Complexity, Holography & QFT: A Progress Report

with Alice Bernamonti, Dean Carmi, Horacio Casini, Shira Chapman, Jason Cotler, Federico Galli, Minyong Guo, Juan Hernandez, Ro Jefferson, Luis Lehner, Hugo Marrochio, Eric Poisson, Pratik Rath, Shan-Ming Ruan, Joan Simon, Rafael Sorkin & Sotaro Sugishita
Holographic Entanglement Entropy:

\[ S(A) = \min_{\partial V \sim \Sigma} \frac{A_V}{4G_N} \]

(Ryu & Takayanagi ‘06)

• holographic EE is a fruitful forum for bulk-boundary dialogue:

Susskind: Entanglement is not enough!
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- “to understand the rich geometric structures that exist behind the horizon and which are predicted by general relativity.”

- recall $S_{EE}$ only probes the eigenvalues of the density matrix

\[ S_{EE} = -Tr [\rho_A \log \rho_A] = - \sum \lambda_i \log \lambda_i \]

- would like a new probe which is “sensitive to phases”

\[ |TFD\rangle \sim \sum_{\alpha} e^{-E_{\alpha}/(2T)-iE_{\alpha}(t_L+t_R)} \times |E_{\alpha}\rangle_L |E_{\alpha}\rangle_R \]
Complexity:

• computational complexity: how difficult is it to implement a task? eg, how difficult is it to prepare a particular state?

• quantum circuit model:

\[ j\tilde{A}_T i = U j\tilde{A}_R i \]

unitary operator built from set of simple gates

simple reference state eg, \[ |00000 \cdots 0\rangle \]
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  - simple reference state eg, \( |00000 \cdots 0\rangle \)

  tolerance: \( j j \tilde{A}_T i i U j \tilde{A}_R i j^2 \cdot " \)

- \textbf{complexity} = minimum number of gates required to prepare the desired target state (ie, need to find optimal circuit)

- does the answer depend on the choices??
**Holographic Complexity: A Tale of Two Dualities**

- **complexity=volume**: evaluate proper volume of extremal codim-one surface connecting Cauchy surfaces in boundary theory (cf holo EE) 
  (Stanford & Susskind)

  $$C_V(\Sigma) = \max_{\Sigma = \partial B} \left[ \frac{\mathcal{V}(B)}{G_N \ell} \right]$$

- **complexity=action**: evaluate gravitational action for Wheeler-DeWitt patch = domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT 
  (Brown, Roberts, Swingle, Susskind & Zhao)

  $$C_A(\Sigma) = \frac{I_{WDW}}{\pi \hbar}$$

- **complexity=volume**: evaluate spacetime volume of WDW patch (Couch, Fischler & Nguyen)

  $$\mathcal{V}(\Sigma) = \frac{V_{WDW}}{G_N}$$
Holographic Complexity:

\[ \text{Complexity} = \text{Volume} \]

\[ \text{Complexity} = \text{Action} \]

**WHY COMPLEXITY??**

- connection of complexity=volume to AdS/MERA
- linear growth (at late times)
  \[ \left. \frac{dC_V}{dt} \right|_{t \to \infty} = \frac{8\pi}{d - 1} M \text{ (planar)} \]
  \[ \left. \frac{dC_A}{dt} \right|_{t \to \infty} = \frac{2M}{\pi} \text{ (d = boundary dimension)} \]
- “switchback” effect (out of time-order correlators)
Complexity of Formation:

\[ | \text{TFD} \rangle = Z^{-1/2} \sum_{\alpha} e^{-E_{\alpha}/(2T)} |E_{\alpha}\rangle_L |E_{\alpha}\rangle_R \]

• additional complexity involved in forming thermofield double state compared to preparing two copies of vacuum state? (Brown et al)

\[ \Delta C = C(|\text{TFD}\rangle) - C(|0\rangle \otimes |0\rangle) \]

\[ \Delta C_A = -2 \times r_{\text{max}} \]
Complexity of Formation:

- additional complexity involved in forming thermofield double state compared to preparing two copies of vacuum state?

\[
\Delta C = C(\ket{\text{TFD}}) - C(\ket{0} \otimes \ket{0})
\]

\[
\Delta C_A = \frac{d - 2}{d \pi} \cot \left( \frac{\pi}{d} \right) S + \cdots
\]

both complexity, but different microscopic rules?

\[
\Delta C_V = 4\sqrt{\pi} \frac{(d - 2) \Gamma(1 + \frac{1}{d})}{(d - 1) \Gamma(\frac{1}{2} + \frac{1}{d})} S + \cdots
\]

\(d = 2\)

\((d = \text{boundary dimension})\)
Questions?

• What is “holographic complexity”?
  ➢ what is boundary dual of these gravitational observables?
  ➢ QFT/path integral description of “complexity” in boundary CFT?

• is there a privileged role for (states on) null Cauchy surfaces?
  ➢ provide distinguished reference states?

• is there a “renormalized holographic complexity”?
  ➢ what is boundary dual of these gravitational observables?

• ambiguities?
  ➢ provide distinguished reference states?

• more boundary terms: higher codim. intersections; “complex” joint contributions; boundary “counterterms”

• why is complexity of formation positive?

• $C_A$ contribution of spacetime singularity?

What does “complexity” mean in a quantum field theory?
Complexity in Quantum Field Theory??

• computational complexity: how difficult is it to implement a task? eg, how difficult is it to prepare a particular state?

• quantum circuit model:

\[ |\psi\rangle = U |\psi_0\rangle \]

unitary operator built from set of simple gates

simple reference state eg, \(|00000\cdots0\rangle\)

tolerance:

\[ \left| |\psi\rangle - |\psi\rangle_{\text{Target}} \right|^2 \leq \varepsilon \]

• **complexity** = minimum number of gates required to prepare the desired target state (ie, need to find optimal circuit)
Quantum Field Theory:

- free scalar field theory (in $d$ spacetime dimensions)

\[ H = \frac{1}{2} \int d^{d-1}x \left[ \pi(x)^2 + \nabla \phi(x)^2 + m^2 \phi(x)^2 \right] \]

\[ = \frac{1}{2} \sum_n \left[ \frac{p(n)^2}{\delta^2} + m^2 \right] \left[ x(n)^2 + \sum_i \left( x(n+i) - x(n) \right)^2 \right] \]

- $p(n) = \pm^{d=2}/4(n)$
- $x(n) = \pm^{d=2} \dot{A}(n)$
- $M = 1 = \pm$

\[ \Omega^2 = 1/\delta^2 \]

\[ \omega^2 = m^2 \]
Reference state: \[ \tilde{\mathcal{R}}(x_i) = \exp \left( \sum_i \frac{1}{2} \left( x_i^2 \right) \right) \]

Gates/Unitaries:
- factorized Gaussian: all lattice sites disentangled
  \[ x_i; \tilde{\mathcal{P}}_j(x_j) = 0 \quad [x_i; p_j] = i \pm j \]

- natural operators:
  \[ Q_{ij} = \exp[i^2 x_i p_j] \quad \text{“shift } x_j \text{ by } \epsilon x_i” \quad \text{ (entangling)} \]
  \[ Q_{ii} = \exp \left( \frac{i}{2} (x_i p_i + p_i x_i) \right) = \exp[i^2 x_i p_i] \quad \text{“rescale } x_i \text{ to } e^{\epsilon x_i}” \quad \text{ (scaling)} \]

Target state: \[ \text{ground state; thermofield double state} \]

How do we find optimal circuit??

- follow approach of Mike Nielsen (e.g., Hamiltonian control theory)

- circuit depth: \[ D_1 = \sum |\alpha_{i,j,n}| \]

(see also: Chapman, Eisert, Heller, Jefferson, Marrochio & Pastawski)
(see also: Yang, Kim, Niu, Yang & Zhang)
“What is the minimal size quantum circuit required to exactly implement a specified n-qubit unitary operation, $U$, without the use of ancilla qubits?”

**Nielsen approach:**

- work with smooth functions on a smooth space (rather than discrete)

$$\tilde{\mathcal{A}}(x_1; x_2) = U \tilde{\mathcal{A}}_{\text{lin}}(x_1; x_2)$$

where

$$U(s) = P \exp \int_0^s ds \ Y^I(s) \ O_I$$

with

$$\Delta s = \epsilon$$

and

$$s : \text{position label}$$

- consider trajectories:

$$U(s) = P \exp \int_0^s ds \ Y^I(s) \ M_I$$

where

$$O_{ij} = \frac{i}{2} (x_i p_j + p_j x_i)$$

and

velocity: $Y^I(s) = \text{Tr} @ U(s) U^{-1}(s) M_I$
“What is the minimal size quantum circuit required to exactly implement a specified n-qubit unitary operation, U, without the use of ancilla qubits?”

**Nielsen approach:**

- work with smooth functions on a smooth space (rather than discrete)
- consider trajectories:

  \[ U(s) = P \exp \left( \int_0^s ds \ Y^I(s) M_I \right) \]

  where \( U(s = 0) = 1; \ U(s = 1) = U_{\text{fin}} \)

  velocity: \( Y^I(s) = \text{Tr} \left( \dot{U}(s) U^{-1}(s) M_I \right) \)

- analogy with motion of a particle determined by minimizing an action minimizing the cost function/action:

  \[ D = \int_0^1 ds \ \sum_{I \neq J} |Y^I(s) Y^J(s)|^2 \]

  [ \( F_1 \rightarrow F_2 \) \( \rightarrow F_q \) ]

  extremal path \( U(s) \) is geodesic in a Riemannian geometry

- Calculate, Calculate, Calculate, ...
Ground state complexity:

\[ \tilde{A}_\text{M}(x_k) \cdot \exp \{ i \frac{1}{2} \int_0^X j x_k^2 \} \rightarrow \tilde{A}_\text{T}(x_k) \cdot \exp \{ i \frac{1}{2} M \int_0^X j x_k^2 \} \]

where for periodic square lattice:

\[ !^2 = m^2 + \sum_i \sin^2 \frac{1}{N} k_i; \quad k_i = 0; 1; \ldots; N \]

QFT:\n\[ !^0 \]

- reference state introduces scale
- \( \omega_0 \sim 1/R \): complexity is superextensive
- \( \omega_0 = 1/\ell_0 \): complexity depends on new (unphysical?) scale
- \( \omega_0 \sim 1/\delta \): IR contributions depend on UV cutoff

Complexity = Action:

\[ \mathbb{R} = \mathbb{L} = \]

- normalizing null normals introduces scale
- \( \ell \sim R \): complexity is superextensive
- \( \ell = \ell_0 \): complexity depends on new (unphysical?) scale
- \( \ell \sim \delta \): IR contributions depend on UV cutoff, eg, \( dC_A = dt \)

Dean Carmi, RCM & Pratik Rath
Complexity of Formation:

\[ \Delta C = C(|TFD\rangle) - 2C_{vac} \]

- **thermofield double state:**

  \[ j^{\mu}F^{\nu}D_i^{\lambda} \sim e^{i n_R^-} n_R^- = 2j_{n_R}^L j_{n_R}^R \]

  \[ \Phi C = V \frac{d^{d_i} 1 \kappa}{(2^{1/4})^{d_i - 1}} \log \frac{1 + e^{-i \theta R = 2 \Pi}}{1_i e^{-i \theta R = 2}} \]

  \[ \Phi C_{m=0} = VT^{d_i} \frac{1 - d_i}{(2^{1/4})^{d_i - 1}} i 2^{d_i} i \frac{\phi}{(d_i - 1)^3 (d)} = \frac{2^{d_i} i}{d} S_{m=0} \]

- Compare to holography: \( \Delta C \propto S > 0 \); UV finite; independent of \( \omega_0 \)

  **but** in \( d = 2 \), \( \Delta C_{m=0} \propto RT \) vs \( \Delta C_{holo} \propto c \)

- Consider massive theory:

  \[ \Phi C_m = \frac{2^{d_i} i}{d} S_m \kappa 1 + \# \frac{m^2}{T^2} + \Phi \Phi \Phi \]

• thermofield double state: \( j^{\mu}F^{\nu}D_i^{\lambda} \sim e^{i n_R^-} n_R^- = 2j_{n_R}^L j_{n_R}^R \)
Complexity of Time-dependent TFD:

- time-dependent TFD (with $t_L = t_R = t/2$):

$$j \text{TFD}(t) \sim e^{i n_R^{-1}} e^{i (n_R + \frac{1}{2})} t j n_R \ L j n_R \ R$$

- saturates with $C \sim e^{N^2}$ at $\tau \sim e^{N^2}$

> Free: state remains Gaussian
> vs
> Holography: explore full Hilbert space (chaotic)
Conclusions/Questions:

- Complexity model for free scalar shows surprising similarities to holographic proposals for complexity of boundary CFT states.

- Possible extensions of QFT model:
  - Complexity for excited QFT states? in interacting QFT’s?
  - Appropriate gate set? appropriate cost functions?

- Geometry of “states” versus geometry of “unitaries”?

  - Preliminary suggestions:
    - Chapman, Heller, Marrochio & Pastawski (1707.08582)

- Complexity for gauge theories? Hashimoto, Iizuka & Sugishita (1707.03840)

- Concrete connection to “holographic complexity”?
  - QFT/path integral description of “complexity” in boundary CFT?
  - What is boundary dual of these gravitational observables?

- Need/want a better idea?

  - Build $\rho_A \to S_{EE} = -\sum \lambda_n \log \lambda_n$ → Replica Trick

  - Build optimal $U \to C = \# \text{ gates}$ → ???

  - Preliminary suggestions:
    - Caputa et al (1703.00456; 1706.07056); Czech (1706.00965)
Path Integral Complexity(??):

- Optimize resources/geometry for path integral construction of CFT states

  \[ \text{cost functional} = \text{Liouville action for } d=2 \text{ CFTs} \]

- Network elements ≠ unitaries

  Bhattacharyya, Caputa, Das, Kundu, Miyaji, Takayanagi, Watanabe (1703.00456; 1706.07056; 1804.01999)

- Similar optimization trading between “euclideanons” and scaling unitaries

  Czech (1706.00965); Sully & Vidal (unpublished)

- Holographic spacetimes = “quantum circuits” based on path-integrals

  Takayanagi (1808.09072)

  See also: Milsted & Vidal (1807.02501, 1812.00529)
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• need/want a better idea?
  ❖ Replica Trick

  build \( \rho_A \rightarrow S_{EE} = -\sum \lambda_n \log \lambda_n \) ➔ Replica Trick
  build optimal \( U \rightarrow \mathcal{C} = \# \text{ gates} \) ➔ ????

Lots to explore!

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