Numerics for elliptic Feynman integrals

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I: **Review**: **Numerics for multiple polylogarithms**

II: **Example**: **The two-loop electron self-energy**

III: **Numerics**: **Single-scale elliptic Feynman integrals**

Motivation

- Precision physics for heavy particles (e.g. Higgs, top, W/Z-bosons).
- If we allow masses, already two-loop two-point functions go beyond multiple polylogarithms.
- What functions occur beyond multiple polylogarithsm?
- Is there an efficient method to evaluate these functions numerically everywhere?

Outline

Efficient numerical evaluation methods rely on three ingredients:

1. An (iterated) integral representation, used to transform the arguments into the region of convergence.

2. A (nested) sum representation defined in the region of convergence, which can be truncated and gives a numerical approximation.

3. Methods, which accelerate the convergence of the truncated series.

Numerical evaluation of the dilogarithm

The dilogarithm:

$$\mathrm{Li}_2(x) \quad = \quad \int_0^x \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \quad = \quad \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Map into region $|x| \leq 1$ and $-1 \leq \mathrm{Re}(x) \leq 1/2$, using

$$\mathrm{Li}_2(x) = -\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{\pi^2}{6} - \frac{1}{2}(\ln(-x))^2, \qquad \mathrm{Li}_2(x) = -\mathrm{Li}_2(1-x) + \frac{\pi^2}{6} - \ln(x)\ln(1-x).$$

Accelaration using Bernoulli numbers B_j:

$$\mathrm{Li}_2(x) \quad = \quad \sum_{j=0}^{\infty} \frac{B_j}{(j+1)!}(-\ln(1-x))^{j+1},$$

't Hooft, Veltman, '79

Multiple polylogarithms

Definition based on nested sums:

$$\mathrm{Li}_{m_1,m_2,\ldots,m_k}(x_1,x_2,\ldots,x_k) \quad = \quad \sum_{n_1>n_2>\ldots>n_k>0} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Definition based on iterated integrals:

$$G(z_1,\ldots,z_k;y) \quad = \quad \int_0^y \frac{dt_1}{t_1-z_1} \int_0^{t_1} \frac{dt_2}{t_2-z_2} \ldots \int_0^{t_{k-1}} \frac{dt_k}{t_k-z_k}$$

Conversion:

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) \quad = \quad (-1)^k G_{m_1,\ldots,m_k}\left(\frac{1}{x_1},\frac{1}{x_1x_2},\ldots,\frac{1}{x_1\ldots x_k};1\right)$$

Short hand notation:

$$G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y) \quad = \quad G(\underbrace{0,\ldots,0}_{m_1-1},z_1,\ldots,z_{k-1},\underbrace{0\ldots,0}_{m_k-1},z_k;y)$$

Numerical evaluation of multiple polylogarithms

Use the integral representation

$$G_{m_1,\ldots,m_k}(z_1,z_2,\ldots,z_k;y) \quad = \quad \int_0^y \left(\frac{dt}{t}\circ\right)^{m_1-1} \frac{dt}{t-z_1} \left(\frac{dt}{t}\circ\right)^{m_2-1} \frac{dt}{t-z_2} \ldots \left(\frac{dt}{t}\circ\right)^{m_k-1} \frac{dt}{t-z_k}$$

to transform all arguments into a region, where we have a converging power series expansion:

$$G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y) =$$
$$\sum_{j_1=1}^{\infty} \ldots \sum_{j_k=1}^{\infty} \frac{1}{(j_1+\ldots+j_k)^{m_1}} \left(\frac{y}{z_1}\right)^{j_1} \frac{1}{(j_2+\ldots+j_k)^{m_2}} \left(\frac{y}{z_2}\right)^{j_2} \ldots \frac{1}{(j_k)^{m_k}} \left(\frac{y}{z_k}\right)^{j_k}.$$

Use the Hölder convolution to accelerate the convergent series:

$$G(z_1,\ldots,z_w;1) \quad = \quad \sum_{j=0}^{w} (-1)^j G\left(1-z_j,1-z_{j-1},\ldots,1-z_1;\frac{1}{2}\right) G\left(z_{j+1},\ldots,z_w;\frac{1}{2}\right).$$

J. Vollinga, S.W., '04

The two-loop electron self-energy

As a proof-of-concept consider the two-loop electron self-energy (Sabry, '62).

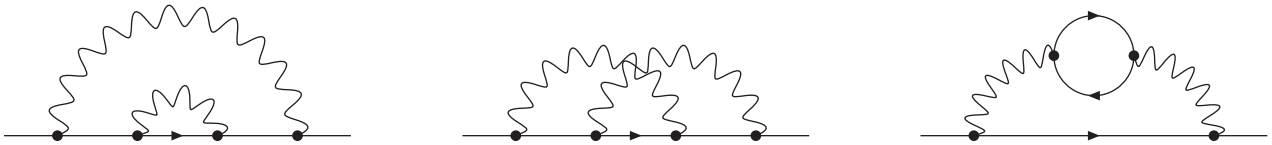
This is a quantity of physical relevance, although

- the self-energy is not gauge-invariant,
- the renormalised self-energy is renormalisation-scheme dependent,
- the bare self-energy is not a pure function.

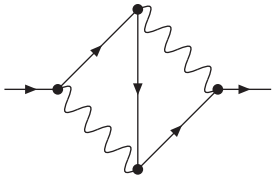
For concreteness, consider in Feynman gauge the ε^0-term within dimensional regularisation and the onshell-scheme.

Diagrams

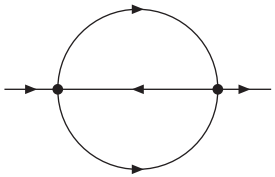
There are three Feynman diagrams contributing to the two-loop electron self-energy in QED with a single fermion:



All master integrals are (sub-) topologies of the kite graph:



One sub-topology is the sunrise graph with three equal non-zero masses:



The two-loop electron self-energy

Write for the two-loop electron self-energy

$$\Sigma^{(2)}_{\text{bare}} = \left(\frac{\alpha}{4\pi}\right)^2 \left(\Sigma^{(2)}_{\text{bare},V}\, \not{p} + \Sigma^{(2)}_{\text{bare},S}\, m\right),$$

Write the vector and the scalar part as a linear combination of master integrals I_j:

$$\Sigma^{(2)}_{\text{bare},V} = \sum_{j=1}^{8} c_j^V\, I_j, \qquad \Sigma^{(2)}_{\text{bare},S} = \sum_{j=1}^{8} c_j^S\, I_j.$$

Laurent expansion in ε:

$$\Sigma^{(2)}_{\text{bare},V} = \sum_{k=-2}^{\infty} \varepsilon^k\, \Sigma^{(2,k)}_{\text{bare},V}, \qquad \Sigma^{(2)}_{\text{bare},S} = \sum_{k=-2}^{\infty} \varepsilon^k\, \Sigma^{(2,k)}_{\text{bare},S}.$$

Pole terms and counterterms simple. Focus on $\Sigma^{(2,0)}_{\text{bare},V}$ and $\Sigma^{(2,0)}_{\text{bare},S}$.

The master integrals

There are 8 master integrals (Remiddi, Tancredi, '16).

Strategy for the master integrals:

- Derive the differential equations for the master integrals.
- Transform the system by a non-algebraic redefinition of the master integrals to an ε-form.
- Solve the system in terms of iterated integrals.

The differential equations

Set $x = p^2/m^2$.

Singularities of the differential equation are at $x \in \{0, 1, 9, \infty\}$.
Call these points the cusps.

The system of differential equations contains the second-order irreducible differential operator (Broadhurst, Fleischer, Tarasov '93)

$$L = x(x-1)(x-9)\frac{d^2}{dx^2} + \left(3x^2 - 20x + 9\right)\frac{d}{dx} + x - 3$$

Let ψ_1 and ψ_2 be two independent solutions of the homogeneous equation

$$L\,\psi = 0.$$

Periods

ψ_1 and ψ_2 can be taken as periods of the elliptic curve

$$E \quad : \quad w^2 - z(z+4)\left[z^2 + 2(1+x)z + (1-x)^2\right] \; = \; 0.$$

Set

$$k^2 \; = \; \frac{16\sqrt{x}}{(1+\sqrt{x})^3(3-\sqrt{x})}, \qquad k'^2 \; = \; 1-k^2.$$

In a neighbourhood of $x = 0$ the periods may be taken as

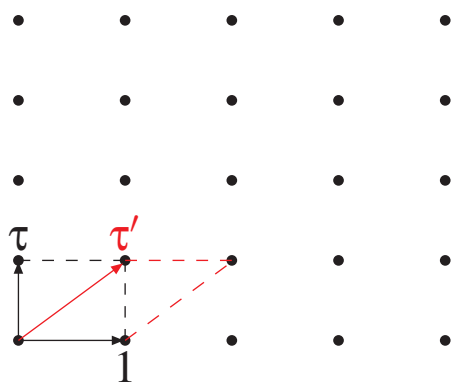
$$\psi_{1,0} \; = \; \frac{4K(k)}{(1+\sqrt{x})^{\frac{3}{2}}(3-\sqrt{x})^{\frac{1}{2}}}, \qquad \psi_{2,0} \; = \; \frac{4iK(k')}{(1+\sqrt{x})^{\frac{3}{2}}(3-\sqrt{x})^{\frac{1}{2}}},$$

The complete elliptic integral $K(k)$ can be computed efficiently from the arithmetic-geometric mean

$$K(k) \quad = \quad \frac{\pi}{2\,\mathrm{agm}(k',1)}.$$

Bases of lattices

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).
Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$ where $\tau = \psi_2/\psi_1$.



Change of basis:
$$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

In terms of τ and τ':
$$\tau' = \frac{a\tau + b}{c\tau + d}$$

Modular forms

Denote by $\mathbb{H}$ the complex upper half plane. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

(i) f transforms under Möbius transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

(ii) f is holomorphic on $\mathbb{H}$,

(iii) f is holomorphic at ∞.

Congruence subgroups

Apart from $\mathrm{SL}_2(\mathbb{Z})$ we may also look at congruence subgroups, for example

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N,\ b, c \equiv 0 \bmod N \right\}$$

Modular forms for congruence subgroups: Require "nice" transformation properties only for subgroup Γ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

Relevant to this talk: Modular forms for $\Gamma_1(6)$.

Variables

Recall

$$x \quad = \quad \frac{p^2}{m^2}.$$

Set

$$\tau_0 \; = \; \frac{\psi_{2,0}}{\psi_{1,0}}, \qquad q_0 \; = \; e^{2i\pi\tau_0}.$$

Change variable from x to τ_0 (or q_0) (Bloch, Vanhove, '13).

The differential equation for the master integrals reads then

$$\frac{d}{d\tau_0}\vec{I} \quad = \quad \varepsilon\, A(\tau_0)\, \vec{I},$$

where $A(\tau_0)$ is an ε-independent 8×8-matrix whose entries are modular forms.

Adams, S.W., '17, '18

Iterated integrals

For ω_1, ..., ω_k differential 1-forms on a manifold M and $\gamma : [0,1] \to M$ a path, write for the pull-back of ω_j to the interval $[0,1]$

$$f_j(\lambda)\, d\lambda \quad = \quad \gamma^* \omega_j .$$

The iterated integral is defined by (Chen '77)

$$I_\gamma(\omega_1, \ldots, \omega_k; \lambda) \quad = \quad \int_0^\lambda d\lambda_1 f_1(\lambda_1) \int_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \ldots \int_0^{\lambda_{k-1}} d\lambda_k f_k(\lambda_k) .$$

Example 1: Multiple polylogarithms (Goncharov '98)

$$\omega_j \quad = \quad \frac{d\lambda}{\lambda - z_j}.$$

Example 2: Iterated integrals of modular forms (Brown '14): $f_j(\tau)$ a modular form,

$$\omega_j \quad = \quad 2\pi i \, f_j(\tau) \, d\tau.$$

Iterated integrals of modular forms

Modular forms have a q-expansion. Using

$$2\pi i\, d\tau \;=\; \frac{dq}{q}$$

we may integrate term-by-term and obtain the q-expansion of the master integrals.

For example, for the ε^2-term of the sunrise integral one finds

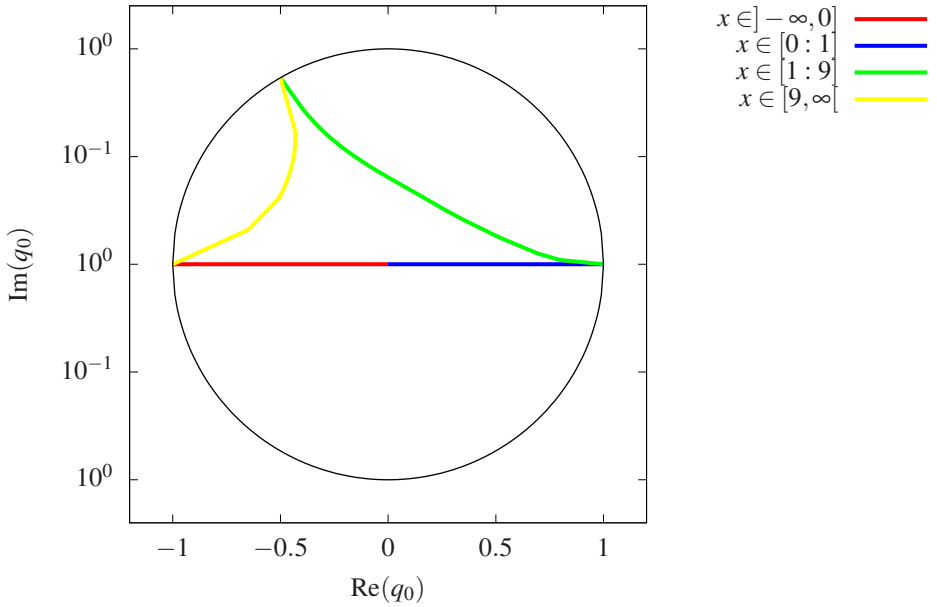
$$I_6^{(2)} \;=\; 3\,\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) - 3\sqrt{3}\left[q - \frac{5}{4}q^2 + q^3 - \frac{11}{16}q^4 + \frac{24}{25}q^5 - \frac{5}{4}q^6 + \frac{50}{49}q^7 - \frac{53}{64}q^8 + q^9\right] + O\left(q^{10}\right)$$

We may truncate the q-series and evaluate the resulting polynomial numerically.

Convergence

We defined q_0 such that $q_0 = 0$ for $x = 0$.

For which values $x \in \mathbb{R}$ do we have $|q_0| < 1$?



We have $|q_0| < 1$ for $x \in \mathbb{R} \backslash \{1, 9, \infty\}$.

Bogner, Schweitzer, S.W., '17

Expansion parameters

Recall that the periods ψ_1 and ψ_2 are not uniquely determined.

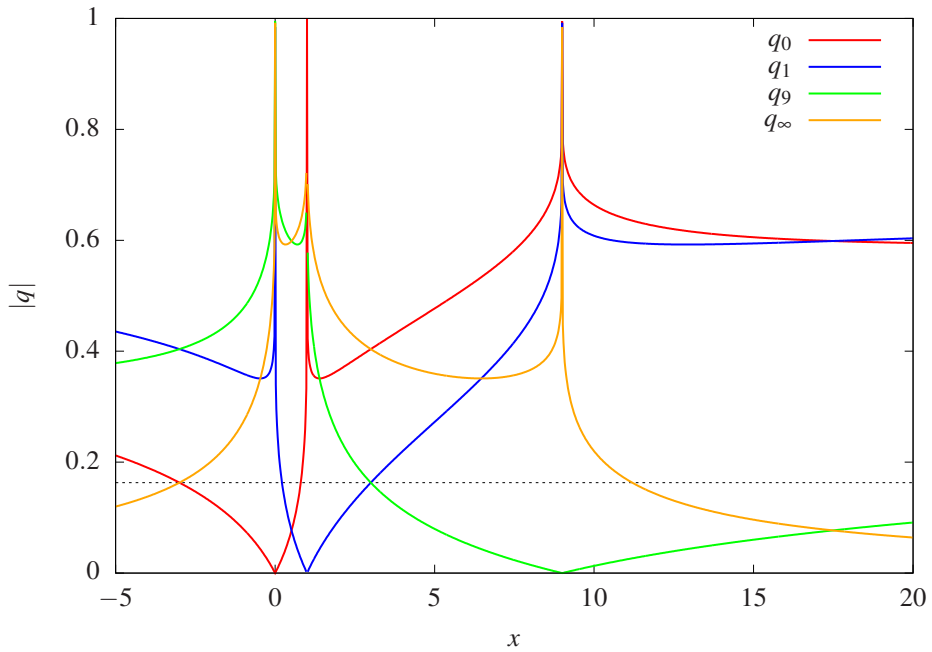
Can define q_0, q_1, q_9 and q_∞ such that

- the integration kernels are modular forms of $\Gamma_1(6)$,
- $q_j = 0$ for $x = j$.

Adams, S.W. '17, Hönemann, Tempest, S.W. '18

Expansion parameters

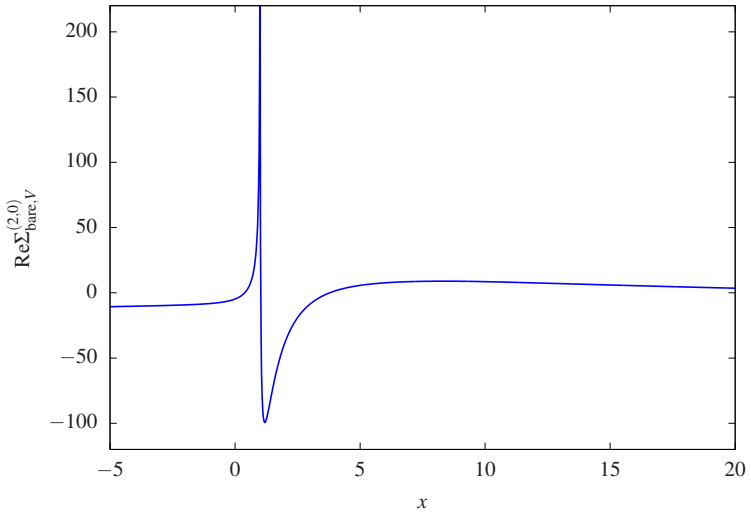
The absolute values of the variables q_j for $j \in \{0, 1, 9, \infty\}$ as a function of x.

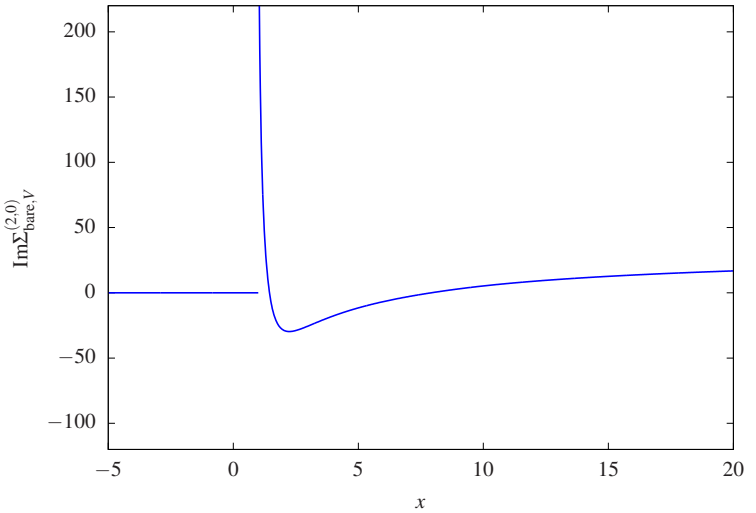


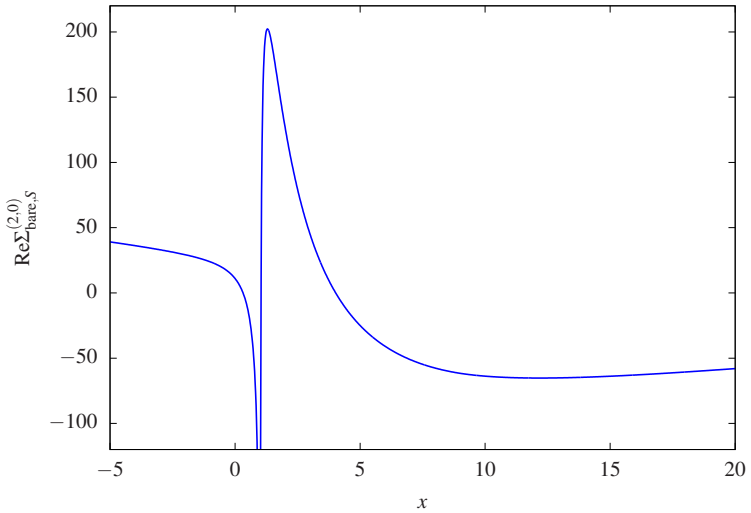
There is always a choice such that $|q_j| \lesssim 0.163$ for all values of x.

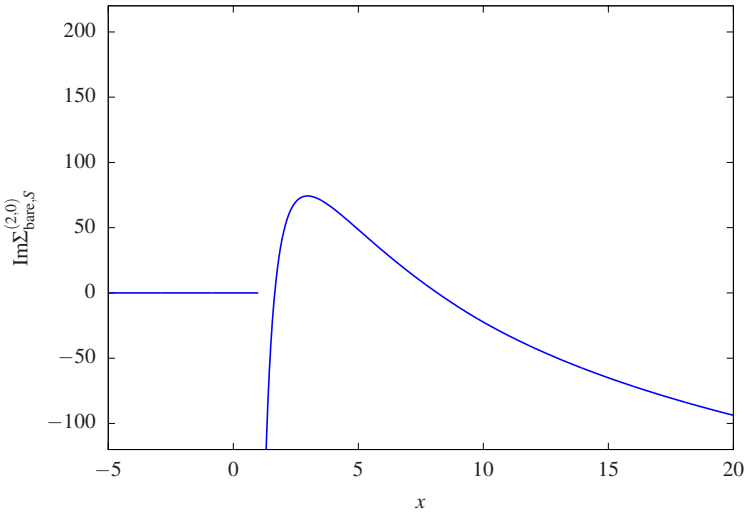
Truncation of the q-series to order $O(q^{30})$ gives for the finite part of the self-energy a relative precision better than 10^{-20} for all real values p^2/m^2.

Numerical results









Wrap-up

- Periods are calculated from the arithmetic-geometric mean.
- Master integrals are calculated from truncated q-expansions.

Conclusions

- Loop integrals with masses important for top, W/Z- and H-physics.
- May involve elliptic sectors from two loops onwards.
- There is a class of Feynman integrals evaluating to iterated integrals of modular forms.
- These can be evaluated efficiently from their q-expansions.
- We may expect more results in the near future.

Back-up

Relative precision for $\Sigma^{(2,0)}_{\mathrm{bare},S}$ (similar for $\Sigma^{(2,0)}_{\mathrm{bare},V}$):