Analytics from Numerics: 5-Point QCD Amplitudes at Two Loops

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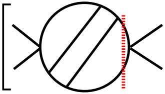
based on [1812.04586]

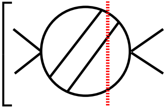
Introduction

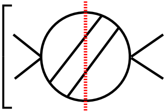
Why Analytic 5-Point Two-Loop Amplitudes?

- NNLO corrections often require two-loop amplitudes.
- A laboratory for complex multi-scale computations.
- Analytic results are faster and more flexible.
- Allows study of interesting mathematical properties.

$\delta\sigma_n^{\mathrm{NNLO}} =$

$\int d\Phi_n$ [ $+ \cdots$]

$+ \int d\Phi_{n+1}$ [ $+ \cdots$]

$+ \int d\Phi_{n+2}$ [ $+ \cdots$]

Progress in Five-Point NNLO Relevant Calculations

Integrals:

- Planar [Gehrmann, Henn, lo Presti 15], [Papadopoulos, Tommasini, Wever 15]
- Non-planar [Abreu, Dixon, Herrmann, B.P., Zeng 18], [Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia 18]

IBP Tables:

- Planar [Chawdry, Lim, Mitov 18], [Boels, Jin, Luo 18]
- Some non-planar [Böhm, Georgoudis, Larsen, Schönemann, Zhang 18]

Amplitudes:

- Numeric 5-parton [Badger, Brønnum-Hansen, Bayu Hartanto, Peraro 17, 18] [Abreu, Ita, Febres Cordero, B.P., Sotnikov, Zeng 17, 18]
- Analytic -++++ [Badger, Brønnum-Hansen, Bayu Hartanto, Peraro 18]
- Analytic 5-gluon [Abreu, Dormans, Febres Cordero, Ita, B.P. 18]

How To Approach?

Traditionally:

- Use FORM to generate integrand from diagrams.
- Construct IBP tables using Laporta algorithm.

But:

- Gigabytes of expressions for diagrams/relations. [Chawdry, Lim, Mitov '18] [Boels, Jin, Luo '18]
- Simplifying multivariate final result hard.

Alternatively:

- Reconstruct expression from numerical samples.
- Exploit numerical progress. [Abreu et al '17/'18]
- Directly target final results.
- Sidestep large expressions and IBP tables.
- Low memory needs for numerical evaluations.

Analytic Reconstruction

Numerical Unitarity @ Two Loops

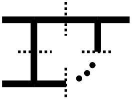
[Abreu, Ita, Jaquier, Febres Cordero, BP '17]

- Take an ansatz for loop-amplitude integrand, decomposing into master (M_Γ) and surface (S_Γ) integrands.

$$\bar{\mathcal{A}}(\ell_l, \vec{x}) = \sum_{\text{Topologies}\,\Gamma} \; \sum_{i \in M_\Gamma \cup S_\Gamma} \frac{c_{\Gamma,i}(\vec{x})\, m_{\Gamma,i}(\ell_l)}{\prod_{\text{props}\,j} \rho_j}.$$

[Ita '15]

- Numerically fix $c_{\Gamma,i}(\vec{x})$ on finite field* from on-shell data.



$$= \sum_{\substack{\Gamma' \geq \Gamma,\\ i \in M_{\Gamma'} \cup S_{\Gamma'}}} \frac{c_{\Gamma',i}(\vec{x})\, m_{\Gamma',i}(\ell_l^\Gamma)}{\prod_{\text{props}\,j} \rho_j}.$$

[BDDK '94, '95]

- Insert master integrals, expand $\Rightarrow$ integrated amplitude.
- Amplitude naturally splits into rational functions of external kinematics $c_{\Gamma,i}(\vec{x})$ and special functions (master integrals).

*See also [Peraro '16]

Basics of Functional Reconstruction

- Coefficients are rational in twistor parameters x_i. [Hodges '09]

$$s_{12} = x_4 , \quad s_{23} = x_2 x_4 , \quad s_{34} = x_4 \left(\frac{(1+x_1)x_2}{x_0} + x_1(x_3-1) \right) ,$$
$$s_{45} = x_3 x_4 , \quad s_{51} = x_1 x_4 (x_0 - x_2 + x_3) ,$$
$$\mathrm{tr}_5 = 4i\varepsilon(p_1, p_2, p_3, p_4)$$
$$= x_4^2 \left(x_2(1+2x_1) + x_0 x_1 (x_3-1) - \frac{x_2(1+x_2)(x_2-x_3)}{x_0} \right) ,$$

- Rational functions can be reconstructed from samples on finite fields. [Peraro '16]

$$f(\vec{x}) = \frac{\sum_{0\leq|\alpha|\leq R} a_\alpha \vec{x}^\alpha}{1 + \sum_{1\leq|\beta|\leq R'} b_\beta \vec{x}^\beta}$$

- Total degree R, n-variate polynomial. Many unknowns.
- $R = 50, n = 4$ $\Rightarrow \sim 320000$ terms.

$$\#\ \text{terms} = \binom{R+n}{n}$$

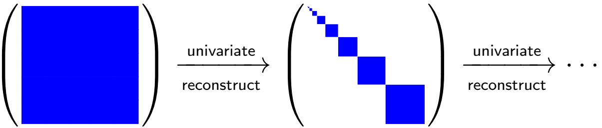
System too large for random sampling and linear solving.

Reconstruction Algorithm

[Peraro '16]

- Key idea: univariate reconstruct to block diagonalize.
- Blocks now simpler functions of fewer variables $\Rightarrow$ recurse.

$$x_4 = 1, \quad x_i = c_i + d_i t, \quad \Rightarrow \quad f(\vec{x}(t)) = \frac{\sum_{i=0}^{R} a_i t^i}{1 + \sum_{j=1}^{R'} b_j t^j}.$$



- Efficient approach, evaluation time/ansatz size now dominant.
- Simultaneous reconstruction of multiple functions possible.
- Cluster capable: can predetermine evaluation points.

5-Gluon Amplitudes

Remainders and Pentagon Functions [Abreu, Dormans, Ita, Febres Cordero, BP '18]

- UV/IR poles are universal. Finite remainder is new. [Catani 98]
- Insert one-loop to $O(\epsilon^2)$.
- Simplifies all-plus amplitude [Gehrmann, Henn, lo Presti '15]

$$\mathcal{R}^{(2)}_{\pm++++} = \bar{\mathcal{A}}^{(2)}_{\pm++++} - \bar{\mathcal{A}}^{(1)}_{\pm++++} \sum_{i=1}^{5} \frac{(-s_{i,i+1})^{-\epsilon}}{\epsilon^2} + \mathcal{O}(\epsilon),$$

$$\mathcal{R}^{(2)}_{-\mp\pm++} = \bar{\mathcal{A}}^{(2)}_{-\mp\pm++} - \left(\frac{5\,\tilde{\beta}_0}{2\epsilon} + \mathbf{I}^{(1)}\right) S_\epsilon \bar{\mathcal{A}}^{(1)}_{-\mp\pm++} + \left(\frac{15\,\tilde{\beta}_0^2}{8\epsilon^2} + \frac{3}{2\epsilon}\left(\tilde{\beta}_0 \mathbf{I}^{(1)} - \tilde{\beta}_1\right) - \mathbf{I}^{(2)}\right) S_\epsilon^2 + \mathcal{O}(\epsilon)\,.$$

- Express in basis B of ~ 400 "pentagon functions" h_i. [Gehrmann, Henn, lo Presti '18]
- Basis of polylogs satisfying physical conditions.

$$\bar{\mathcal{A}}^{(2)} = \sum_{i\in B} \sum_{k=-4}^{0} \epsilon^k\, \tilde{c}_{k,i}(\vec{x}) h_i(\vec{x}) + \mathcal{O}(\epsilon),$$

$$\mathcal{R}^{(2)} = \sum_{i\in B} r_i(\vec{x})\, h_i(\vec{x})\,.$$

Further Simplifications and Effect

[Abreu, Dormans, Ita, Febres Cordero, BP '18]

- Denominators are (empirically) products of symbol letters w_j.
- Determine q_{ij} from slice $\vec{x}(t)$.

$$r_i(\vec{x}) = \frac{n_i(\vec{x})}{\prod_{j\in A} w_j(\vec{x})^{q_{ij}}}\,.$$

- Expect cancellations between coefficients in singular regions.
- Search for linear combinations of r_i where N_k factorizes w_j.

$$\sum_{i\in B} a_{i,k} r_i(\vec{x}) = \frac{N_k(\vec{x}, a_{i,k})}{\prod_{j\in A} w_j(\vec{x})^{q'_{kj}}}\,,$$

helicity	$\tilde{c}_{k,i}(t)$	$r_i(t)$	$n'_i(t)$	w_j's in denominator
$+++++$	t^{34}/t^{28}	t^{10}/t^4	t^{10}	3
$-++++$	t^{50}/t^{42}	t^{35}/t^{28}	t^{35}	14
$--+++$	t^{70}/t^{65}	t^{50}/t^{45}	t^{40}	17
$-+-++$	t^{84}/t^{82}	t^{68}/t^{66}	t^{53}	20

Implementation and Results

[Abreu, Dormans, Ita, Febres Cordero, BP '18]

Calculation:

- Implemented in flexible C++ framework - "Caravel".
- For $-+-++$: $\sim$ 4mins/eval, $\sim$250k required points.
- Calculation over two finite fields of cardinality $O(2^{31})$.
- Rational reconstruction prefering composite denominators.

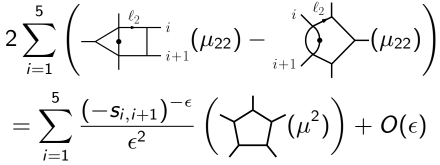
Analytic expressions:

- One-loop amplitudes as linear combination of master integrals.
- Two-loop remainders, in terms of pentagon functions.
- Valid in Euclidean region.
- 45MB total for all amplitudes, no simplification attempted.

Structure of Results

[Abreu, Dormans, Ita, Febres Cordero, BP '18]

- All-plus and single-minus remainders, no weight 3, 4.
- Revealed interesting cross-order identity.



Compact irreducible weight 4:

- Part of $--+++$ with $\varepsilon(p_1, p_2, p_3, p_4)$ letters.
- Only even functions.
- Also simple in $-+-++$.

$$\mathcal{R}_{\varepsilon,4}^{--+++} = -2x_0 x_1 \left(f_{4,11}^{1} + f_{4,11}^{5}\right) C,$$

$$C = \frac{6+x_1(6(2+x_1)+5x_0(3+(3+2x_0)x_1))}{3(1+x_1)^3}.$$

Conclusions

- Alternative method for analytic computations: reconstruction from numerical samples over finite fields.
- We have analytically computed the leading-colour 5-gluon two-loop amplitudes.
- Natural next step - all 5-parton amplitudes at leading colour.
- Interesting structures revealed by computation.

Finite Fields Crash Course (I)

- Take integers $\mathbb{F}_p = \{0, \ldots, p-1\}$, where p is prime.
- Perform multiplication/addition/subtraction modulo p,

$$5+7 \mod 11 = 1 \qquad 5 \times 7 \mod 11 = 2 \qquad 5-7 \mod 11 = 9.$$

- Every $a \in \mathbb{F}_p$ has a multiplicative inverse, a^{-1} so $\mathbb{F}_p$ is a field,

$$5^{-1} \mod 11 = 9.$$

- All rational operations possible, but (e.g.) no square roots.
- From $\mathbb{Q}$ to $\mathbb{F}_p$

$$a = \frac{r}{s} \in \mathbb{Q} \quad \rightarrow \quad a \bmod p \equiv r \cdot (s^{-1} \bmod p) \bmod p$$

Finite Fields Crash Course (II)

- Is there an inverse map?
- Consider $a = \frac{r}{s}$ where $r, s < \sqrt{n}$ and we know $a \mod n$.
- Find r and s, using a "Rational Reconstruction" algorithm. [Wang, '81]
- p not big enough? Use Chinese Remainder Theorem.

$$\{a \mod n_1, a \mod n_2, \ldots\} \xrightarrow{CRT} a \mod (n_1 \cdot n_2 \cdots)$$

Exact numerics with speed of integer operations.