Recent progress on the calculation of two-loop five-point Master Integrals

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FEYNMAN

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Space-Time Approach to Quantum Electrodynamics

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Department of Physics, Cornell University, Ithaca, New York
(Received May 9, 1949)

In this paper two things are done. (1) It is shown that a considerable simplification can be attained in writing down matrix elements for complex processes in electrodynamics. Further, a physical point of view is available which permits them to be written down directly for any specific problem. Being simply a and presumably consistent, method is therefore available for the calculation of all processes involving electrons and photons.
The simplification in writing the expressions results from an emphasis on the over-all space-time view resulting from a study of the solution of the equations of electrodynamics. The relation

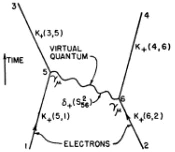
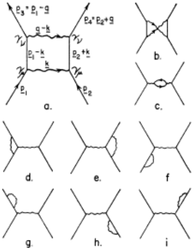


FIG. 1. The fundamental interaction Eq. (4). Exchange of one quantum between two electrons.

D. More Complex Problems

Matrix elements for complex problems can be set up in a manner analogous to that used for the simpler cases. We give three illustrations; higher order corrections to the Møller scatter-



T. Gehrmann, J. M. Henn and N. A. Lo Presti, Phys. Rev. Lett. **116** (2016) no.6, 062001 [arXiv:1511.05409 [hep-ph]].

T. Gehrmann, J. M. Henn and N. A. Lo Presti, arXiv:1807.09812 [hep-ph].

D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang and S. Zoia, arXiv:1812.11160 [hep-ph].

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1604**, 078 (2016) [arXiv:1511.09404 [hep-ph]].

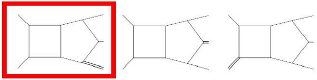


Figure 1. The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

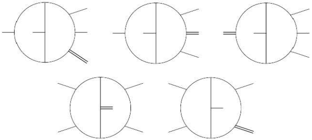


Figure 2. The five non-planar families with one external massive leg.

$$
\begin{aligned}
\mathbf{G} = {} & \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathrm{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
& + \varepsilon^{0}\left(\sum \mathcal{G}_{ab}\mathrm{M}_a\mathrm{M}_b\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a\mathrm{M}_a\mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
& + \varepsilon\left(\sum \mathcal{G}_{abc}\mathrm{M}_a\mathrm{M}_b\mathrm{M}_c\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab}\mathrm{M}_a\mathrm{M}_b\mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a\mathrm{M}_a\mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
& + \varepsilon^{2}\left(\sum \mathcal{G}_{abcd}\mathrm{M}_a\mathrm{M}_b\mathrm{M}_c\mathrm{M}_d\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc}\mathrm{M}_a\mathrm{M}_b\mathrm{M}_c\mathbf{b}_0^{(-1)} \right. \\
& \left. + \sum \mathcal{G}_{ab}\mathrm{M}_a\mathrm{M}_b\mathbf{b}_0^{(0)} + \sum \mathcal{G}_a\mathrm{M}_a\mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right) \qquad (3.6)
\end{aligned}
$$

Best today

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2),[0]}_{--+++}$	12.5	27.7526	-23.773	-168.117	-175.207±0.004
$P^{(2),[0]}_{--+++}$	12.5	27.7526	-23.773	-168.116	—
$\widehat{A}^{(2),[0]}_{-+-++}$	12.5	27.7526	2.5029	-35.8094	69.661±0.009
$P^{(2),[0]}_{-+-++}$	12.5	27.7526	2.5028	-35.8086	—

TABLE II. The numerical evaluation of $\widehat{A}^{(2),[0]}(1,2,3,4,5)$ using $\{x_1 = -1, x_2 = 79/90, x_3 = 16/61, x_4 = 37/78, x_5 = 83/102\}$ in Eq.(6). The comparison with the universal pole structure, P, is shown. The **+++++** and **-++++** amplitudes vanish to $\mathcal{O}(\epsilon)$ for this $(d_s-2)^0$ component.

S. Badger, C. BrG'Ennum-Hansen, H. B. Hartanto and T. Peraro, "A first look at two-loop five-gluon scattering in QCD," arXiv:1712.02229 [hep-ph].

Best today

$\mathcal{A}^{(2)}/\mathcal{A}^{(0)}$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1^-, 2^-, 3^+, 4^+, 5^+)$	12.5000000	25.46246919	-1152.843107	-4072.938337	-3637.249567
$(1^-, 2^+, 3^-, 4^+, 5^+)$	12.5000000	25.46246919	-6.121629624	-90.22184215	-115.7836685

TABLE II. Numeric results truncated to 10 significant figures for the two-loop split and alternating MHV amplitudes, normalized to the tree level, at the kinematic point of eq. (IV.1).

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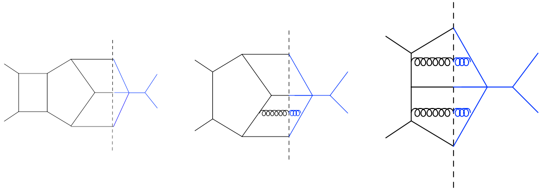
What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$



Perturbative QCD at NNLO

What do we need for an NNLO calculation ?

$$\begin{aligned}
\sigma_{NNLO} \quad \rightarrow \quad & \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) & \quad VV \\
+ \quad & \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) & \quad RV \\
+ \quad & \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) & \quad RR
\end{aligned}$$

$RV + RR \rightarrow$ antenna-S, colorfull-NNLO, sector-improved residue subtraction, nested soft-collinear, local analytic sector subtraction, projection to born, q_T, N-jetiness

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OPP at two loops

- Write the "OPP-type" equation at two loops

$$\frac{N(l_1, l_2; \{p_i\})}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$$\sum \frac{\Delta_{i_1 i_2 \ldots i_m}(l_1, l_2; \{p_i\})}{D_{i_1} D_{i_2} \ldots D_{i_m}} \rightarrow \text{spurious} \oplus \text{ISP} - \text{irreducible integrals}$$

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ISP-irreducible integrals → use IBPI to Master Integrals

Libraries in the future: `QCD2LOOP`, `TwOLOop`

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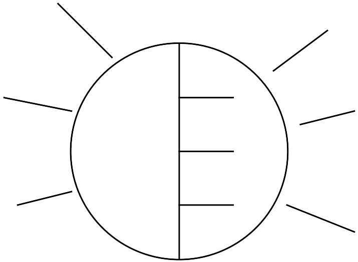
Two-loop graph



IBPI: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \ldots D_N$, allows to express all scalar products $D_i = (\{k,l\} + p_i)^2 - M_i^2$
- $$F[a_1,\ldots,a_N] = \int d^dk d^dl \frac{1}{D_1^{a_1}\ldots D_N^{a_N}}$$
  $$\int d^dk d^dl \;\; \frac{\partial}{\partial\{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, \upsilon^\mu\}}{D_1^{a_1}\ldots D_N^{a_N}} \right) = 0$$
- IBP Laporta: `FIRE`, `AIR`, `Reduze`, `Kira` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations
- Or numerical: `SecDec`, Weinzierl

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$$F[a_1, \ldots, a_N] = \int d^d k d^d l \frac{1}{D_1^{a_1} \ldots D_N^{a_N}}$$

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Differential Equations Approach

The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum C_{b_1,\ldots,b_n} F[b_1, \ldots, b_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f(\varepsilon, \{x_i\}) = \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\})$$
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The Simplified Differential Equations Approach

C. G. Papadopoulos, JHEP **1407** (2014) 088

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

- Introduce one parameter

$$G_{11\dots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k+x\,p_1)^2\,(k+p_1+p_2)^2\dots(k+p_1+p_2+\dots+p_n)^2}$$

- Factorizing external momenta dependence:

$$x : (q_1 = xp_1, q_2 = p_{12} - xp_1, \dots) \to x \otimes (q_1 = p_1, q_2 = p_2, \dots)$$

- Now the integral as a function of x, allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\dots1}(x) = -\frac{1}{x} G_{11\dots1}(x) + xp_1^2 G_{12\dots1} + \frac{1}{x} G_{02\dots1}$$

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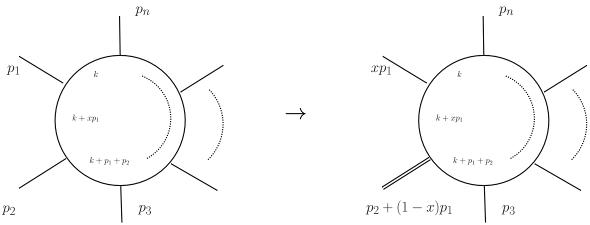

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5BOX - ONE LEG OFF-SHELL: ALL FAMILIES

C. G. Papadopoulos, D. Tommasini and C. Wever, arXiv:1511.09404 [hep-ph].

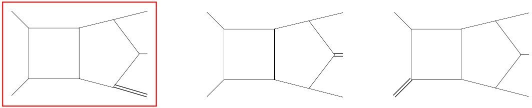

FIGURE: The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

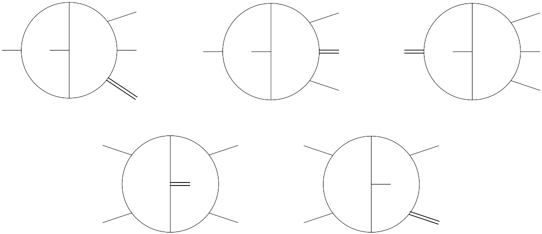

FIGURE: The five non-planar families with one external massive leg.

$$p(q_1)p'(q_2) \to V(q_3)j_1(q_4)j_2(q_5), \quad q_1^2 = q_2^2 = 0, \quad q_3^2 = M_3^2, \quad q_4^2 = q_5^2 = 0.$$

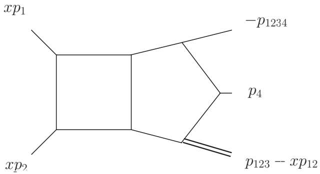

FIGURE: The parametrization of external momenta in terms of x for the planar pentabox of the family P_1. All external momenta are incoming.

$$s_{12} := p_{12}^2, \quad s_{23} := p_{23}^2, \quad s_{34} := p_{34}^2, \quad s_{45} := p_{45}^2 = p_{123}^2, \quad s_{51} := p_{15}^2 = p_{234}^2,$$

$$q_1^2 = q_2^2 = q_4^2 = q_5^2 = 0 \quad q_3^2 = (s_{45} - s_{12}x)(1-x)$$
$$q_{12}^2 = s_{12}x^2 \;\; q_{23}^2 = s_{45}(1-x) + s_{23}x \;\; q_{34}^2 = (s_{34} - s_{12}(1-x))\,x \;\; q_{45}^2 = s_{45} \;\; q_{51}^2 = s_{51}x$$

5box - one leg off-shell: P1

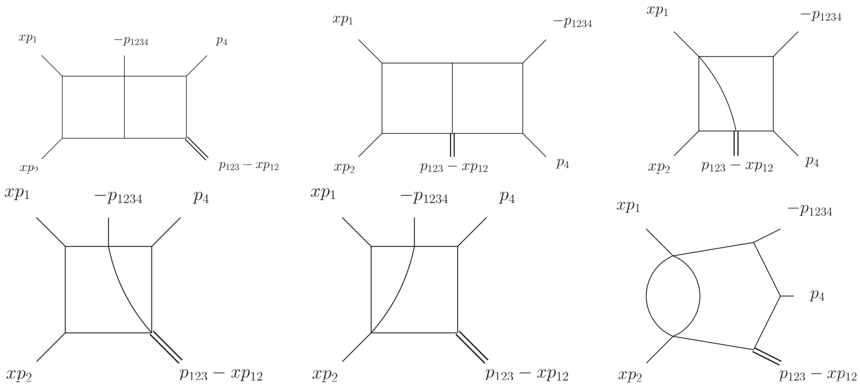

FIGURE: The five-point Feynman diagrams, besides the pentabox itself in Figure 7, that are contained in the family P_1. All external momenta are incoming.

5BOX - ONE LEG OFF-SHELL: P1

$$G^{P_1}_{a_1\cdots a_{11}}(x,s,\epsilon) := e^{2\gamma_E\epsilon}\int \frac{d^dk_1}{i\pi^{d/2}}\frac{d^dk_2}{i\pi^{d/2}}\frac{1}{k_1^{2a_1}(k_1+xp_1)^{2a_2}(k_1+xp_{12})^{2a_3}(k_1+p_{123})^{2a_4}}$$
$$\times\frac{1}{(k_1+p_{1234})^{2a_5}k_2^{2a_6}(k_2-xp_1)^{2a_7}(k_2-xp_{12})^{2a_8}(k_2-p_{123})^{2a_9}(k_2-p_{1234})^{2a_{10}}(k_1+k_2)^{2a_{11}}},$$

$P_1(74)$: {10000000101, 01000000101, 00100000101, 10000001001, 01000000011, 00100000011, 10100001100, 10100001010, 10100101000, 01000101001, 10100100100, 10100000102, 10100000101, 10100000011, 10000001102, 10000001101, 10000001011, 01000100101, 01000001101, 01000001011, 00100100102, 00100100101, 11100000101, 11100000011, 11000001102, 11000001101, 11000001012, 11000001011, 11000000111, 10100000112, 10000001111, 01100100102, 01100100101, 01100100011, 01100000111, 01000101102, 01000101101, 01000101011, 01000100111, 01000001111, 00100100111, 10100101100, 10100100101, 10100001101, 10100001011, 10100000111, 111m0000111, 110000m1111, 11000001111, 10100101110, 10100100111, 10100001111, 011001m0111, 01100100111, 010m0101111, 01000101111, 11100100101, 11100001101, 11100001011, 11100000111, 111m0101101, 111001m1101, 11100101101, 1110m101011, 11100101011, 111m0100111, 11100100111, 111000m1111, 111m0001111, 11100001111, 111001m0111, 11100101111, 111001m1111, 111m0101111},

m= −1

5BOX P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\}, \varepsilon, x\right)\mathbf{G}$$

$$\partial_x \mathbf{G}' = \mathbf{M}'\mathbf{G}' \qquad \mathbf{M}' = \mathbf{T}\mathbf{M}\mathbf{T}^{-1} + (\partial_x \mathbf{T})\,\mathbf{T}^{-1} \qquad \mathbf{G}' = \mathbf{T}\mathbf{G}$$

$$(M_D)_{IJ} = \delta_{IJ} M_{II}(\varepsilon = 0), I, J = 1 \ldots 74$$

$$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G},\ \mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right) \text{ and } \mathbf{M} \to \mathbf{S}^{-1}(\mathbf{M} - \mathbf{M}_D)\,\mathbf{S}.$$

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x - l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

$$1-\frac{s_{34}-s_{51}}{s_{12}},\quad \frac{s_{45}-s_{23}}{s_{12}},\quad -\frac{s_{51}}{s_{12}},\quad \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\quad \frac{s_{45}}{s_{34}+s_{45}},$$

$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

5BOX P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\{s_{ij}\},\varepsilon,x\right)\mathbf{G}$$

$$\partial_x \mathbf{G}' = \mathbf{M}'\mathbf{G}' \qquad \mathbf{M}' = \mathbf{T}\mathbf{M}\mathbf{T}^{-1} + \left(\partial_x \mathbf{T}\right)\mathbf{T}^{-1} \qquad \mathbf{G}' = \mathbf{T}\mathbf{G}$$

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$\mathbf{G} \to \mathbf{S}^{-1}\mathbf{G}$, $\mathbf{S} = \exp\left(\int dx\, \mathbf{M}_D\right)$ and $\mathbf{M} \to \mathbf{S}^{-1}\left(\mathbf{M} - \mathbf{M}_D\right)\mathbf{S}$.

$$M_{IJ} = N_{IJ}\left(\varepsilon\right)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{\left(x-l_i\right)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

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$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

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$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}+s_{34}-s_{51}\right)},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

5box P1 - DE

$$\partial_x \mathbf{G} = \mathbf{M}\left(\left\{s_{ij}\right\}, \varepsilon, x\right) \mathbf{G}$$

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$$0,\quad 1,\quad \frac{s_{45}}{s_{45}-s_{23}},\quad \frac{s_{45}}{s_{12}},\quad 1-\frac{s_{34}}{s_{12}},\quad 1+\frac{s_{23}}{s_{12}},$$

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$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},\quad \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}\left(s_{23}-s_{45}-s_{51}\right)},$$

$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}\left(s_{23}+s_{34}-s_{51}\right)},\quad \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\quad \frac{s_{45}}{s_{12}+s_{23}},$$

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$$M_{IJ} = N_{IJ}\left(\varepsilon\right)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1} \frac{C_{IJ;ijk}\varepsilon^k}{\left(x - l_i\right)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1} \tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

Letters (20):

$$0, \quad 1, \quad \frac{s_{45}}{s_{45} - s_{23}}, \quad \frac{s_{45}}{s_{12}}, \quad 1 - \frac{s_{34}}{s_{12}}, \quad 1 + \frac{s_{23}}{s_{12}},$$

$$1 - \frac{s_{34} - s_{51}}{s_{12}}, \quad \frac{s_{45} - s_{23}}{s_{12}}, \quad -\frac{s_{51}}{s_{12}}, \quad \frac{s_{45}}{-s_{23} + s_{45} + s_{51}}, \quad \frac{s_{45}}{s_{34} + s_{45}},$$

$$\frac{s_{12}s_{23} - 2s_{12}s_{45} - s_{12}s_{51} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} \pm \sqrt{\Delta_1}}{2s_{12}\left(s_{23} - s_{45} - s_{51}\right)}, \quad \frac{s_{12}s_{23} - s_{12}s_{45} - s_{12}s_{51} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} \pm \sqrt{\Delta_2}}{2s_{12}\left(s_{23} - s_{45} - s_{51}\right)},$$

$$\frac{s_{12}s_{23} - s_{12}s_{51} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} \pm \sqrt{\Delta_1}}{2s_{12}\left(s_{23} + s_{34} - s_{51}\right)}, \quad \frac{s_{12}s_{45} \pm \sqrt{\Delta_3}}{s_{12}s_{34} + s_{12}s_{45}}, \quad \frac{s_{45}}{s_{12} + s_{23}},$$

5BOX P1 - DE

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j}+\sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\int_0^x dt\,\frac{1}{(t-a_n)^2}\mathcal{G}(a_{n-1},\ldots,a_1,t) \qquad \int_0^x dt\;t^m\;\mathcal{G}(a_{n-1},\ldots,a_1,t)$$

Fuchsian

$N_{IJ}(\varepsilon) = \delta_{IJ} n_J(\varepsilon),\; G_I \to n_I(\varepsilon)\, G_I$

$$M_{IJ} = \left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j}+\sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right).$$

$$\mathbf{G} \to (\mathbf{I}-\mathbf{K}_i)\,\mathbf{G}, \quad \mathbf{M} \to (\mathbf{M}-\partial_x\mathbf{K}_i-\mathbf{K}_i\mathbf{M})\,(\mathbf{I}-\mathbf{K}_i)^{-1} \quad i=1,2,3$$

$$\partial_x \mathbf{G} = \left(\varepsilon\sum_{a=1}^{19}\frac{\mathbf{M}_a}{(x-l_a)}\right)\mathbf{G}$$

R. N. Lee, JHEP **1504** (2015) 108 [arXiv:1411.0911 [hep-ph]].
O. Gituliar and V. Magerya, Comput. Phys. Commun. **219** (2017) 329 [arXiv:1701.04269 [hep-ph]].
C. Meyer, Comput. Phys. Commun. **222** (2018) 295 [arXiv:1705.06252 [hep-ph]].

5BOX P1 - DE

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5BOX P1 - DE

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$\mathbf{M}(\varepsilon = 0)$ contains $(x - l_i)^{-2}$ and x^0

$$(\mathbf{K}_1)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon = 0))_{IJ} & I,J \neq 69,74 \\ 0 & I,J = 69,74 \end{cases}$$

$$(\mathbf{K}_2)_{IJ} = \begin{cases} \int dx (\mathbf{M}(\varepsilon = 0))_{IJ} & I,J \neq 74 \\ 0 & I,J = 74 \end{cases}$$

$$(\mathbf{K}_3)_{IJ} = \int dx (\mathbf{M}(\varepsilon = 0))_{IJ}$$

M.A. Barkatou and E.Pflügel, Journal of Symbolic Computation, **44** (2009),1017

$$\partial_x \mathbf{G} = \left(\varepsilon \sum_{a=1}^{19} \frac{\mathbf{M}_a}{(x - l_a)} \right) \mathbf{G}$$

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Fuchsian
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C. Meyer, Comput. Phys. Commun. **222** (2018) 295 [arXiv:1705.06252 [hep-ph]].

5box P1 - solution

- Solution:

$$
\begin{aligned}
\mathbf{G} &= \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
&+ \varepsilon^{0}\left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
&+ \varepsilon\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
&+ \varepsilon^{2}\left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)}\right. \\
&+ \left.\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned}
$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$
\mathbf{G} \underset{x \to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}
$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

  $\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a \varepsilon}$, n_a positive integers, $x \to l_a$.

- Analytic continuation: F polynomial

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072 [arXiv:1409.6114 [hep-ph]].

- Solution:

$$\begin{aligned}
\mathbf{G} &= \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
&+ \varepsilon^{0}\left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
&+ \varepsilon\left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
&+ \varepsilon^{2}\left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(-1)} \right. \\
&+ \left. \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned}$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$\mathbf{G} \underset{x\to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

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5BOX P1 - SOLUTION

- Solution:

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\begin{aligned}
\mathbf{G} &= \varepsilon^{-2}\mathbf{b}_0^{(-2)} + \varepsilon^{-1}\left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(-2)} + \mathbf{b}_0^{(-1)}\right) \\
&+ \varepsilon^{0}\left(\sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(-1)} + \mathbf{b}_0^{(0)}\right) \\
&+ \varepsilon\left(\sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(-1)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)}\right) \\
&+ \varepsilon^{2}\left(\sum \mathcal{G}_{abcd}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{M}_d\mathbf{b}_0^{(-2)} + \sum \mathcal{G}_{abc}\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{b}_0^{(-1)}\right. \\
&+ \left.\sum \mathcal{G}_{ab}\mathbf{M}_a\mathbf{M}_b\mathbf{b}_0^{(0)} + \sum \mathcal{G}_a\mathbf{M}_a\mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)}\right)
\end{aligned}
$$

$\mathbf{b}_0^{(k)}$, $k = -2, \ldots, 2$ representing the x-independent boundary terms in the limit $x = 0$ at order ε^k

$$
\mathbf{G} \underset{x\to 0}{\sim} \sum_{k=-2}^{2} \varepsilon^k \sum_{n=0}^{k+2} \mathbf{b}_n^{(k)} \log^n(x) + \text{subleading terms.}
$$

$\mathcal{G}_{a,b,\ldots} = \mathcal{G}(l_a, l_b, \ldots; x)$ with $a, b, c, d = 1, \ldots, 19$.

- Uniform transcendental: UT multi- vs one-parameter DE

  $\mathbf{M}_a$ depend on kinematics, but eigenvalues not: $(x - l_a)^{-n_a\varepsilon}$, n_a positive integers, $x \to l_a$.

- Analytic continuation: F polynomial

C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072 [arXiv:1409.6114 [hep-ph]].

THE $x = 1$ LIMIT

$$\mathbf{G} = \sum_{n\geq -2} \varepsilon^n \sum_{i=0}^{n+2} \frac{1}{i!} \mathbf{c}_i^{(n)} \log^i (1-x)$$

$\mathbf{c}_i^{(n)}$ are finite in the limit $x = 1$

$$\mathbf{c}_i^{(n)} = \mathbf{M}_2 \mathbf{c}_{i-1}^{(n-1)} \quad i \geq 1$$

$$\mathbf{G}_{reg} = \sum_{n\geq -2} \varepsilon^n \mathbf{c}_0^{(n)}.$$

$$\mathbf{G} = \mathbf{G}_{reg} + \frac{\left((1-x)^{-2\varepsilon} - 1\right)}{(-2\varepsilon)} \mathbf{X} + \frac{\left((1-x)^{-\varepsilon} - 1\right)}{(-\varepsilon)} \mathbf{Y}$$

$$\mathbf{X} = \sum_{n\geq -1} \varepsilon^n \mathbf{X}^{(n)} \quad \mathbf{Y} = \sum_{n\geq -1} \varepsilon^n \mathbf{Y}^{(n)}.$$

$$(-1)^n {\mathbf{M}_2}^n = {\mathbf{M}_2}^2 \left(2^{n-1} - 1\right) + \mathbf{M}_2 \left(2^{n-1} - 2\right), \quad n \geq 1.$$

minimal polynomial $x(x+1)(x+2)$ of the matrix $\mathbf{M}_2$

$$\mathbf{G}_{x=1} = \left(\mathbf{I} + \frac{3}{2}\mathbf{M}_2 + \frac{1}{2}\mathbf{M}_2^2\right) \mathbf{G}_{trunc}$$

$\mathbf{G}_{trunc} \equiv \mathbf{G}_{reg}(x=1)$.

The Feynman trick

Internal Reduction

Papadopoulos, G. Costas and Wever, Christopher, LL2018

$$\frac{1}{\cdots\left[(k+p_1)^2-m_1^2\right]\left[(k+p_2)^2-m_2^2\right]\cdots}=\int_0^1 dx\frac{1}{\cdots\left[(k+q)^2-M^2\right]^2\cdots}$$

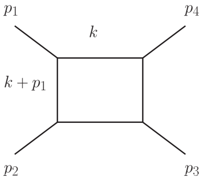
with

$$q=xp_1+(1-x)\,p_2$$

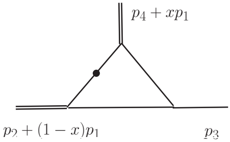
and

$$M^2=xm_1^2+(1-x)\,m_2^2-x\,(1-x)\,(p_1-p_2)^2$$

.



$$= \int_0^1 dx$$



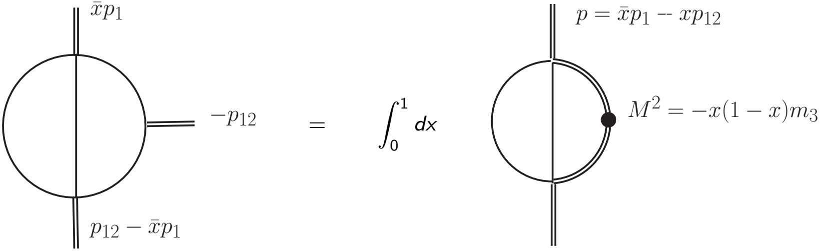
$$\mathrm{Tri} = -\frac{2\,(d-3)}{S_3\,(S_2 - S_3)}\mathrm{Bub}_1 + \frac{2\,(d-3)}{S_2\,(S_2-S_3)}\mathrm{Bub}_2 = \frac{2}{\varepsilon}\left[\frac{(-s)^{-1-\varepsilon}(1-x)^{-1-\varepsilon}}{s\,(1-x)-tx} - \frac{(-t)^{-1-\varepsilon}x^{-1-\varepsilon}}{s\,(1-x)-tx}\right]$$

with $d = 4 - 2\epsilon$, $S_2 = (p_2 + (1-x)p_1)^2 = (1-x)s$ and $S_3 = (p_4 + xp_1)^2 = xt$, $s = (p_1+p_2)^2$, $t = (p_2+p_3)^2$.

$$\mathrm{Box} = \int_0^1 dx \;\; \mathrm{Tri}$$
$$= \frac{2}{\varepsilon^2}\frac{1}{st}\left[(-s)^{-\varepsilon}\,{}_2F_1\left(1,-\varepsilon;1-\varepsilon;\frac{s+t}{t}\right) + (-t)^{-\varepsilon}\,{}_2F_1\left(1,-\varepsilon;1-\varepsilon;\frac{s+t}{s}\right)\right]$$

Eq. (4.18) in Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412** (1994) 751 [hep-ph/9306240].

Internal Reduction



E. Remiddi and L. Tancredi, Nucl. Phys. B **907**, 400 (2016) [arXiv:1602.01481 [hep-ph]].

C. G. Papadopoulos, JHEP **1407**, 088 (2014) [arXiv:1401.6057 [hep-ph]].

Inverting ?

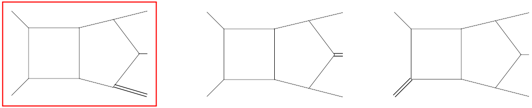


FIGURE: The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.

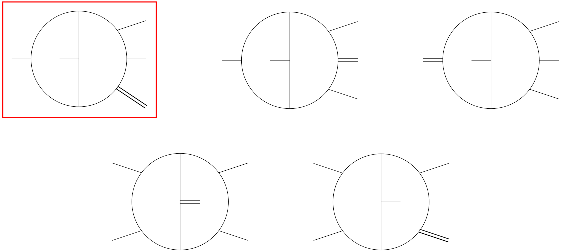


FIGURE: The five non-planar families with one external massive leg.

Internal Reduction

J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1405**, 090 (2014) [arXiv:1402.7078 [hep-ph]].
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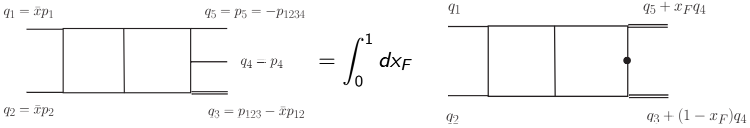

- Physical vs Euclidean
- Re-deriving DE in x with UT canonical basis: rational relation between the two parametrizations

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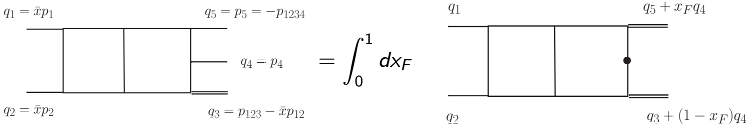

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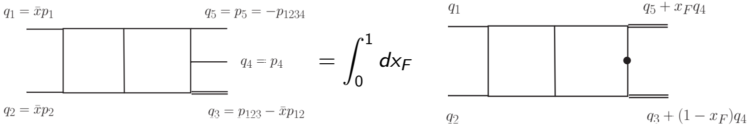

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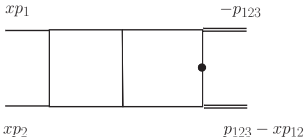


$$s = x^2 S_{12}, \quad t = xS_{23} + (1-x)\,q, \quad M_3^2 = (1-x)\,(q - S_{12}x)\,, \quad M_4^2 = q$$

$$\{l_i\} = \left\{ \frac{q}{S_{12}}, \frac{q}{q - S_{23}}, 1 + \frac{S_{23}}{S_{12}}, \frac{q - S_{23}}{S_{12}} \right\}$$

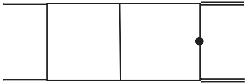
$$\partial_x \mathbf{I} = \sum_{i=1}^{4} \frac{\mathbf{M}_i}{x - l_i} \mathbf{I} \qquad \mathbf{I} \to G\left(l_i, l_j, \ldots ; x\right)$$

$$M_4^2 = m_3^2\,(1 - x_F) + s_{34} x_F, \quad M_3^2 = (1 - x_F)\,s_{45}, \quad s = s_{12}, \quad t = s_{23}\,(1 - x_F) + s_{51} x_F$$

$$x_F \to \frac{(1-x)\left(s_{12} - x m_3^2\right) + x s_{45}}{x\,(1-x)\left(s_{34} - m_3^2\right) + x s_{45}}$$

- Agreement with on-shell ($x = 1$) case

T. Gehrmann, J. M. Henn and N. A. Lo Presti, arXiv:1807.09812



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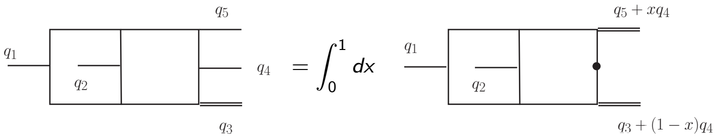

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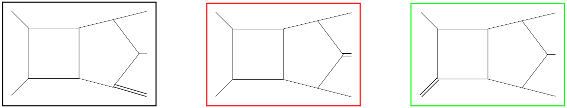
F. Caola, J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1409** (2014) 043 [arXiv:1404.5590 [hep-ph]].
C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501**, 072 (2015) [arXiv:1409.6114 [hep-ph]].
S. Abreu, B. Page and M. Zeng, arXiv:1807.11522 [hep-th].
D. Chicherin, T. Gehrmann, J. M. Henn, N. A. Lo Presti, V. Mitev and P. Wasser, arXiv:1809.06240 [hep-ph].
D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang and S. Zoia, arXiv:1812.11160 [hep-ph].

5BOX



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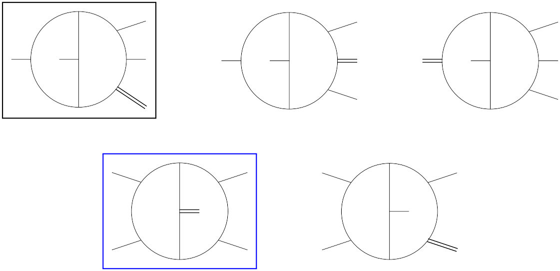

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Summary

1. New concepts (SDE, IR) and tools (B-IBP) $\rightarrow$ reduce P+NP 5box MI and derive DE in x
2. Baikov representation of cut integrals $\rightarrow$ IBP, DE, establish the class of functions, MPL or EI
3. Deriving canonical DE
4. Solving and verifying by recycling known results: Internal Reduction
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6. All 8-denominator MI, arbitrary internal masses, number of legs and external kinematics

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H. Frellesvig and C. G. Papadopoulos, JHEP **1704** (2017) 083 [arXiv:1701.07356 [hep-ph]].

A. Primo and L. Tancredi, Nucl. Phys. B **916** (2017) 94

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M. Harley, F. Moriello and R. M. Schabinger, JHEP **1706** (2017) 049

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J. Bosma, K. J. Larsen and Y. Zhang, arXiv:1712.03760 [hep-th].

J. Boehm, A. Georgoudis, K. J. Larsen, M. Schulze and Y. Zhang, arXiv:1712.09737 [hep-th].

J. Bfhm, A. Georgoudis, K. J. Larsen, H. Schfnemann and Y. Zhang, arXiv:1805.01873 [hep-th].

S. Abreu, B. Page and M. Zeng, arXiv:1807.11522 [hep-th].

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