Laporta algorithm for multi-loop vs multi-scale problems

(with Philipp Maierhöfer and Peter Uwer)

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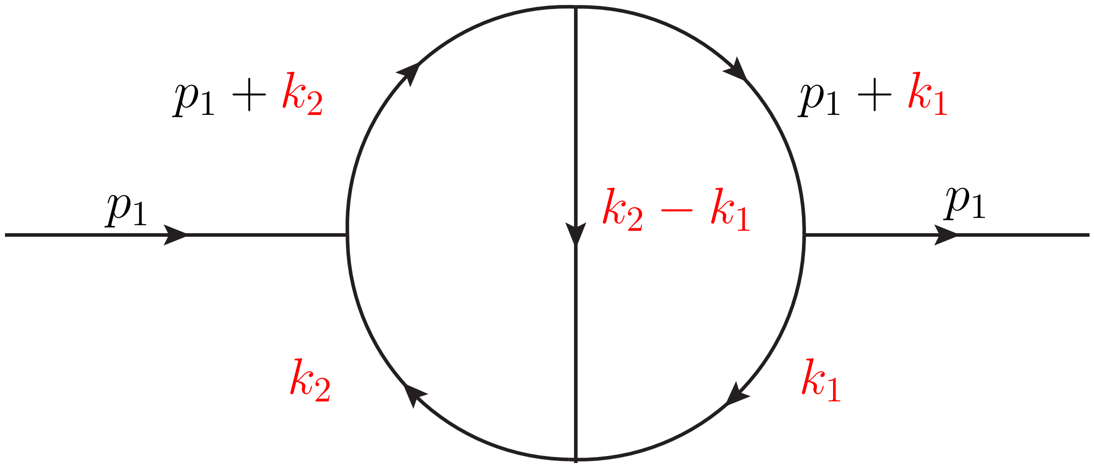
Outline

Integration-by-parts identities applications

- Integration-by-parts (IBP)[Chetyrkin,Tkachov,1981] and Lorentz invariance [Gehrmann,Remiddi,2000] identities for scalar Feynman integrals are very important in quantum field theoretical computations (multi-loop computations)
- Reduce the number of Feynman integrals to compute, which appear in scattering amplitude computations
- Compute the integrals analytically or numerically with the method of differential equations [Kotikov,1991;Remiddi,1997;Henn,2013;Argeri et al.,2013;Lee,2015;Meyer,2016] or difference equations[Laporta,2000;Lee,2010] (require basis change and IBP reductions)
- Use the method of sector decomposition [Heinrich,2008] (pySecDec [Borowka et al.,2018] and Fiesta4 [Smirnov,2016]) or use the linear reducibility of the integrals (HyperInt [Panzer,2014]) to compute the Feynman integrals analytically or numerically (require basis change and IBP reductions).

Scalar Integrals



$$I(a_1, \ldots, a_5) = \int \frac{d^d k_1 d^d k_2}{[k_1{}^2]^{a_1}[(p_1 + k_1)^2]^{a_2}[k_2{}^2]^{a_3}[(p_1 + k_2)^2]^{a_4}[(k_2 - k_1)^2]^{a_5}}$$

- Integral depends explicitly on the exponents a_f
- Loop momenta: $k_1,\ k_2$, $L = 2$
- Number of the propagators: $N = 5$

IBP Identities

$$I(a_1,\dots,a_5)=\int\frac{d^dk_1\,d^dk_2}{[k_1{}^2]^{a_1}[(p_1+k_1)^2]^{a_2}[k_2{}^2]^{a_3}[(p_1+k_2)^2]^{a_4}[(k_2-k_1)^2]^{a_5}}$$

Integration-by-parts (IBP) identities:

$$\begin{aligned}\int d^d\boldsymbol{k_1}\dots d^d\boldsymbol{k_L}\frac{\partial}{\partial(\boldsymbol{k_i})_\mu}\left((q_j)_\mu\frac{1}{[P_1]^{a_1}\dots[P_N]^{a_N}}\right)&=0\\ c_1(\{a_f\})I(a_1,\dots,a_N\mathbf{-1})+\dots+c_m(\{a_f\})I(a_1\mathbf{+1},\dots,a_N)&=0\end{aligned}$$

$q_j=p_1,\dots,p_E,k_1,\dots,k_L$

Express all integrals with the same set of propagators but with different exponents a_f as a linear combination of some basis integrals (master integrals).

- Gives relations between the scalar integrals with different exponents a_f
- Number of $L(E+L)$ IBP equations, $i=1,\dots,L$ and $j=1,\dots,E+L$
- a_f = symbols: Look for recursion relations, LiteRed [Lee,2012]
- a_f = integers: Sample a system of equations, Laporta algorithm [Laporta,2000]

Laporta Algorithm [Laporta,2000]

Scalar integrals $I(a_1,\ldots,a_5)$ with integer values a_f

Sample system of IBP equations, Reduze [Studerus,Manteuffel,2012] language

- $r=\sum_{f=1}^{N} a_f$ mit $a_f>0,\ f=1,\ldots,N$
- $s=-\sum_{f=1}^{N} a_f$ mit $a_f<0,\ f=1,\ldots,N$
- Seed integrals: $r\in[r_{\min},r_{\max}],\ \ s\in[s_{\min},s_{\max}]$
- $S=\sum_{i=1}^{N}\theta_j\times 2^{j-1}\theta_j=1$ for each $a_f>0$ else $\theta_j=0$
- T topology number

Fire [Smirnov,2008] language

- Avoid reductions of scalar integrals $\notin(r,s)$

- Different public implementations: Air [Lazopoulos,Anastasiou,2004], FIRE [Smirnov,2008] and Reduze [Studerus,Manteuffel,2012] and Kira [Maierhöfer, Usovitsch, Uwer,2017]
- Kira is more powerful the less LiteRed succeeds

Kira version 1.2

Kira is an implementation of the Laporta algorithm

Get Kira gitlab at: https://gitlab.com/kira-pyred/kira.git

- New equation generator which is $\sim 10^L$ faster than Kira 1.1 multi-loop
- Improved parallelization - no openMP
- Compiles on your Mac / New build system: Meson
- Get relations from higher sectors – minimize the number of master integrals
- Start a reduction with a preferred list of master integrals
- Focus the reduction only to a subset of master integrals — set all other coefficients to zero, since Kira 1.0 and 1.1
- New flexible seed notation is introduced, while the old is preserved
- Choose between 8 different integral Laporta orderings
- Coefficient simplifications are based on heuristics
- New feature: Algebraic reconstruction multi-scale
- New feature: User defined system of equations
- Release notes: arXiv:1812.01491

gg→H at 3-loops: integralfamilies.yaml


```
integralfamilies:
  - name: Xhiggs3l1_mmmmmmm00
    loop_momenta: [ l1, l2, l3 ]
    top_level_sectors: [511] # important option
    propagators:
      - [ "l1", "m^2" ]
      - [ "l2", "m^2" ]
      - [ "l3", "m^2" ]
      - [ "l1 - q1", "m^2" ]
      - [ "l2 - q1 - q2", "m^2" ]
      - [ "l1 - l2", 0 ]
      - [ "-l2 + l3 + q1 + q2", 0 ]
      - [ "l1 - l2 + l3", "m^2" ]
      - [ "l1 - l2 + l3 + q2", "m^2" ]
      - { bilinear: [ [ "l1", "l3" ], 0 ] }
      - { bilinear: [ [ "l2", "q1" ], 0 ] }
      - { bilinear: [ [ "l3", "q1" ], 0 ] }
```

gg→H at 3-loops: Old v.s. new jobs.yaml interface

```
jobs:
 - reduce_sectors:
    sector_selection: # Old
     select_recursively: # Old
      - [Xhiggs3l1_mmmmmmm00,511] # Old
    identities: # Old
     ibp: # Old
      - { r: [t, 10], s: [0, 4] } # Old
    reduce: # New
     - {r: 10, s: 4} # New
    select_integrals: # important option
     select_mandatory_recursively: # important option
      - {r: 10, s: 4, d: 1} # important option
```

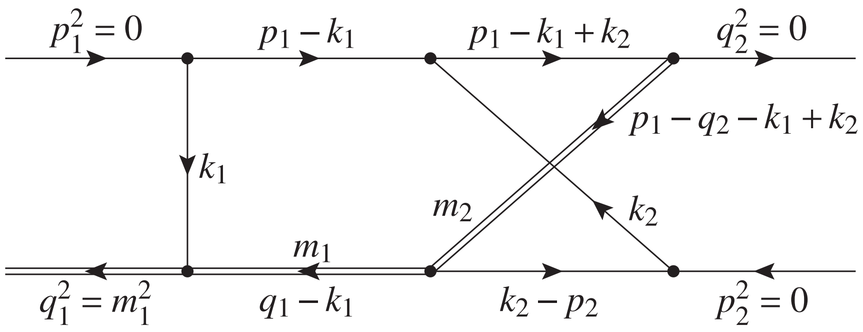
- Kira implicitly knows from integralfamilies.yaml that the user wants to reduce the topology named: Xhiggs3l1_mmmmmmm00
- From `top_level_sectors: [511]` Kira assumes that the user wants to reduce the sector: 511

Reduction of a gg→H at 3-loops non-planar topology

Algorithm	**Kira 1.1** (32 cores)	**Kira 1.2** (16 cores)
Generate system of equations	7.9 h	-
Reduce numerically	3.6 h	-
Generate and reduce numerically	-	3.4 h
Build triangular form (thread pools)	26 h	4.8 h
Backward substitution (heuristics)	18.8 d	4.1 d

- Seed specification: {r: 10, s: 4, d: 1}
- Speedup comes from less calls to Fermat: 382.502.520 x 5 (Kira 1.1) v.s. 981 (Kira 1.2)
- After the numerical reduction over the finite field (integers modulo 64 Bit prime number) is finished, you know the master integrals

Algebraic coefficient simplification



Type	$T_{\text{Kira 1.1}}^{m_2^2=\frac{3}{14}m_1^2}$	$T_{\text{Kira 1.2}}^{m_2^2=\frac{3}{14}m_1^2}$	$T_{\text{Kira 1.1}}$	$T_{\text{Kira 1.2}}$	$T_{\text{Reduze 2}}^{m_2^2=\frac{3}{14}m_1^2}$	$T_{\text{FIRE 5}}^{m_2^2=\frac{3}{14}m_1^2}$
default	2.4 h	1 h	-	11.5 h	2.7 d	23.5 h
A	35.3 min	28.4 min	10 h	5.8 h	-	22.4 h

- default: `select_mandatory_recursively: [{r: 7, s: 4}]`
- A: `select_mandatory_recursively: [{r: 7,s: 4,d: 0}]`
- Reduze 2 A. von Manteuffel and C. Studerus (2012), FIRE 5 A. V. Smirnov (2014) in C++ and using the same Fermat executable.

Algebraic reconstruction

Backward substitution gives: $I(\{a_i\}) = \sum_j^M C_j M_j$, M_j master integral

- $C_j = \sum_{i=1}^N c_i$,
- $N \approx \mathcal{O}(10^2) - (10^5)$
- Naiv sum gives a snow ball effect: Intermediate sum grows to more complicated terms then the final result.
- One solution since Kira 1.0 is to constantly sort the terms c_i and the intermediate sums in their string length.

Second solution since Kira 1.2 is the algebraic reconstruction

- Sample $\sum_{i=1}^N c_i$ by setting at least one parameter $\{\frac{s}{m_1^2}, \frac{t}{m_1^2}, \frac{m_{i\neq 1}^2}{m_1^2}, \ldots\}$ to integer numbers
- Interpolate the final result from these samples

Implementation part 1

Dependence on at least 2 parameters, e.g.: $\{D, x\}$, $x = \frac{s}{m_1^2}$

- Sample once $C(D, x)$ for numeric value in D
- Get $C(x)$ rational function
- Get the degree of the polynomials (numerator and denominator) of $C(x)$ in x: d_N and d_D
- Interpolate the numerator and denominator in x individually with Newtonian approach
- Use $C(x)$ later as a reference point to eliminate sign and numeric prefactor ambiguities
- Original work in this field is based on, see arXiv: 1805.01873 1712.09737 1511.01071 by Yang Zhang and his collaborators

Implementation part 2

Sample $C(D,x)$ $\max(d_N+2, d_D+2)$ for numeric values x_j in x

- Get multiple functions $C(D,x) \to \{C(D,x_j)\}$
- Test that all numerators and denominators have the same number of terms, if not, resample

Interpolate the numerator and the denominator of $C(D,x)$ individually, by using the Newtonian interpolation formula

- $C(D,x) = \sum_{i=1}^{d_N+1, d_D+1} a_i \prod_{j=1}^{i-1} (x - x_j)$
- $a_1 = C(D, x_1)$
- $a_2 = \frac{C(D,x_2)-a_1}{x_2-x_1}$
- $a_3 = (\frac{C(D,x_3)-a_1}{x_3-x_1} - a_2)\frac{1}{x_3-x_2}$
- ...
- $a_{d_N+1} = ((\frac{C(D,x_{d_N+1})-a_1}{x_{d_N+1}-x_1} - a_2)\frac{1}{x_{d_N+1}-x_2} - \cdots - a_{d_N})\frac{1}{x_{d_N+1}-x_{d_N}}$

Implementation part 3

- To activate the algebraic reconstruction use:
  `algebraic_reconstruct: true`
- Kira decides based on heuristics to use the algebraic reconstruction algorithm or not
- Heuristics are: Number of terms in a sum, length of the biggest coefficients
- All implementation details are "hidden under the hood" — await improvements and more benchmarks (code is public)
- at present algebraic reconstruction kicks in only for the coefficients during the backward substitution
- Next Kira version will include the algebraic reconstruction of the whole reduction
- Possible usage: Treat coefficients of the master integrals individually

Summary and Outlook

- Kira version 1.2 is available: https://gitlab.com/kira-pyred/kira.git and includes:
- Fast equation generator
- Improved parallelization
- New flexible seed notation, while the old is preserved
- New feature: Algebraic reconstruction
- Todo list:
- Algebraic reconstruction for the whole system, parallelization across different machines.
- Kira is an all-rounder best in all disciplines: multi-loop, multi-scale and user defined system of equations reductions