \frac{\beta(s)}{2\sin\left(2\pi Q_0\right)} \cdot \oint_s \delta k(s_0)\,\beta(s_0)\cos\left[2\left(\phi(s)-\phi(s_0)-\pi Q_0\right)\right] \cdot ds_0}$$

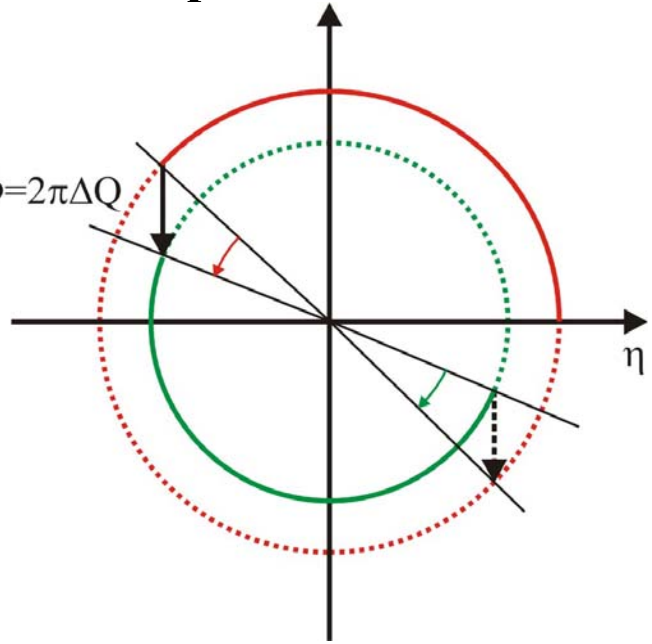
4.2.6. Optical Resonances

Dipole errors will give a large closed orbit displacement when the tune is close to an integer number which will be unstable in case of an integer Q.

Gradient errors will produce an average tune shift ΔQ and an amplitude modulation of the beta function which will explode for half integer Q values.

These phenomena are called resonances. Due to the turn by turn modulation of the tune, there exist regions of instability called stop bands around the resonance conditions. The width of these stop bands are given by the tune modulation amplitude:

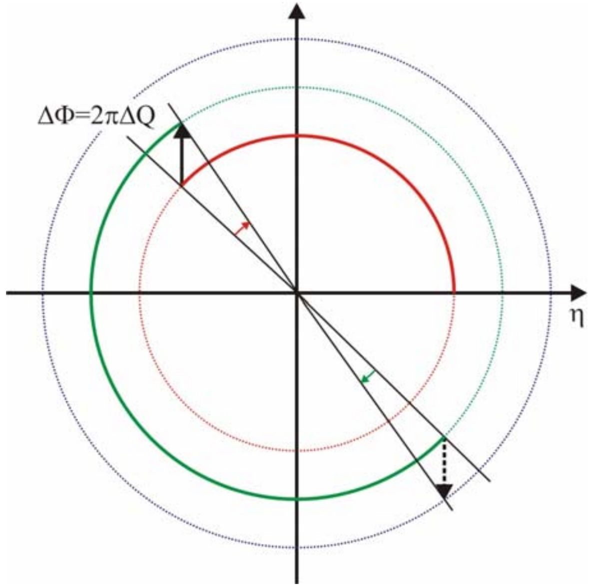
Dipole Errors:



No average tune shift $\Delta Q = 0$

Tune modulation amplitude dQ

Gradient Errors:



Average tune shift $\Delta Q = \dfrac{1}{4\pi}\beta\,\delta(kl)$

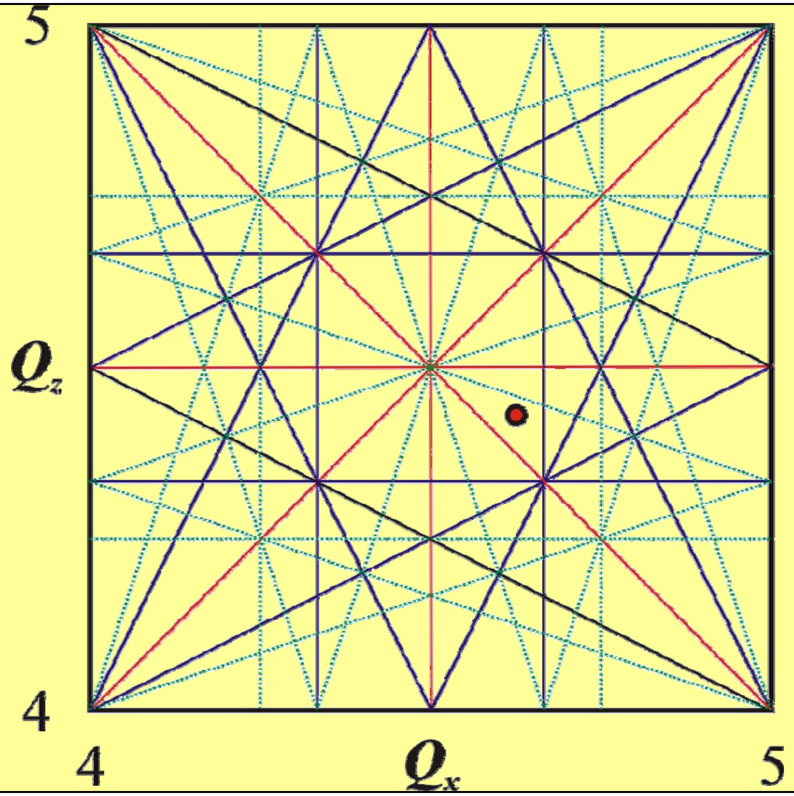
Tune modulation amplitude $dQ = \Delta Q$

Any particle whose unperturbed Q lies in the stop band width dQ will lock into resonance and is lost.

We may generalize and give a list of resonances and their driving multipoles:

resonance type	**driving multipole**
integer resonance: $Q = n$	dipole errors
half-integer resonance $2 \cdot Q = n$	quadrupole errors
third-integer resonance $3 \cdot Q = n$	sextupole errors

...



Due to betatron coupling, perturbations may depend on the betatron amplitude in both planes. These coupling terms lead to the generalized resonance condition

$$\boxed{j \cdot Q_x + k \cdot Q_z = N}$$

where $j+k$ indicates the order of the resonance.

4.2.7. Chromaticity

The variation of tunes is called **chromaticity** and is defined by the factor ξ in

$$\boxed{\Delta Q_{x,z} = \xi_{x,z} \cdot \frac{\Delta p}{p_0}}$$

We distinguish between natural chromaticity created by the chromatic aberration of quadrupole magnets and perturbations derived from non linear perturbations in the particles trajectories (e.g. produced by sextupole magnets).

<u>Natural Chromaticity:</u>

The quadrupole strength scales with the particles momentum:

$$\Delta k = -k \cdot {\Delta p}/{p_0}$$

and the tune shift can therefore be calculated from:

$$\boxed{\Delta Q_{x,z} = \underbrace{-\frac{1}{4\pi}\int \beta_{x,z}(\tilde{s}) \cdot k_{x,z}(\tilde{s}) \cdot d\tilde{s}}_{=\xi_{x,z}} \cdot \frac{\Delta p}{p_0}}$$

Chromaticity produced by sextupoles:

A beam of particles moving on a dispersion orbit through a sextupole magnet is "focused" by the nonlinear field due to horizontal displacement $x = D \cdot \Delta p / p_0$. We can derived a position dependent focusing strength from

$$\frac{e}{p} \vec{B}_{sext} = m\, x\, z\, \hat{e}_x + \frac{1}{2} m \left(x^2 - z^2 \right) \hat{e}_z$$

giving a dispersion dependent k_x and k_z to:

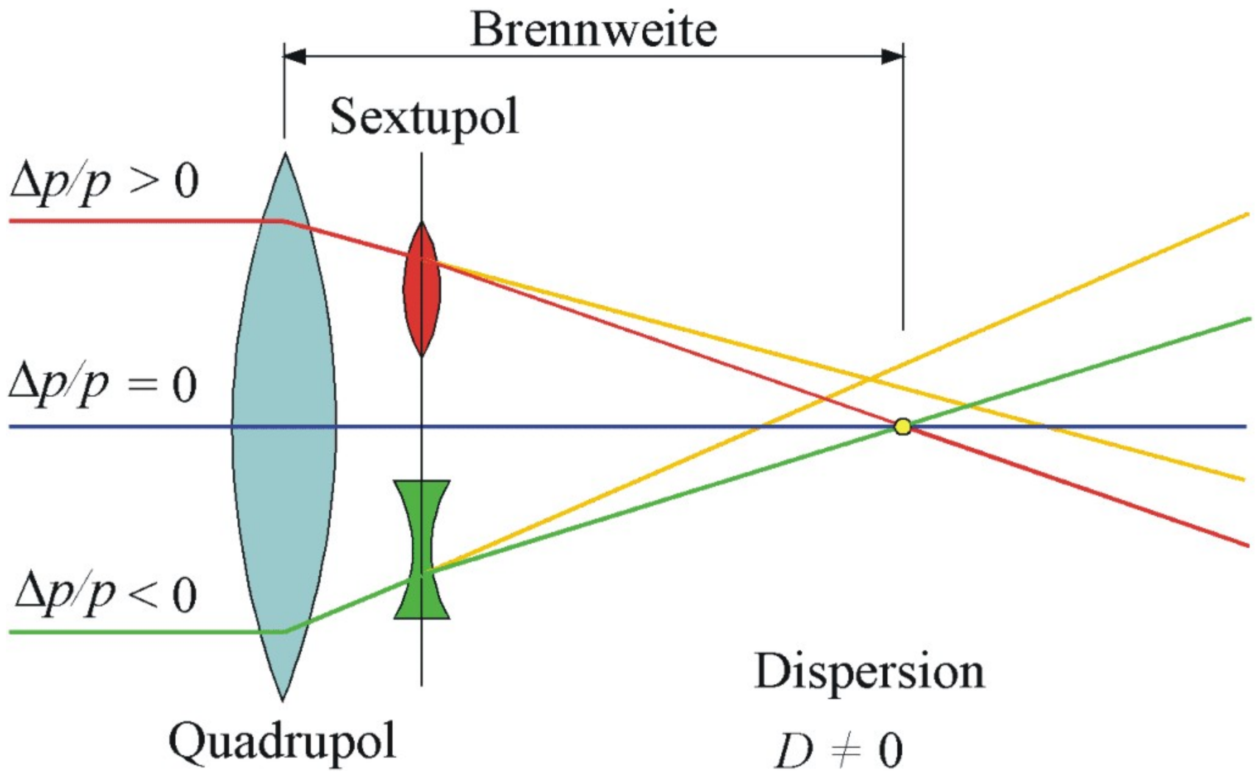
$$k_x = \frac{e}{p} \cdot \frac{\partial B_z}{\partial x} = m \cdot x = m \cdot D \cdot \frac{\Delta p}{p_0}$$

$$k_z = \frac{e}{p} \cdot \frac{\partial B_x}{\partial z} = m \cdot x = m \cdot D \cdot \frac{\Delta p}{p_0}$$

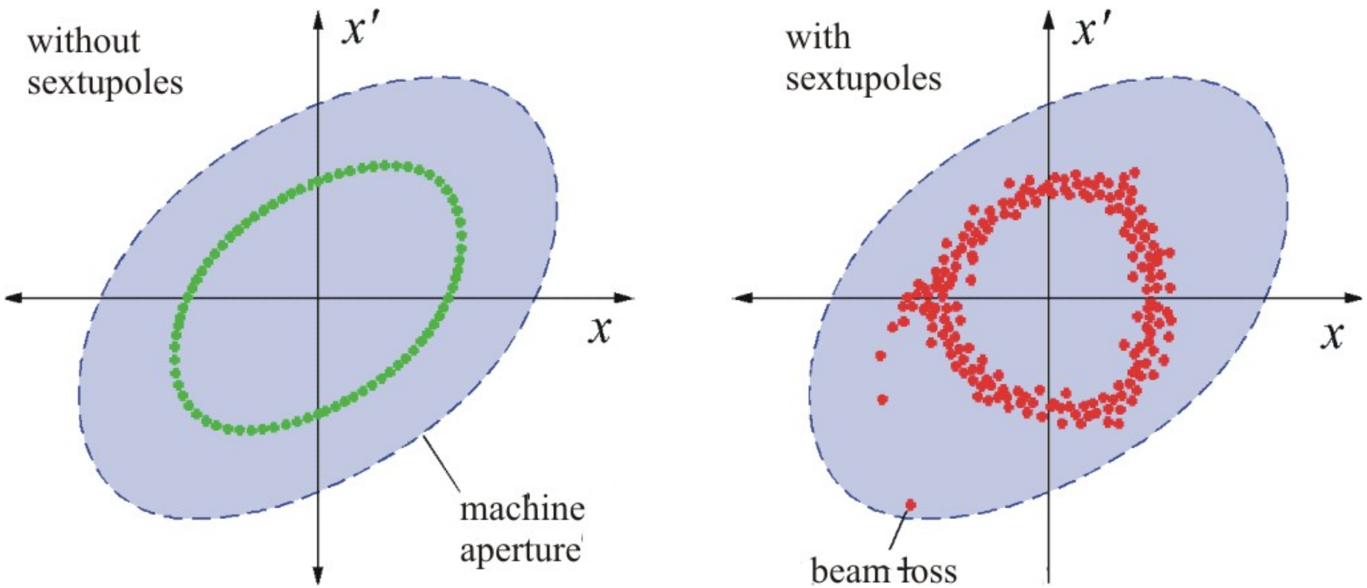
This adds to the natural chromaticity and gives in total:

$$\boxed{\xi_{x,z} = -\frac{1}{4\pi} \int \left[k_{x,z}(\tilde{s}) - m(\tilde{s})\, D(\tilde{s}) \right] \cdot \beta_{x,z}(\tilde{s})\, d\tilde{s}}$$

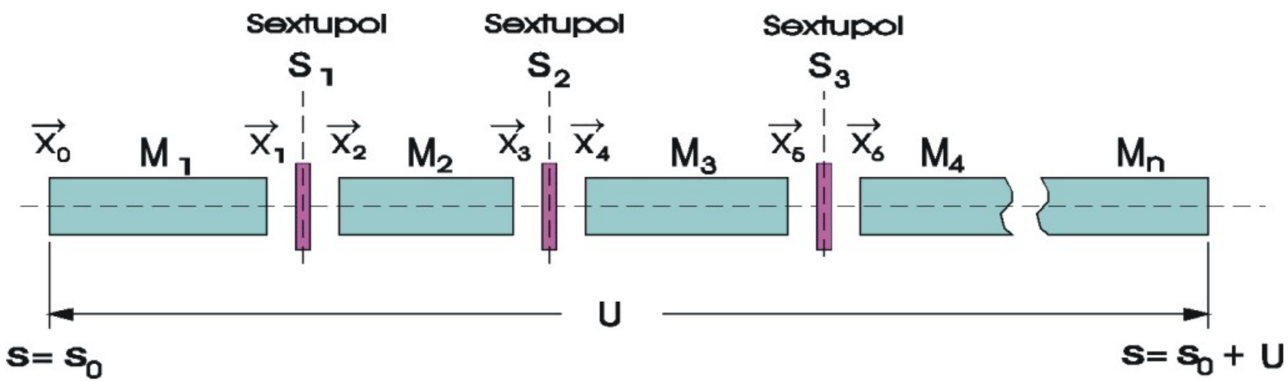
In order to avoid a large tune spread, chromaticity has to be corrected by the use of additional sextupole magnets right after focusing and defocusing quadrupoles where the horizontal dispersion does not vanish:



This correction will have an influence on the stability of the beam and th maximum aperture given by nonlinear effects (so called dynamic aperture):



The dynamic aperture can be calculated from a tracking of the particles orbit through the accelerator where the nonlinear effect of sextupole magnets has to be treated as step by step correction in linear beam matrix optics:



The orbit vector is transformed from s_0 to s_1 by matrix transformation

$$\vec{X}_1 = \mathbf{M}_1 \cdot \vec{X}_0$$

A sextupole of length l will produce an angular kick in the horizontal and vertical orbit of

$$\Delta x_1' = \frac{1}{2} m l \cdot \left(x_1^2 - z_1^2 \right)$$

$$\Delta z_1' = m l \cdot x_1 z_1$$

which gives an orbit vector right after the sextupole of

$$\vec{X}_2 = \begin{pmatrix} x_1 \\ x_1' + \Delta x_1' \\ z_1 \\ z_1' + \Delta z_1' \end{pmatrix}$$

By this method a randomly chosen distribution of start vectors $\vec{X}_0$ is tracked through the accelerator for many revolutions and the resulting dynamic aperture is derived from the phase space representation.

4.3. Longitudinal Beam Dynamics

4.3.1. Equation of Motion in Phase Space

From the discussion of the momentum compaction (chapter 4.3.7.) we have obtained for the relative variation of the travel time $\Delta T/T_0$ and the angular revolution frequency $\Delta\omega/\omega_0$:

$$\frac{\Delta T}{T_0} = -\frac{\Delta\omega}{\omega_0} = -\left(\frac{1}{\gamma^2} - \alpha_c\right)\frac{\Delta p}{p_0} = -\eta \cdot \frac{\Delta p}{p_0}$$

The revolution frequency ω_0 is linked to the RF frequency ω_{RF} by the number h of circulating bunches, which is called the **harmonic number**. Using this relation we obtain for the phase shift $\Delta\varphi = \varphi - \varphi_0$ with respect to a reference particle (with reference phase φ_0):

$$\Delta\varphi = \omega_{RF} \cdot \Delta T = h \cdot \omega_0 \cdot \Delta T$$

The phase shift per revolution can be linked to the relative momentum deviation by using the η-parameter:

$$\left(\Delta\varphi\right)_{rev} = -\eta\, h\, \omega_0\, T_0 \frac{\Delta p}{p_0} = -2\pi\, h\, \eta \frac{\Delta p}{p_0}$$

and may be expressed in terms of the relative energy deviation using

$$E^2 = m_0^2 c^4 + p^2 c^2 \;\Rightarrow\; 2E \cdot dE = 2 p c^2 \cdot dp \;\Rightarrow\; dE = \beta c \cdot dp \;\Rightarrow\; \frac{dE}{E} = \beta^2 \frac{dp}{p}$$

which gives:

$$\left(\Delta\varphi\right)_{rev} = -\frac{2\pi\, h\, \eta}{\beta^2} \frac{\Delta E}{E_0}$$

So far, we have expressed the phase shift $\left(\Delta\varphi\right)_{rev}$ per revolution in terms of $\Delta E = E - E_0$. In order to relate this to the energy gain per turn produced by acceleration, we first have to divide by the revolution time T_0 to get the change of the phase shift per unit time $\Delta\dot{\varphi}$:

$$\frac{d}{dt}\Delta\varphi = \frac{(\Delta\varphi)_{rev}}{T_0} = -\frac{2\pi h\eta}{\beta^2 T_0 E_0}\cdot\Delta E$$

We then have to built the second derivative to express this variation in terms of the energy gain $(\Delta E)_{rev}$ per turn

$$(\Delta E)_{rev} = eU(\varphi) - W(E) = eU_0 \sin\varphi - W(E)$$

where $W(E)$ represents the radiation losses per turn due to synchrotron radiation and $U(\varphi)$ is the acceleration voltage for a given phase φ. The energy gain per turn $(\Delta E)_{rev}$ is linked to the energy deviation ΔE with respect to the reference particle by

$$\frac{d}{dt}\Delta E = \frac{1}{T_0}\cdot(\Delta E)_{rev}$$

This gives

$$\frac{d^2\Delta\varphi}{dt^2} + \frac{2\pi h\eta}{\beta^2 T_0 E_0}\cdot\frac{d\,\Delta E}{dt} = 0$$

and we finally obtain

$$\frac{d^2\Delta\varphi}{dt^2} + \frac{2\pi h\eta}{\beta^2 T_0^2 E_0}\cdot\left[eU_0\sin\left(\varphi_0+\Delta\varphi\right)-W(E)\right] = 0$$

4.3.2. Small Oscillation Amplitudes

For small deviations $\Delta\varphi$ from the synchronous phase we can expand the acceleration voltage into a Taylor series and get

$$\frac{d}{dt}\Delta E = \frac{\Delta E}{T_0} \approx \frac{1}{T_0}\left\{eU(\varphi_0)+e\frac{dU(\varphi_0)}{d\varphi}\cdot\Delta\varphi - W\left(E_0\right)-\frac{dW\left(E_0\right)}{dE}\cdot\Delta E\right\}$$

At equilibrium we have $eU(\varphi_0)=W(E_0)$ and obtain the phase equation

$$\frac{d^2\Delta\varphi}{dt^2} + 2\cdot\underbrace{\left(\frac{1}{2T_0}\cdot\frac{dW(E_0)}{dE}\right)}_{=\alpha_S}\cdot\frac{d\Delta\varphi}{dt} + \underbrace{\left(\frac{2\pi h\eta e}{\beta^2 T_0^2 E_0}\cdot U_0\cos\varphi_0\right)}_{=\Omega_S^2}\cdot\Delta\varphi = 0$$

Particles orbiting in a circular accelerator therefore perform longitudinal oscillations with the angular frequency Ω_S, which are called **synchrotron oscillations**. These phase oscillations are damped or antidamped depending on the sign of the damping decrement α_S. For small oscillation amplitudes the movement can be described by a damped harmonic oscillator. In most cases we find the damping time much longer than the phase oscillation period

$$\tau_S = \frac{1}{\alpha_S} \ll \frac{2\pi}{\Omega_S} = \frac{1}{Q_S}$$

and the synchrotron tune Q_S, defined by the number of longitudinal oscillations per turn, much smaller than the transverse tunes Q_X, Q_Z.

The oscillations are stable for a real angular frequency Ω_S and therefore for a positive product $\eta \cdot \cos\varphi_0$. From $\eta = 1/\gamma^2 - 1/\gamma_{tr}^2$ and the equilibrium condition $eU_0 \sin\varphi_0 = W(E_0) > 0$ we derive the condition for stable phase focusing:

$$0 < \varphi_0 < \frac{\pi}{2} \qquad \text{for} \qquad \gamma < \gamma_{tr}$$

$$\frac{\pi}{2} < \varphi_0 < \pi \qquad \text{for} \qquad \gamma > \gamma_{tr}$$

Neglecting the small damping term the equation of motion reads

$$\frac{d^2 \Delta\varphi}{dt^2} + \Omega_S^2 \cdot \Delta\varphi = 0$$

and is solved by a harmonic oscillation

$$\Delta\varphi = \widehat{\Delta\varphi} \cdot \cos\left(\Omega_S t + \phi\right)$$

Building the first derivative and relating $\Delta\dot{\varphi}$ to the relative energy deviation $\Delta E / E_0$,

we obtain for the amplitude $\widehat{\Delta\varphi}$ of the oscillation

$$\Delta\dot{\varphi} = -\Omega_S \cdot \widehat{\Delta\varphi} \cdot \sin\left(\Omega_S t + \phi\right) = -\frac{2\pi h \eta}{\beta^2 T_0} \cdot \frac{\Delta E}{E_0} = \eta\, \omega_{RF} \cdot \frac{\Delta p}{p_0}$$

$$\Rightarrow \qquad \widehat{\Delta\varphi} = \frac{\eta\,\omega_{RF}}{\beta^2\,\Omega_S}\cdot\left(\frac{\Delta E}{E_0}\right)_{\max} = \frac{\eta\,\omega_{RF}}{\Omega_S}\cdot\left(\frac{\Delta p}{p_0}\right)_{\max}$$

All particles of a beam perform incoherent phase oscillations about a common reference point and generate thereby the appearance of a steady longitudinal distribution of particles which we call a particle bunch.

The total bunch length l_b can be determined from the maximum longitudinal excursion of particles from the bunch center and is twice the amplitude of the phase variation.:

$$\frac{l_b}{2} = \frac{\lambda_{RF}}{2\pi}\cdot\widehat{\Delta\varphi} = \frac{c}{h\,\omega_0}\cdot\widehat{\Delta\varphi}$$

Using the equation derived above, this gives

$$l_b = 2\cdot\frac{c\sqrt{2\pi}}{\beta\,\omega_0}\cdot\sqrt{\frac{\eta\,E_0}{h\,e\,U_0\cos\varphi_0}}\cdot\left(\frac{\Delta E}{E_0}\right)$$

4.3.3. Large Amplitude Oscillations

We will ignore the small damping term for the following discussions. This allows us to rewrite the equation of motion (without any further approximation) to

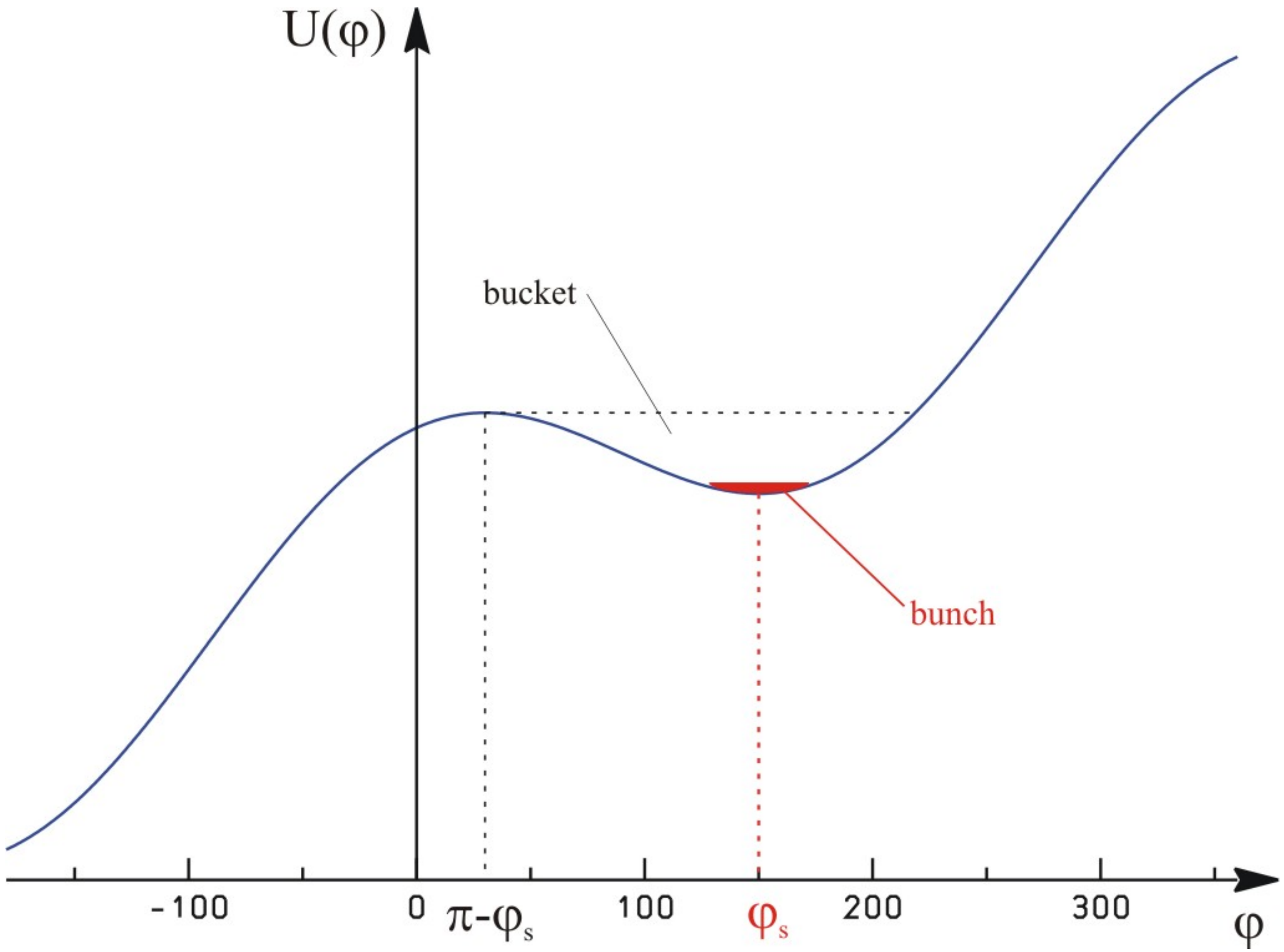
$$\boxed{\ddot{\varphi} + \frac{\Omega_S^2}{\cos\varphi_0}\left[\sin\varphi - \sin\varphi_0\right] = 0}$$

with the synchrotron frequency Ω_S defined above and $\varphi = \varphi_0 + \Delta\varphi$.

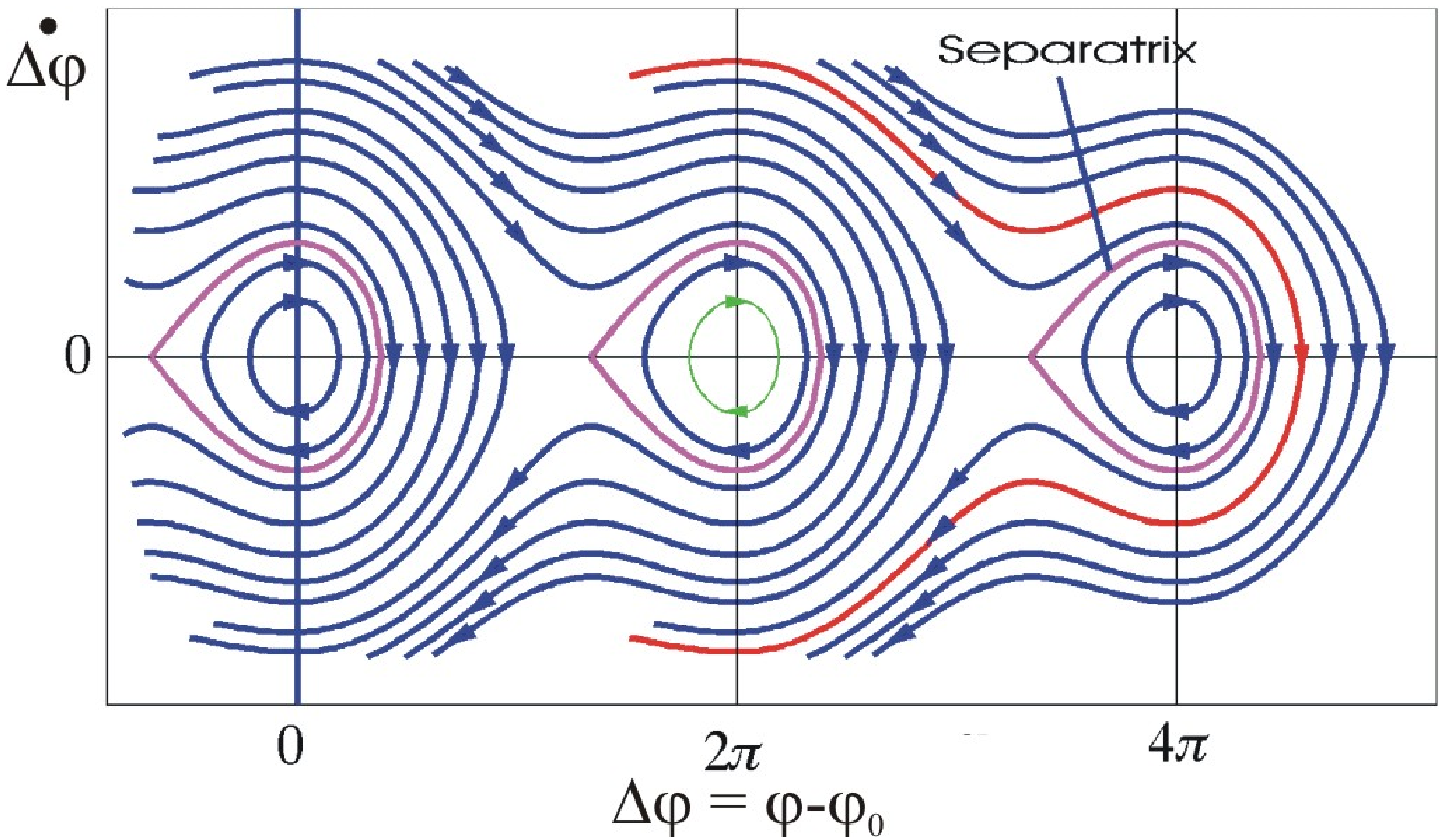
This can easily be integrated to the potential equation

$$\underbrace{\frac{\dot{\varphi}^2}{2}}_{\text{kinetic energy}} + \underbrace{\left\{-\frac{\Omega_S^2}{\cos\varphi_0}\left[\cos\varphi + \varphi\sin\varphi_0\right]\right\}}_{\text{potential energy}} = \text{const.}$$

The potential energy function corresponds to the sum of a linear function and a sinusoidal one. An oscillation can only take place if the particle is trapped in the potential well:



$\varphi_1^{max} = \pi - \varphi_0$ is an extreme elongation corresponding to a stable motion. The corresponding curve in phase space is called separatrix and the area delimited by this curve is called the RF bucket. Part of this area is filled with particles, forming the bunch.

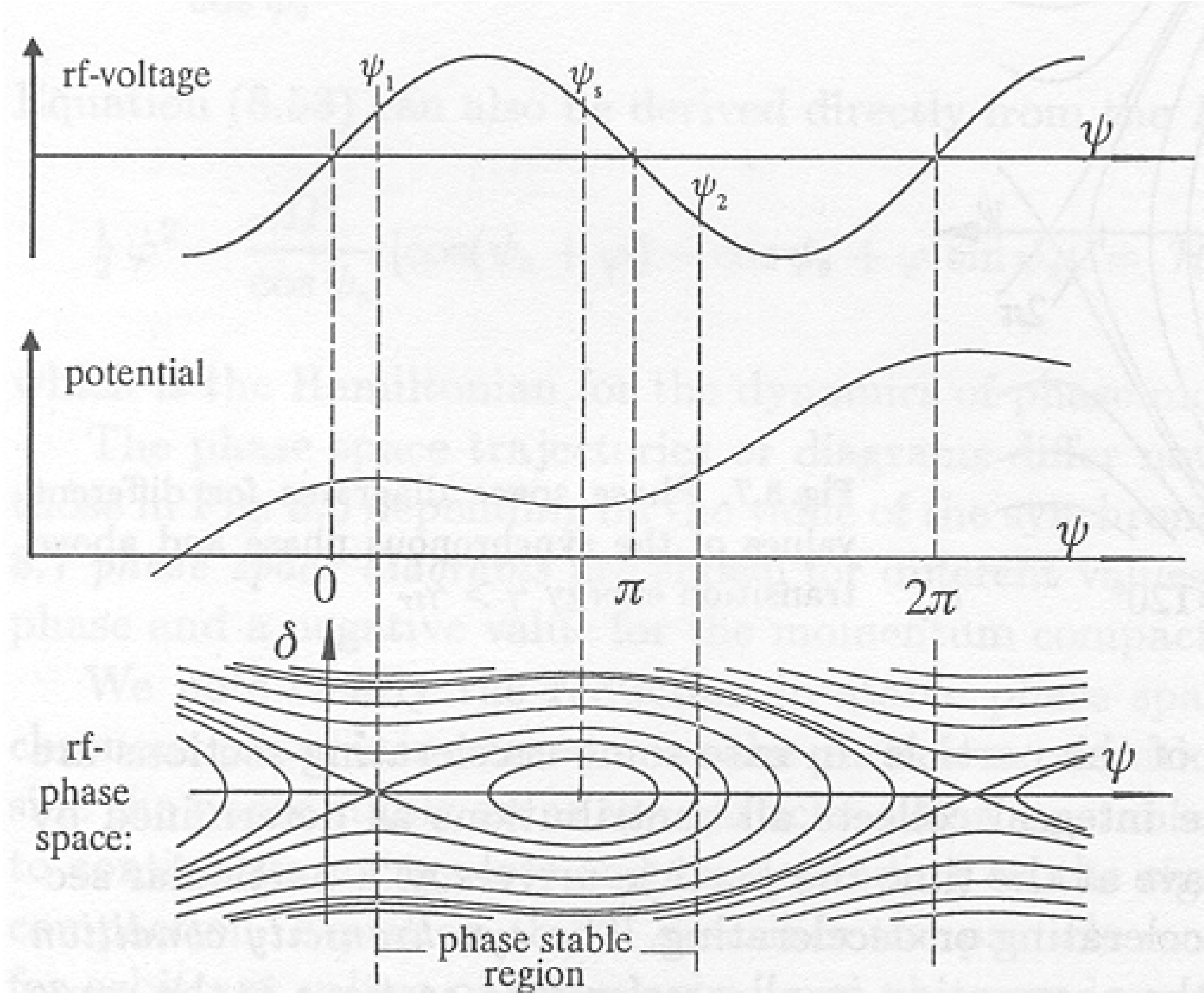


The equation of the separatrix is

$$\boxed{\frac{\dot{\varphi}^2}{2} - \frac{\Omega_S^2}{\cos\varphi_0}\left[\cos\varphi + \varphi\cdot\sin\varphi_0\right] = -\frac{\Omega_S^2}{\cos\varphi_0}\left[\cos(\pi-\varphi_0) + (\pi-\varphi_0)\cdot\sin\varphi_0\right]}$$

The other extreme elongation $\varphi_2^{\max}$ (second value for which $\dot{\varphi} = 0$), is such that

$$\cos\varphi_2^{\max} + \varphi_2^{\max} \cdot \sin\varphi_0 \;=\; \cos\left(\pi - \varphi_0\right) + \left(\pi - \varphi_0\right) \cdot \sin\varphi_0$$



From the equation of motion it is also seen that $\dot{\varphi}$ reaches a maximum when $\ddot{\varphi} = 0$ corresponding to $\varphi = \varphi_0$. This gives the maximum stable values of $\dot{\varphi}$ and the maximum energy spread $\Delta E_{\max}$, which is called the **RF acceptance**:

$$\dot{\varphi}^2_{\max} = 2\Omega_S^2 \left[2 - \left(\pi - 2\varphi_0 \right) \cdot \tan\varphi_0 \right]$$

$$\left(\frac{\Delta E}{E_0} \right)_{\max} = \pm \beta \sqrt{ \frac{eU_0}{\pi h \eta E_0} \cdot \left[2\cos\varphi_0 - \left(\pi - 2\varphi_0 \right) \cdot \sin\varphi_0 \right] }$$

In accelerator physics one usually defines an **over voltage factor** q by

$$\boxed{q = \frac{\text{maximum RF voltage}}{\text{desired energy gain}} = \frac{eU_0}{eU_0 \sin\varphi_0} = \frac{1}{\sin\varphi_0}}$$

Using this factor, we can rewrite the RF acceptance to

$$\left(\frac{\Delta E}{E_0}\right)_{\max} = \beta\sqrt{\frac{2eU_0\sin\varphi_0}{\pi h\eta E_0}\cdot\left(\sqrt{q^2-1}-\arccos\frac{1}{q}\right)} \leq \beta\sqrt{\frac{2eU_0}{\pi h\eta E_0}}$$

Using $\eta=\left(\gamma^{-2}-\alpha_C\right)$, $\alpha_C \approx 1/Q_x{}^2$ and $\omega_{RF} = h\cdot\omega_0$ we finally note the important scaling:

$$\boxed{\left(\frac{\Delta E}{E_0}\right)_{\max} \sim \frac{1}{\sqrt{\omega_{RF}}}}, \qquad \boxed{\left(\frac{\Delta E}{E_0}\right)_{\max} \overset{\gamma\gg 1}{\sim} Q_S}, \qquad \boxed{\left(\frac{\Delta E}{E_0}\right)_{\max} \sim \sqrt{\frac{eU_0\sin\varphi_0}{E_0}}}$$