Karlsruhe Institute of Technology

Charm, bottom and top masses

Matthias Steinhauser | 11th FCC-ee workshop, CERN

January 8-11, 2019

TTP KARLSRUHE

Outline

I. m_c and m_b from low-moment sum rules

II. $m_t^{\overline{\mathrm{MS}}}$ — M_t^{OS} to 4 loops and beyond

Precise m_c and m_b

- m_c, m_b: fundamental parameters of the Standard Model
- B decays: $\Gamma \sim m_b^5$
- Spectroscopy $M(\Upsilon(1S)) = 2m_b - \frac{4\alpha_s^2}{9} m_b + \dots$
- Higgs decay ➪ FCC-ee [talk by M. Spira]

$$\Gamma(H \to b\bar{b}) = \frac{G_F m_H}{4\sqrt{2}\pi} m_b^2(m_H) \left(1 + \mathcal{O}(\alpha_s) + \dots\right)$$
$$\Gamma(H \to c\bar{c}) = \frac{G_F m_H}{4\sqrt{2}\pi} m_c^2(m_H) \left(1 + \mathcal{O}(\alpha_s) + \dots\right)$$

- $\Delta\alpha_{\rm had}$ via Adler function [talk by F. Jegerlehner]
- Yukawa unification: at the GUT scale

$\frac{\delta m_t}{m_t} \approx \frac{\delta m_b}{m_b}$ ➪ $\delta m_t = 1$ GeV ➪ $\delta m_b \lesssim 25$ MeV

Methods to determine m_c and m_b

- non-relativistic sum rules

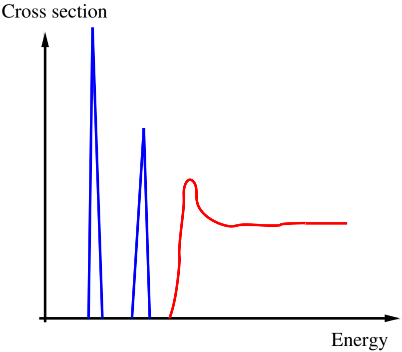
  [Beneke,Maier,Piclum,Rauh'16; Penin,Zerf'14; . . .]
- $\Upsilon(1S)$ bounds state energy

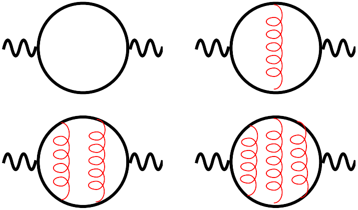
  [Peset,Pineda,Segovia'18; Kiyo,Mishima,Sumino'16; . . .]
- lattice

  [Fermilab Lattice+MILC+TUMQCD'18; HPQCD'18; . . .]
- . . .
- relativistic sum rules ("low-n SRs", "SVZ SRs")

  [Chetyrkin et al.; Dehnadi et al.'15; . . .]

Relativistic sum rules





[Chetyrkin,Kühn,Sturm; Boughezal, Czakon, Schutzmeier; Marquard; Schröder; Lee; . . .]

$$\Pi_Q(q^2) \sim \sum \bar{C}_n \left(\frac{q^2}{4m_Q^2}\right)^n$$

$$R_Q = \frac{\sigma(e^+e^- \to Q\bar{Q} + \ldots)}{\sigma(e^+e^- \to \mu^+\mu^-)} = 12\pi \mathrm{Im}\left[\Pi_Q(q^2 = s + i\varepsilon)\right]$$

$$\mathcal{M}_n^{\mathrm{exp}} \equiv \int \frac{\mathrm{d}s}{s^{n+1}} R_Q(s) \qquad \mathcal{M}_n^{\mathrm{th}} = \frac{12\pi^2}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}q^2}\right)^n \Pi_Q(q^2)\bigg|_{q^2=0}$$

R^{exp}

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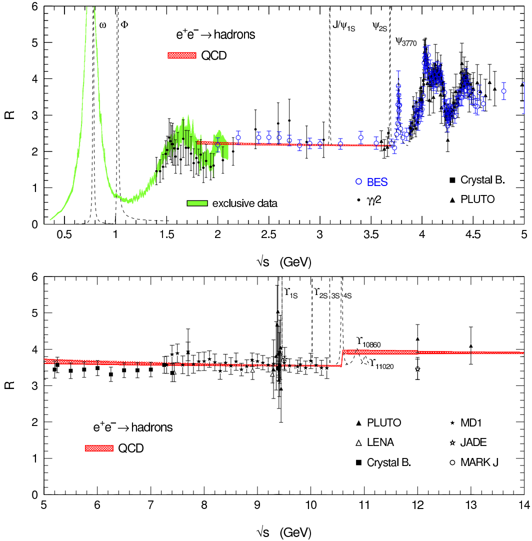


$$\mathcal{M}_n^{\text{exp}} = \mathcal{M}_n^{\text{res}} + \mathcal{M}_n^{\text{thresh}} + \mathcal{M}_n^{\text{cont}}$$

[Davier,Eidelman,Höcker,Zhang'02]

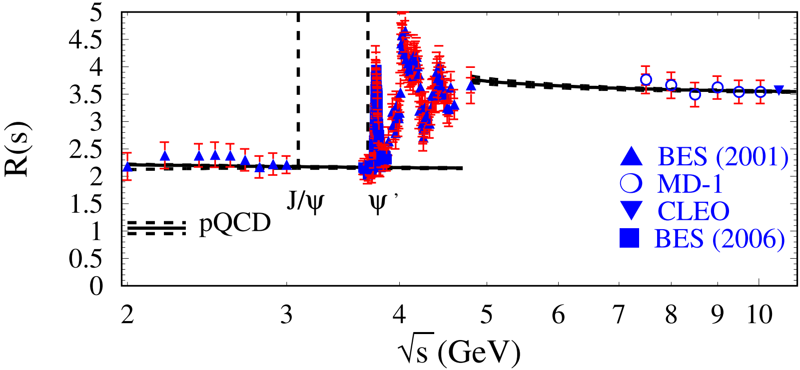
$$\mathcal{M}^{\rm exp} = \mathcal{M}^{\rm res} + \mathcal{M}^{\rm thresh} + \mathcal{M}^{\rm cont}$$

- $\mathcal{M}^{\rm res}$: $R^{\rm res}(s) = \frac{9\pi M_R \Gamma_{ee}}{\alpha^2}\left(\frac{\alpha}{\alpha(s)}\right)^2 \delta(s - M_R^2)$

	J/Ψ	$\Psi(2S)$	
M_Ψ(GeV)	3.096900(6)	3.686097(25)	
Γ_{ee}(keV)	5.57(8)	2.34(4)	[BES III'16]
$(\alpha/\alpha(M_\Psi))^2$	0.957785	0.95554	

$$\mathcal{M}^{\rm exp} = \mathcal{M}^{\rm res} + \mathcal{M}^{\rm thresh} + \mathcal{M}^{\rm cont}$$

- $\mathcal{M}^{\rm res}$: $\quad R^{\rm res}(s) = \frac{9\pi M_R \Gamma_{ee}}{\alpha^2} \left(\frac{\alpha}{\alpha(s)}\right)^2 \delta(s - M_R^2)$
- $\mathcal{M}^{\rm thresh}$: $\quad 3.73 \text{ GeV} \leq \sqrt{s} \leq 4.8 \text{ GeV}$, BES01,06



$$\mathcal{M}^{\rm exp} = \mathcal{M}^{\rm res} + \mathcal{M}^{\rm thresh} + \mathcal{M}^{\rm cont}$$

- $\mathcal{M}^{\rm res}$: $R^{\rm res}(s) = \frac{9\pi M_R \Gamma_{ee}}{\alpha^2} \left(\frac{\alpha}{\alpha(s)}\right)^2 \delta(s - M_R^2)$
- $\mathcal{M}^{\rm thresh}$: $3.73\ \text{GeV} \leq \sqrt{s} \leq 4.8\ \text{GeV}$, BES01,06
- $\mathcal{M}^{\rm cont}$: $\sqrt{s} \geq 4.8\ \text{GeV}$ ➪ pQCD

rhad: [Harlander,Steinhauser'02]

m_c

$$\mathcal{M}_n^{\rm th} \stackrel{!}{=} \mathcal{M}_n^{\rm exp}$$

$$m_c(\mu) \quad = \quad \frac{1}{2}\left(\frac{\bar{C}_n}{\mathcal{M}_n^{\rm exp}}\right)^{1/(2n)}$$

latest development: $\bar{C}_4$ analytically to 4 loops [Marquard,Maier'17]

n	$m_c(3\text{ GeV})$	exp	α_s	μ	np	total
1	993	7	4	2	1	8
2	982	4	7	5	1	10
3	982	3	8	6	1	10
4	1003	2	5	28	1	29

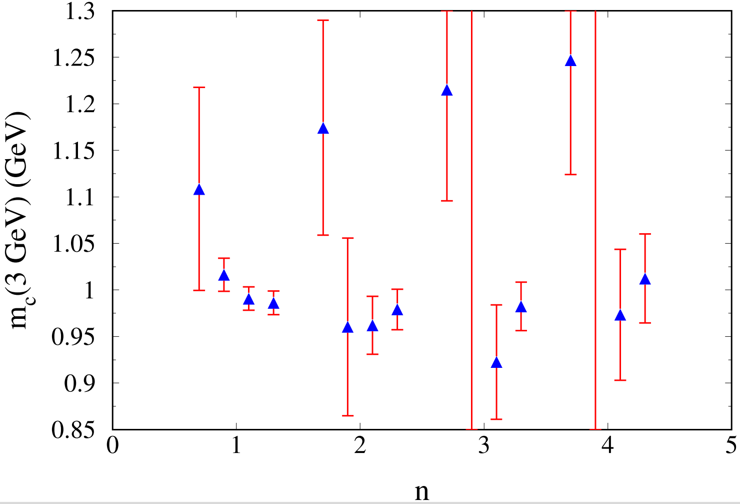
$m_c(3\text{ GeV}) = 0.993(8)\text{ GeV}$ $\quad$ $m_c(m_c) = 1.279(8)\text{ GeV}$

[Kühn,Steinhauser,Sturm'07; Chetyrkin,Kühn,Maier,Maierhöfer,Marquard,Steinhauser,Sturm'09'17]

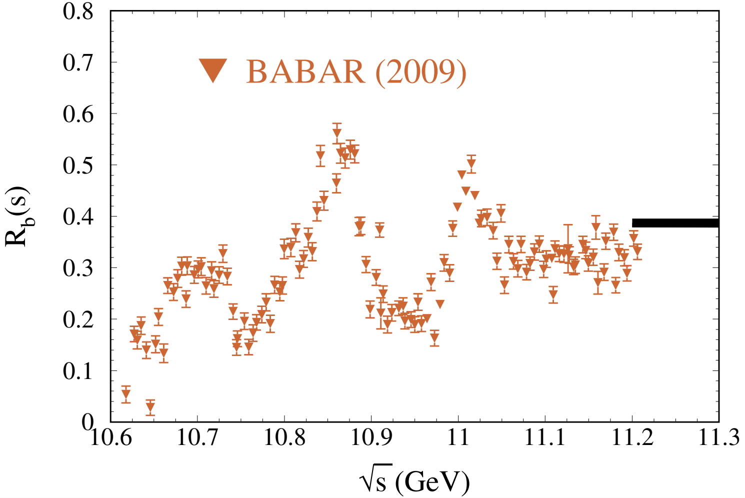
[Uncertainties: $\delta\mathcal{M}_n^{\rm exp}$ | $\alpha_s(M_Z) = 0.1181 \pm 0.0011$ | $\mu = (3 \pm 1)$ GeV]

[np: gluon condensate $\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle$, NLO [Broadhurst et al.'94]]

m_c



b quark



BABAR (2009)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$R_b(s)$
10.6
10.7
10.8
10.9
11
11.1
11.2
11.3
$\sqrt{s}$ (GeV)

m_b

$$\mathcal{M}_n^{\mathrm{th}} \stackrel{!}{=} \mathcal{M}_n^{\mathrm{exp}}$$

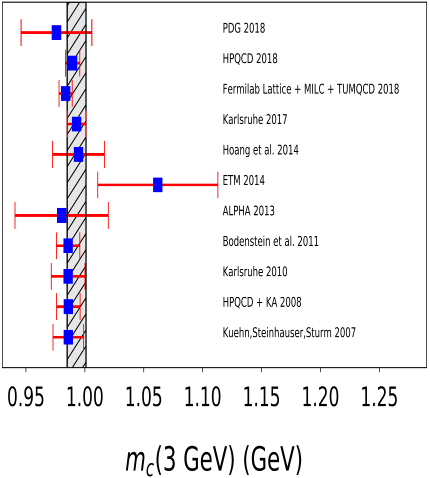
n	$m_b(10\text{ GeV})$	exp	α_s	μ	total	$m_b(m_b)$
1	3597	14	7	2	16	4151
2	3610	10	12	3	16	4163
3	3619	8	14	6	18	4172
4	3631	6	15	20	26	4183

$m_b(10\text{ GeV}) = 3.610(16)\text{ GeV}$ $m_b(m_b) = 4.163(16)\text{ GeV}$

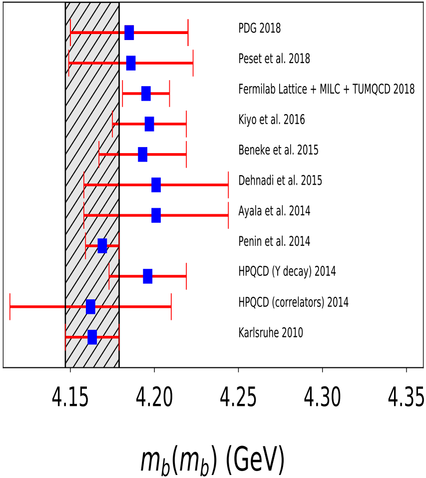
[Kühn,Steinhauser,Sturm'07; Chetyrkin,Kühn,Maier,Maierhöfer,Marquard,Steinhauser,Sturm'10]

[Uncertainties: $\delta\mathcal{M}_n^{\mathrm{exp}}$ | $\alpha_s(M_Z) = 0.1189 \pm 0.0020$ ([Bethke'06]; $\delta\alpha_s \times 2$) | $\mu = (10 \pm 5)$ GeV]

Comparison

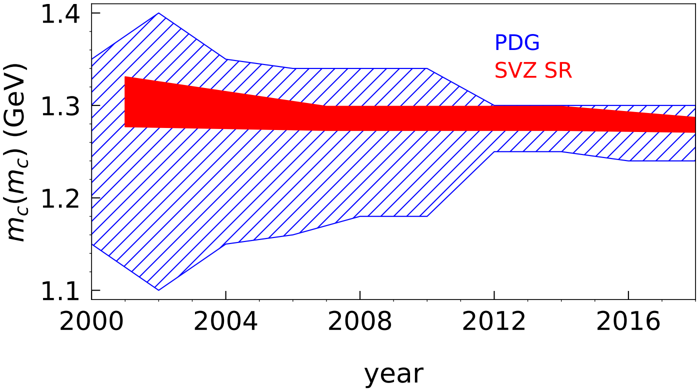


PDG 2018
HPQCD 2018
Fermilab Lattice + MILC + TUMQCD 2018
Karlsruhe 2017
Hoang et al. 2014
ETM 2014
ALPHA 2013
Bodenstein et al. 2011
Karlsruhe 2010
HPQCD + KA 2008
Kuehn,Steinhauser,Sturm 2007
0.95
1.00
1.05
1.10
1.15
1.20
1.25
m_c(3 GeV) (GeV)

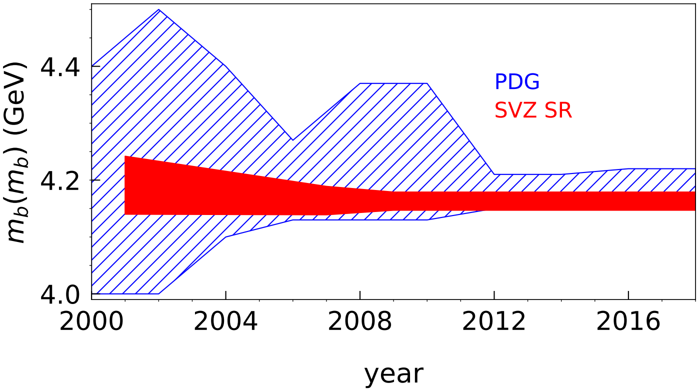


PDG 2018
Peset et al. 2018
Fermilab Lattice + MILC + TUMQCD 2018
Kiyo et al. 2016
Beneke et al. 2015
Dehnadi et al. 2015
Ayala et al. 2014
Penin et al. 2014
HPQCD (Υ decay) 2014
HPQCD (correlators) 2014
Karlsruhe 2010
4.15
4.20
4.25
4.30
4.35
$m_b(m_b)$ (GeV)

m_c



m_b



Top quark mass

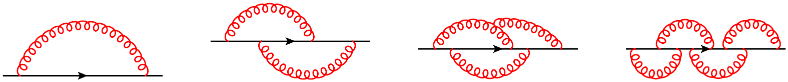
often one reads:
“The accuracy of the top quark pole mass is of $\mathcal{O}$(few 100 MeV).”

Goal: make this estimate more precise

Not in this talk:

- no beyond-QCD corrections
- “Which mass is extracted at LHC?”

$$m^{\overline{\rm MS}} = m^{\rm OS}\frac{Z_m^{\rm OS}}{Z_m^{\overline{\rm MS}}}$$



- $Z_m^{\overline{\rm MS}}$ known to 4 loops [Chetyrkin'97; Larin,van Ritbergen,Vermaseren'97] (5 loops: [Baikov,Chetyrkin,Kühn'14; Luthe,Maier,Marquard,Schröder'17])
- QCD up to 3 loops: [Tarrach'81] [Gray,Broadhurst,Grafe,Schilcher'90] [Chetyrkin,Steinhauser'99; Melnikov, v. Ritbergen'00; Marquard,Mihaila,Piclum,Steinhauser'07]
- QCD 4 loops: [Lee,Marquard,Smirnov,Smirnov,Steinhauser'13] [Marquard,Smirnov,Smirnov,Steinhauser'15; Marquard,Smirnov,Smirnov,Steinhauser,Wellmann'16]
- electroweak corrections [Hempfling,Kniehl'94; Jegerlehner,Kalmykov'03; Faisst,Kühn,Veretin'04; Martin'05; Eiras,Steinhauser'05; Kniehl,Pikelner,Veretin'15; Martin'16]

Calculation

1. reduction to MIs `FIRE5` [Smirnov'14] and `crusher` [Marquard,Seidel], analytic
2. compute 386 MIs (analytic and numerical methods) `FIESTA` [Smirnov'15]

$\overline{\rm MS}$−OS relation up to 4 loops

$$
\begin{aligned}
m_t^{\rm OS} &= m_t^{\overline{\rm MS}} \left[1 + 0.4244\,\alpha_s + 0.8345\,\alpha_s^2 + 2.375\,\alpha_s^3 + (8.615 \pm 0.017)\,\alpha_s^4\right] \\
&= 163.508 + 7.529 + 1.606 + 0.496 + (0.195 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

$$
\begin{aligned}
m_b^{\rm OS} &= m_b^{\overline{\rm MS}} \left[1 + 0.4244\,\alpha_s + 0.9401\,\alpha_s^2 + 3.045\,\alpha_s^3 + (12.685 \pm 0.025)\,\alpha_s^4\right] \\
&= 4.163 + 0.398 + 0.199 + 0.145 + (0.136 \pm 0.0003)\ \text{GeV}
\end{aligned}
$$

$$
\begin{aligned}
m_c^{\rm OS} &= m_c^{\overline{\rm MS}}(3\ \text{GeV}) \\
&\quad \times \left(1 + 1.133\,\alpha_s + 3.119\,\alpha_s^2 + 10.981\,\alpha_s^3 + (51.419 \pm 0.102)\,\alpha_s^4\right) \\
&= 0.986 + 0.284 + 0.198 + 0.177 + (0.211 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

Beyond 4 loops?

Asymptotic behaviour

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$$m_t^{\mathrm{OS}} = m_t^{\overline{\mathrm{MS}}}(\mu_m) \left(1 + \sum_{n\geq 1} c_n \, \alpha_s^n(\mu)\right)$$

IR renormalon predicts large-n behaviour:

$$c_n \underset{n\to\infty}{\longrightarrow} N \frac{\mu}{m(\mu_m)} \tilde{c}_n^{(\mathrm{as})}$$

$b = \beta_1/(2\beta_0)$

$$\tilde{c}_n^{(\mathrm{as})} = (2\beta_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{s_1}{n+b} + \frac{s_2}{(n+b)(n+b-1)} + \cdots\right).$$

- $s_1, s_2, b, \beta_0, \ldots$ known
- N unknown

[Beneke,Braun'94; Beneke'94; Beneke'95]

Determine N

$$m^{\rm OS} = m^{\overline{\rm MS}}(\mu_m)\left(1 + \sum_{n\geq 1} c_n\,\alpha_s^n(\mu)\right) \qquad c_n \underset{n\to\infty}{\longrightarrow} N c_n^{\rm (as)} = N \frac{\mu}{m(\mu_m)} \tilde{c}_n^{\rm (as)}$$

$$N = \lim_{n\to\infty} \frac{c_n(\mu, \mu_m, m^{\overline{\rm MS}}(\mu_m))}{c_n^{\rm (as)}(\mu, m^{\overline{\rm MS}}(\mu_m))}$$

- determine N from exact (3- and) 4-loop c_n
- how close is the 3rd order coefficient to asymptotic value:
  $\Delta_{34} = 2\,\frac{|c_3/c_3^{\rm (as)} - c_4/c_4^{\rm (as)}|}{|c_3/c_3^{\rm (as)} + c_4/c_4^{\rm (as)}|}$
  vary n_c and n_ℓ
  $\lim_{|n_\ell|\to\infty} N = \frac{C_F}{\pi} e^{\frac{5}{6}}$
- determine N for $n_c = 3, n_\ell = 5$ (top) from $c_4/c_4^{\rm (as)}$
  uncertainty from: $0.5 \leq \mu, \mu_m \leq 2$

N

$$\boxed{N = 0.4616^{+0.027}_{-0.070}\,(\mu \text{ and } \mu_m) \pm 0.002\,(c_4)}$$

[Beneke,Marquard,Nason,Steinhauser'16]

see also [Ayala,Cvetic,Pineda'16; Hoang et al.'18]

$$m^{\rm OS} = m^{\overline{\rm MS}}(\mu_m)\left(1 + \sum_{n\geq 1} c_n\,\alpha_s^n(\mu)\right) \qquad c_n \underset{n\to\infty}{\longrightarrow} N\frac{\mu}{m^{\overline{\rm MS}}(\mu_m)}\tilde{c}_n^{\rm (as)}$$

$$\tilde{c}_n^{\rm (as)} = (2\beta_0)^n\,\frac{\Gamma(n+1+b)}{\Gamma(1+b)}\left(1+\frac{s_1}{n+b}+\frac{s_2}{(n+b)(n+b-1)}+\cdots\right).$$

estimate $\geq$ 5 loop contribution: Borel summation

Borel transformation/Borel summation

- $f(\alpha_s) = \sum_{n=1}^{\infty} c_n \alpha_s^n$ ➪ $B[f](t) = \sum_{n=0}^{\infty} c_{n+1} \frac{t^n}{n!}$

  ➪ Borel integral $\int_0^\infty \mathrm{d}t e^{-t/\alpha_s} B[f](t)$

  has same series expansion as $f(\alpha_s)$

- Here: $c_{n+1} \sim (2\beta_0)^n n!$ ➪ $\int_0^\infty \mathrm{d}t e^{-t/\alpha_s}/(1 - 2\beta_0 t)$

- Procedure:
  define integral with principal value prescription
  ambiguity: imag. part of integral / π [Beneke'99]

$$\delta^{(5+)} m^{\mathrm{OS}} = 0.250^{+0.015}_{-0.038}\,(N) \pm 0.001\,(c_4) \pm 0.010\,(\alpha_s) \pm 0.071\text{ (ambiguity) GeV}$$

$m^{\mathrm{OS}} - m^{\overline{\mathrm{MS}}}$

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$$\delta^{(5+)} m^{\mathrm{OS}} = 0.250^{+0.015}_{-0.038}\,(N) \pm 0.001\,(c_4) \pm 0.010\,(\alpha_s) \pm 0.071 \text{ (ambiguity) GeV}$$

Check: Truncate series at minimal term:

$$\delta^{(5+)} m^{\mathrm{OS}} = 0.272^{+0.016}_{-0.041}\,(N) \pm 0.001\,(c_4) \pm 0.011\,(\alpha_s) \pm 0.066 \text{ (ambiguity) GeV}$$

m_c and m_b effects:

- $m_{u,d,s} \ll \Lambda_{\mathrm{QCD}} \ll m_{c,b}$
- typical loop momentum at $\mathcal{O}(\alpha_s^n)$: $m_t e^{-(n-1)}$ [Ball,Beneke,Braun'95]

$$\delta^{(5+)} m^{\mathrm{OS}} = 0.304^{+0.012}_{-0.063}\,(N) \pm 0.030\,(m_{b,c}) \pm 0.009\,(\alpha_s) \pm 0.108 \text{ (ambiguity) GeV}$$

$$\frac{m^{\mathrm{OS}}}{m^{\overline{\mathrm{MS}}}(m)} = 1.06213^{+0.00007}_{-0.00038}\,(N) \pm 0.00018\,(m_{b,c}) \pm 0.00086\,(\alpha_s) \pm 0.00066 \text{(amb.)}$$

[Beneke,Marquard,Nason,Steinhauser'16]

m^{OS} — $m^{\overline{\mathrm{MS}}}$ — threshold masses

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- compare to [Hoang,Lepenik,Preisser'17]: ± 250 MeV
- ambiguity(m^{OS}) < 1 GeV ➪ save for hadron collider
- FCC-ee ➪ threshold masses

$m^{\rm OS}$ — $m^{\overline{\rm MS}}$ — threshold masses

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$m_t^{\overline{\rm MS}}(m_t^{\overline{\rm MS}})$

input #loops	$m^{\rm PS} =$ 168.049	$m^{\rm 1S} =$ 172.060	$m^{\rm RS} =$ 166.290	$m^{\rm RS'} =$ 171.785	
1	164.174	164.904	163.702	164.226	
2	163.580	163.727	163.520	163.591	1-2 GeV
3	163.492	163.519	163.490	163.500	$\lesssim$ 200 MeV
4	163.508	163.508	163.508	163.508	$\lesssim$ 20 MeV

Conclusions

- m_c and m_b from low-moment sum rules
- direct determination of $\overline{\mathrm{MS}}$ quark mass
- $m_c(3\ \mathrm{GeV}) = 993 \pm 8$ MeV
  $m_b(m_b) = 4163 \pm 16$ MeV
- 4-loop $\overline{\mathrm{MS}}$–OS relation
- m_t:
  $\geq$ 5 loops: $+304$ MeV
  irreducible uncertainty: ± 108 MeV