UNSUBTRACTIONS AT NNLO

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LOCAL SUBTRACTION IN THE UV

UNSUBTRACTION IN THE IR

▸ LTD: open loops to trees
▸ FDU: mapping of V→R kinematics
Local Subtraction in the UV

Unsubtraction in the IR

- LTD: open loops to trees
- FDU: mapping of V→R kinematics

- Integrand cancellation of singularities in d=4 space-time dimensions
**LOCAL SUBTRACTION IN THE UV**

- **LTD:** open loops to trees
- **FDU:** mapping of $V \rightarrow R$ kinematics

**UNSUBTRACTION IN THE IR**

- Integrand cancellation of singularities in $d=4$ space-time dimensions
- $V+R$ simultaneous:
**LOCAL SUBTRACTION IN THE UV**

**UNSUBTRACTION IN THE IR**

- **LTD:** open loops to trees
- **FDU:** mapping of $V \rightarrow R$ kinematics

- **Integrand cancellation** of singularities in $d=4$ space-time dimensions
- **$V+R$ simultaneous:**
  - More efficient event generators
**Local Subtraction in the UV**

- **LTD**: open loops to trees
- **FDU**: mapping of V ▶ R kinematics

**Unsubtraction in the IR**

- **Integrand cancellation** of singularities in d=4 space-time dimensions
- **V+R simultaneous:**
  - More efficient event generators
- LTD suitable for **amplitudes**, FDU aimed at **physical observ.**
IT’S ALL ABOUT THE TINY +i0 FROM

PROPAGATORS

\[ G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0} \]

- MATH: the +i0 is a small quantity usually ignored, assuming that the analytical continuation to the physical kinematics is well defined

- PHYS: the +i0 encodes **CAUSALITY** | positive frequencies are propagated forward in time, and negative backward
FROM THE CAUCHY RESIDUE THEOREM

THE LOOP–TREE DUALITY (LTD)

Cauchy residue theorem
in the loop energy complex plane

Feynman Propagator $+i0$:
positive frequencies are propagated forward in time, and negative backward

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0}$$

selects residues with definite positive energy and negative imaginary part (indeed in any other coordinate system)
The Loop-Tree Duality (LTD)

One-loop amplitudes in any relativistic, local and unitary QFT represented as a linear combination of \( N \) single-cut phase-space/dual amplitudes | non-disjoint trees (at higher orders: number of cuts equal to the number of loops)

\[
\int_{\ell_1} N(\ell_1) \prod G_F(q_i) = - \int_{\ell_1} N(\ell_1) \otimes \sum \tilde{\delta}(q_i) \prod_{i \neq j} G_D(q_i; q_j)
\]

- \( \tilde{\delta}(q_i) = i \, 2\pi \theta(q_i, 0) \delta(q_i^2 - m_i^2) \) sets internal line on-shell, positive energy mode

- \( G_D(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - i \epsilon \eta k_{ji}} \) dual propagator, \( k_{ji} = q_j - q_i \)

- best choice \( \eta^\mu = (1, 0) \) : energy component integrated out, remaining integration in Euclidean space
One-loop amplitudes in any relativistic, local and unitary QFT represented as a linear combination of $N$ single-cut phase-space/dual amplitudes $|\text{non-disjoint trees}|$ (at higher orders: number of cuts equal to the number of loops)

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- $G_D(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - i0 \eta \cdot k_{ji}}$ dual propagator, $k_{ji} = q_j - q_i$

- best choice $\eta^\mu = (1, 0)$: energy component integrated out, remaining integration in Euclidean space

- LTD realised by modifying the customary $+i0$ prescription of the Feynman propagators (only the sign matters), it encodes in a compact way the effect of multiple-cut contributions that appear in the Feynman’s Tree Theorem
energy of the on-shell propagator smaller than the energy of the emitted particles
time-like or light-like
this other propagator eventually on-shell
this propagator on-shell
energy of the on-shell propagator smaller than the energy of the emitted particles

- Causally connected in one direction
energy of the **on-shell** propagator smaller than the energy of the emitted particles

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- **Threshold** singularities occur when a second propagator gets on-shell: consistent with **Cutkosky**

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time-like or light-like
SINGULARITIES OF SINGLE-CUT TREES AND THE FOREST

WHEN A BRANCHES GET BROKEN

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- It becomes **collinear (soft)** when a single massless particle is emitted
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Space-like or light-like at energy of the **on-shell** propagator *larger* than the energy of the emitted particle(s)
SINGULARITIES OF SINGLE-CUT TREES AND THE FOREST

WHEN A BRANCHES GET BROKEN

energy of the **on-shell** propagator **smaller** than the energy of the emitted particles

**time-like or light-like**

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Space-like or light-like at energy of the **on-shell** propagator **larger** than the energy of the emitted particle(s)

▸ Causally connected in one direction

▸ **Threshold** singularities occur when a second propagator gets on-shell: consistent with **Cutkosky**

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▸ Virtual particle emitted and absorbed
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Virtual particle emitted and absorbed

Potential threshold and IR singularities cancel in the sum of single-cut trees

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- **Non-singular configurations at very large energies (UV) expected to be suppressed.** If not sufficiently suppressed, **renormalise**

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- Virtual particle emitted and absorbed
- Potential threshold and IR singularities cancel in the sum of single-cut trees
- Non-singular configurations at very large energies (UV) expected to be suppressed. If not sufficiently suppressed, renormalise
- The bulk of the physics is in the “low” energy region of the loop momentum
Motivated by the factorisation properties of QCD: assuming $q_i^\mu$ on-shell, and close to collinear with $p_i^\mu$, we define the momentum mapping

$$p_r^\mu = q_i^\mu,$$

$$p_i'^\mu = p_i^\mu - q_i^\mu + \alpha_i p_j^\mu,$$

$$p_j'^\mu = (1 - \alpha_i) p_j^\mu,$$

$$p_k'^\mu = p_k^\mu, \quad k \neq i, j.$$

All the primed momenta (real process) on-shell and momentum conservation: $p_i'^\mu$ is the emitter, $p_j'^\mu$ the spectator needed to absorb momentum recoil.
Motivated by the factorisation properties of QCD: assuming \( q_i^\mu \) on-shell, and close to collinear with \( p_i^\mu \), we define the momentum mapping
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p_r^\mu = q_i^\mu ,
\]
\[
p_i'^\mu = p_i'^\mu - q_i^\mu + \alpha_i \, p_j'^\mu , \quad \alpha_i = \frac{(q_i - p_i)^2}{2 p_j \cdot (q_i - p_i)} ,
\]
\[
p_j'^\mu = (1 - \alpha_i) \, p_j'^\mu , \quad p_k'^\mu = p_k'^\mu , \quad k \neq i, j
\]

All the primed momenta (real process) on-shell and momentum conservation: \( p_i'^\mu \) is the emitter, \( p_j'^\mu \) the spectator needed to absorb momentum recoil.

Quasi-collinear configurations can also be conveniently mapped such that the massless limit is smooth [Sborlini, Driencourt-Mangin, GR, JHEP 1610, 162]
UV RENORMALISATION: LOCAL SUBTRACTION

- Expand propagators and numerators around a **UV propagator** [Weinzierl, Pittau-Page]

\[
G_F(q_i) = \frac{1}{q_{UV}^2 - \mu_{UV}^2 + i0} \left[ 1 - \frac{2q_{UV} \cdot k_i + k_i^2 - m_i^2 + \mu_{UV}^2}{q_{UV}^2 - \mu_{UV}^2 + i0} + \frac{(2q_{UV} \cdot k_i)^2}{(q_{UV}^2 - \mu_{UV}^2 + i0)^2} \right] + \ldots
\]

- and adjust **subleading** terms to subtract only the pole (**\( \overline{MS} \) scheme**), or to define any other renormalisation scheme. For the scalar two point function

\[
I_{UV}^{cnt} = \int \frac{1}{\epsilon} \frac{1}{(q_{UV}^2 - \mu_{UV}^2 + i0)^2} \left( 1 + d_{UV} \frac{\mu_{UV}^2}{q_{UV}^2 - \mu_{UV}^2 + i0} \right)
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G_F(q_i) = \frac{1}{q_{\text{UV}}^2 - \mu_{\text{UV}}^2 + i0} \left[ 1 - \frac{2q_{\text{UV}} \cdot k_i + k_i^2 - m_i^2 + \mu_{\text{UV}}^2}{q_{\text{UV}}^2 - \mu_{\text{UV}}^2 + i0} + \frac{(2q_{\text{UV}} \cdot k_i)^2}{(q_{\text{UV}}^2 - \mu_{\text{UV}}^2 + i0)^2} \right] + \ldots
\]

\[
q_{\text{UV}} = \ell + k_{\text{UV}} \quad k_i = q_i - q_{\text{UV}}
\]

- and adjust **subleading** terms to subtract only the pole (\textbf{MS} scheme), or to define any other renormalisation scheme. For the scalar two point function

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I_{\text{UV}}^{\text{cnt}} = \int \ell \frac{1}{(q_{\text{UV}}^2 - \mu_{\text{UV}}^2 + i0)^2} \left( 1 + d_{\text{UV}} \frac{\mu_{\text{UV}}^2}{q_{\text{UV}}^2 - \mu_{\text{UV}}^2 + i0} \right)
\]

- dual representation needs to deal with **multiple poles** [Bierenbaum et al., 2010]

\[
I_{\text{UV}}^{\text{cnt}} = \int \ell \frac{\tilde{\delta}(q_{\text{UV}})}{2 \left( q_{\text{UV},0}^{(+)})^2 \right)^2} \left( 1 - \frac{3 d_{\text{UV}} \mu_{\text{UV}}^2}{4 \left( q_{\text{UV},0}^{(+)})^2 \right)} \right) q_{\text{UV},0}^{(+)} = \sqrt{q_{\text{UV}}^2 + \mu_{\text{UV}}^2 - i0}
\]

Hernández-Pinto, Sborlini, GR, JHEP 1602, 044
Expand propagators and numerators around a **UV propagator** [Weinzierl, Pittau-Page]

\[ G_F(q_i) = \frac{1}{q_{UV}^2 - \mu_{UV}^2 + i0} \left[ 1 - \frac{2q_{UV} \cdot k_i + k_i^2 - m_i^2 + \mu_{UV}^2}{q_{UV}^2 - \mu_{UV}^2 + i0} + \frac{(2q_{UV} \cdot k_i)^2}{(q_{UV}^2 - \mu_{UV}^2 + i0)^2} \right] + \ldots \]

\[ q_{UV} = \ell + k_{UV} \quad k_i = q_i - q_{UV} \]

and adjust **subleading** terms to subtract only the pole (**MS** scheme), or to define any other renormalisation scheme. For the scalar two point function

\[ I_{UV}^{cnt} = \int \frac{1}{\epsilon} \frac{1}{(q_{UV}^2 - \mu_{UV}^2 + i0)^2} \left( 1 + d_{UV} \frac{\mu_{UV}^2}{q_{UV}^2 - \mu_{UV}^2 + i0} \right) \]

**dual representation needs to deal with multiple poles** [Bierenbaum et al., 2010]

\[ I_{UV}^{cnt} = \int \frac{\tilde{\delta}(q_{UV})}{\epsilon} \frac{3 d_{UV} \mu_{UV}^2}{2 \left( q_{UV,0}^{(+)})^2 \right)} \left( 1 - \frac{3 d_{UV} \mu_{UV}^2}{4 \left( q_{UV,0}^{(+)})^2 \right)} \right) q_{UV,0}^{(+)} = \sqrt{q_{UV}^2 + \mu_{UV}^2 - i0} \]

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**Integration on the UV on-shell hyperboloid: loop three-momentum unconstrained, but loop contributions suppressed** for loop energies larger than \( \mu_{UV} \)
The dual representation of the renormalised loop cross-section: one single integral in the loop three-momentum

\[
\int_N d\sigma^{(1,R)}_V = \int_N \int_{\vec{\ell}_1} 2 \text{Re} \left\langle \mathcal{M}^{(0)}_N \right| \left( \sum_i \mathcal{M}^{(1)}_N (\tilde{\delta}(q_i)) \right) - \mathcal{M}^{(1)}_{\text{UV}} (\tilde{\delta}(q_{\text{UV}})) \right\rangle
\]

A partition of the real phase-space

\[
\sum_i \mathcal{R}_i(\{p'_j\}_{N+1}) = 1
\]

The real contribution mapped to the Born kinematics + loop three-momentum

\[
\int_{N+1} d\sigma^{(1)}_R = \int_N \int_{\vec{\ell}_1} \sum_i \mathcal{J}_i(q_i) \mathcal{R}_i(\{p'_j\}) \left| \mathcal{M}^{(0)}_{N+1}(\{p'_j\}) \right|^2 \left|_{\{p'_j\}_{N+1} \rightarrow (q_i,\{p_k\}_N)} \right|
\]
The real contribution **mapped** to the **Born kinematics + loop three-momentum** (mappings inspired by the factorization properties of QCD)

- **UV** subtracted locally in **d=4**
\[ \sigma_{\text{NNLO}} = \int_N d\sigma^{(2,R)}_{VV} + \int_{N+1} d\sigma^{(2,R)}_{VR} + \int_{N+2} d\sigma^{(2)}_{RR} \]

where the \textbf{VV} contribution reads

\[ d\sigma^{(2)}_{VV} = \int_{\tilde{\ell}_1} \int_{\tilde{\ell}_2} \sum_{i,j} \left[ 2 \text{Re} \langle \mathcal{M}_N^{(0)} | \mathcal{M}_N^{(2)} (\tilde{\delta}(q_i, q_j)) \rangle + \langle \mathcal{M}_N^{(1)} (\tilde{\delta}(q_i)) | \mathcal{M}_N^{(1)} (\tilde{\delta}(q_j)) \rangle \right] \mathcal{O}(\{p_k\}) \]
At \textbf{NNLO}

\[
\sigma^{\text{NNLO}} = \int_N d\sigma^{(2,R)}_{\text{VV}} + \int_{N+1} d\sigma^{(2,R)}_{\text{VR}} + \int_{N+2} d\sigma^{(2)}_{\text{RR}}
\]

where the \textbf{VV} contribution reads

\[
d\sigma^{(2)}_{\text{VV}} = \int_{\ell_1} \int_{\ell_2} \sum_{i,j} \left[ 2 \text{Re} \left( M_N^{(0)} | M_N^{(2)} (\tilde{\delta}(q_i, q_j)) \right) \right.
\]

\[
+ \left. \langle M_N^{(1)} (\tilde{\delta}(q_i)) | M_N^{(1)} (\tilde{\delta}(q_j)) \rangle \right] \mathcal{O}(\{p_k\})
\]

\textbf{Need the VR and RR contributions mapped to the Born kinematics} + the two independent loop three-momenta
At NNLO

\[ \sigma^{\text{NNLO}} = \int_N d\sigma^{(2,R)}_{VV} + \int_{N+1} d\sigma^{(2,R)}_{VR} + \int_{N+2} d\sigma^{(2)}_{RR} \]

where the VV contribution reads

\[ d\sigma^{(2)}_{VV} = \int_{\tilde{\ell}_1} \int_{\tilde{\ell}_2} \sum_{i,j} \left[ 2 \text{Re} \langle \mathcal{M}_N^{(0)} | \mathcal{M}_N^{(2)} (\tilde{\delta}(q_i, q_j)) \rangle \right] + \langle \mathcal{M}_N^{(1)} (\tilde{\delta}(q_i)) | \mathcal{M}_N^{(1)} (\tilde{\delta}(q_j)) \rangle \bigg] \mathcal{O}\{p_k\} \]

Need the VR and RR contributions mapped to the Born kinematics + the two independent loop three-momenta

Known two-loop amplitudes not suitable: requires LTD unintegrated representation
LTD IN SUCCESSION: PRELIMINARIES

LTD AT TWO-LOOPS (AND BEYOND)

\[ G_F(\alpha_k) = \prod_{i \in \alpha_k} G_F(q_i) \quad G_D(\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{j \neq i} G_D(q_i; q_j) \]

- At one loop:

\[ \int_{\ell_1} \mathcal{N}(\ell_1) G_F(\alpha_1) = - \int_{\ell_1} \mathcal{N}(\ell_1) \otimes G_D(\alpha_1) \]
At two-loops:

\[
\begin{align*}
G_F(\alpha_k) &= \prod_{i \in \alpha_k} G_F(q_i) \\
G_D(\alpha_k) &= \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{j \neq i} G_D(q_i; q_j)
\end{align*}
\]

\[
\int_{\ell_1} \int_{\ell_2} \mathcal{N}(\ell_1, \ell_2) G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) \\
= - \int_{\ell_1} \int_{\ell_2} G_F(\alpha_1) \mathcal{N}(\ell_1, \ell_2) \otimes G_D(\alpha_2 \cup \alpha_3)
\]
At two-loops:

\[
G_F(\alpha_k) = \prod_{i \in \alpha_k} G_F(q_i) \quad \quad \quad G_D(\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{j \neq i} G_D(q_i; q_j)
\]

\[
\int_{\ell_1} \int_{\ell_2} \mathcal{N}(\ell_1, \ell_2) G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)
\]

\[
= - \int_{\ell_1} \int_{\ell_2} G_F(\alpha_1) \mathcal{N}(\ell_1, \ell_2) \otimes G_D(\alpha_2 \cup \alpha_3)
\]

rearrangement of imaginary prescriptions
similar to relation of Feynman with Advanced propagators
LTD IN SUCCESSION: REMAINING PROPAGATORS ARE NOT FEYNMAN

LTD AT TWO-LOOPS (AND BEYOND)

\[ G_F(\alpha_k) = \prod_{i \in \alpha_k} G_F(q_i) \]
\[ G_D(\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{j \neq i} G_D(q_i; q_j) \]

At two-loops:

\[ \int_{\ell_1} \int_{\ell_2} \mathcal{N}(\ell_1, \ell_2) G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) \]
\[ = - \int_{\ell_1} \int_{\ell_2} G_F(\alpha_1) \mathcal{N}(\ell_1, \ell_2) \otimes G_D(\alpha_2 \cup \alpha_3) \]

rearrangement of imaginary prescriptions
similar to relation of Feynman with Advanced propagators

\[ G_D(\alpha_2) G_D(\alpha_3) + G_D(\alpha_2) G_F(\alpha_3) + G_F(\alpha_2) G_D(\alpha_3) \]

two cuts ✔
LTD AT TWO-LOOPS (AND BEYOND)

\[ G_F(\alpha_k) = \prod_{i \in \alpha_k} G_F(q_i) \quad G_D(\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{j \neq i} G_D(q_i; q_j) \]

At two-loops:

\[ \int_{\ell_1} \int_{\ell_2} \mathcal{N}(\ell_1, \ell_2) G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) \]

\[ = - \int_{\ell_1} \int_{\ell_2} G_F(\alpha_1) \mathcal{N}(\ell_1, \ell_2) \otimes G_D(\alpha_2 \cup \alpha_3) \]

rearrangement of imaginary prescriptions similar to relation of Feynman with Advanced propagators

\[ G_D(\alpha_2) G_D(\alpha_3) + G_D(\alpha_2) G_F(\alpha_3) + G_F(\alpha_2) G_D(\alpha_3) - G_D(\alpha_1 \cup \alpha_3) \]

two cuts \(\checkmark\)
At two-loops:

\[
G_F(\alpha_k) = \prod_{i \in \alpha_k} G_F(q_i) \quad \quad G_D(\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{j \neq i} G_D(q_i; q_j)
\]

\[
\int_{\ell_1} \int_{\ell_2} N(\ell_1, \ell_2) G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) = -\int_{\ell_1} \int_{\ell_2} G_F(\alpha_1) N(\ell_1, \ell_2) \otimes G_D(\alpha_2 \cup \alpha_3)
\]

rearrangement of imaginary prescriptions

similar to relation of Feynman with Advanced propagators

\[
G_D(\alpha_2) G_D(\alpha_3) + G_D(\alpha_2) G_F(\alpha_3) + G_F(\alpha_2) G_D(\alpha_3)
\]

two cuts ✔

\[
- G_D(\alpha_1 \cup \alpha_3) \quad \quad - G_D(-\alpha_2 \cup \alpha_1)
\]
OPENING LOOPS WITH A NUMBER OF CUTS EQUAL TO THE NUMBER OF LOOPS

**LTD AT TWO-LOOPS (AND BEYOND)**

- At two-loops (LTD representation):

\[
\int_{\ell_1} \int_{\ell_2} \mathcal{N}(\ell_1, \ell_2) G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) = \int_{\ell_1} \int_{\ell_2} \mathcal{N}(\ell_1, \ell_2) \otimes \left\{ G_D(\alpha_2) G_D(\alpha_1 \cup \alpha_3) + G_D(\alpha_2 \cup \alpha_1) G_D(\alpha_3) - G_F(\alpha_1) G_D(\alpha_2) G_D(\alpha_3) \right\}
\]

With a number of cuts equal to the number of loops the loop amplitude opens to a non-disjoint level like object.
When branches get broken at two-loops

The two-loop forest

- One propagator gets eventually on-shell in the same line where there is a cut propagator: equivalent to the one-loop case

Eventually gets on-shell and generates a singularity of the integrand

- There are two potential singular configurations: one of them cancels in the sum of cuts, the other leads to IR/thresholds
The genuine two-loop case occurs when the singularity is generated in another loop line.

There are four potential singular configurations: two of them cancel in the sum, the other two lead to IR/thresholds (genuine two-loop).

[Driencourt-Mangin, Sborlini, Torres, GR, in preparation]
Simplest two loop amplitude: **proof of concept** for other amplitudes with higher multiplicities, only UV
Simplest two loop amplitude: proof of concept for other amplitudes with higher multiplicities, only UV

Well known numerically/analytically, in general known amplitudes not suitable within LTD/FDU, requires unintegrated amplitude
DUAL AMPLITUDE FOR $H \rightarrow \gamma\gamma$ AT TWO-LOOPS

- Simplest two loop amplitude: **proof of concept** for other amplitudes with higher multiplicities, only UV
- Well known numerically/analytically, in general **known amplitudes not suitable** within LTD/FDU, requires unintegrated amplitude
- **IBP** would modify the **local behaviour** of the integrand: not suitable
Simplest two loop amplitude: **proof of concept** for other amplitudes with higher multiplicities, only UV

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**IBP** would modify the **local behaviour** of the integrand: not suitable

Dual propagators are **linear in the loop momenta**: tensor reduction simpler (reduction to master integrals not necessary)
Simplest two loop amplitude: **proof of concept** for other amplitudes with higher multiplicities, only UV

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Dual propagators are **linear in the loop momenta**: tensor reduction simpler (reduction to master integrals not necessary)

**Universality** also holds at two-loops?
DUAL AMPLITUDE FOR $H \rightarrow \gamma\gamma$ AT ONE-LOOP

- **Universality** and compactness of the dual representation. In four space-time dimensions after **local renormalisation**

$$A^{(1,f)}_{1,R} \bigg|_{d=4} = g_f s_{12} \int \ell \left[ \frac{1}{2\ell_{0}^{(+)}} \left( \frac{\ell^{(+)}}{q_{1,0}} + \frac{\ell^{(+)}}{q_{2,0}} + \frac{2(2\ell \cdot p_{12})^2}{s_{12}^2 - (2\ell \cdot p_{12} - i0)^2} \right) \right. 
\times \left. \frac{M_f^2}{(2\ell \cdot p_1)(2\ell \cdot p_2)} c_1^{(f)} + \frac{3\mu_{\text{UV}}^2}{4(q_{\text{UV},0}^5)} \hat{c}_{23}^{(f)} \right]$$

$$q_{i,0}^{(+)} = \sqrt{q_i^2 + M_f^2 - i0}$$

- The flavour of the internal particles is encoded by **two scalar coefficients**

$$c_1^{(f)} = \left( 2, -4 + \frac{s_{12}}{M_t^2}, 6 - \frac{3s_{12}}{M_W^2} \right)$$

$$\hat{c}_{23}^{f} = \frac{c_{23}^{(f)}}{d - 4} \bigg|_{d=4} = \left( 1, -2, 3 + \frac{s_{12}}{2M_W^2} \right)$$
UNIVERSALITY OF THE DUAL AMPLITUDES

The 22 dual double cuts can be written with 9 generators, for instance

\[ A^{(2,f)}_1(q_i, q_4) = g_f^{(2)} \int_{\ell_1} \int_{\ell_2} \tilde{\delta}(q_i, q_4) \left\{ -\frac{r_f c^{(f)}_1}{D_3 D_{12}} \left( G(D_i, \kappa_i, c^{(f)}_{4,u}) \left( 1 + H(D_3 D_{12}, \kappa_i) \right) + F(D_i, \kappa_4/\kappa_i) \right) \\
+ \left( c^{(f)}_7 \left( \frac{1}{D_i} - \frac{1}{D_3} \left( 1 - \frac{D_3}{D_{12}} \left( 1 - \frac{D_{12}}{D_i} \right) \right) \right) \right) + \frac{1}{D_3} \left( c^{(f)}_8 \left( \frac{1}{D_3} - \frac{1}{D_i} \right) - \frac{1}{D_{12}} \left( c^{(f)}_9 - c^{(f)}_{10} \frac{D_3}{D_i} \right) \right) \\
+ 2 r_f \left[ \frac{1}{D_3 D_{12}} \left( c^{(f)}_1 \left( \frac{1}{D_3 D_3} + \frac{1}{D_i} \left( \frac{1}{D_3} - \frac{1}{D_i} \right) \right) \right) + \frac{c^{(f)}_{14}}{D_3} + \frac{c^{(f)}_{20}}{D_i} - c^{(f)}_{16} \\
+ c^{(f)}_{17} \left( \frac{D_i - D_{12}}{D_3 D_i} \right) \right) - \frac{1}{D_i D_3} \left( c^{(f)}_7 \frac{D_i}{D_{12}} + c^{(f)}_{18} \right) \right] + \{3 \leftrightarrow 12\} \right\} \]

The \( c^{(f)}_i \) are scalar coefficients and depend only on the reduced mass \( r_f = \frac{s_{12}}{M_f^2} \) and the dimension \( d \), while the \( D_i \) are normalized propagators.
DUAL AMPLITUDE FOR $H \rightarrow \gamma\gamma$ AT TWO-LOOPS

$\rho_{t} = s_{12} / M_t^2 \quad s_{12} = M_H^2$

$\rho_{\phi} = s_{12} / M_{\phi}^2$

Analytic expressions from Aglietti, Bonciani, Degrassi, Vicini, JHEP 0701 (2007) 021
CONCLUSIONS

- The **forest is less singular than the individual trees**: essential feature for LTD/FDU to predict cross-sections and other physical observables in $d=4$ space-time dimensions. Potential advantages also for amplitudes.
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- All arises from the tiny

$\pm i0$


Integration along the forward on-shell hyperboloids (light-cones for massless) or positive energy modes

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WHEN A BRANCH GETS BROKEN: SINGULARITIES OF SINGLE CUT TREES

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  - **Light-like distance:** both singular configurations, partial cancellation, IR singularities remain in a compact region
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- needs to be compensated by the contribution from multiple cuts