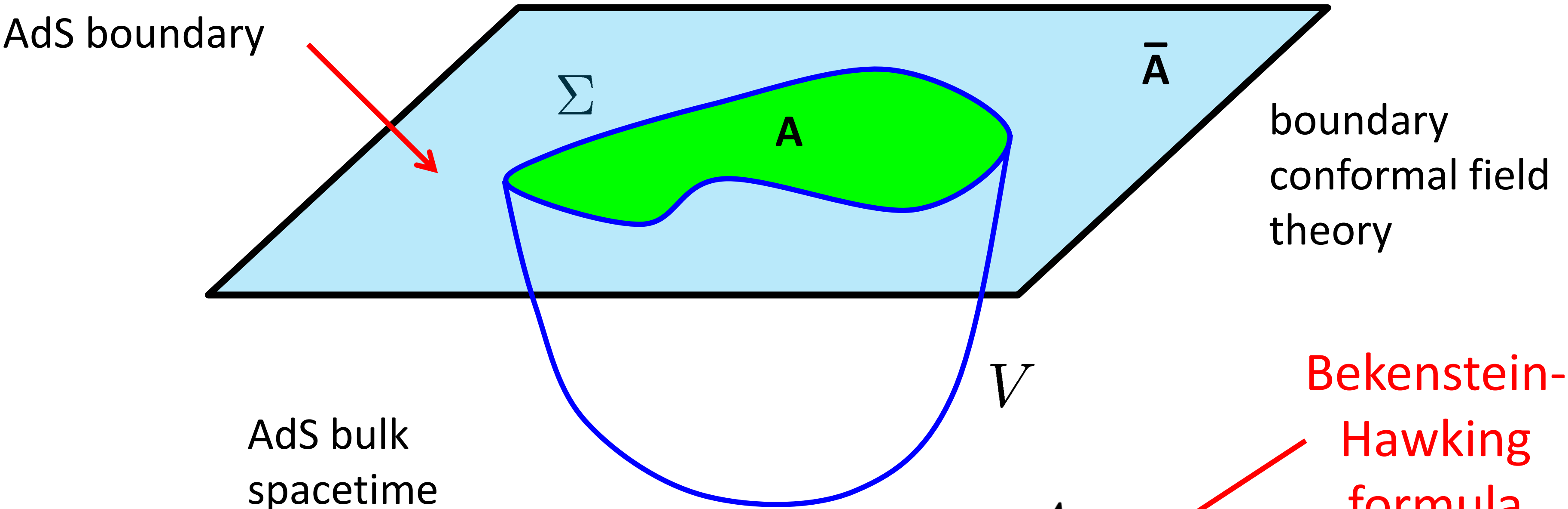
# Complexity, Holography & QFT: A Progress Report

with Alice Bernamonti, Dean Carmi, Horacio Casini, Shira Chapman, Jason Cotler, Federico Galli, Minyong Guo, Juan Hernandez, Ro Jefferson, Luis Lehner, Hugo Marrochio, Eric Poisson, Pratik Rath, Shan-Ming Ruan, Joan Simon, Rafael Sorkin & Sotaro Sugishita

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

(Ryu & Takayanagi `06)

# Holographic Entanglement Entropy:



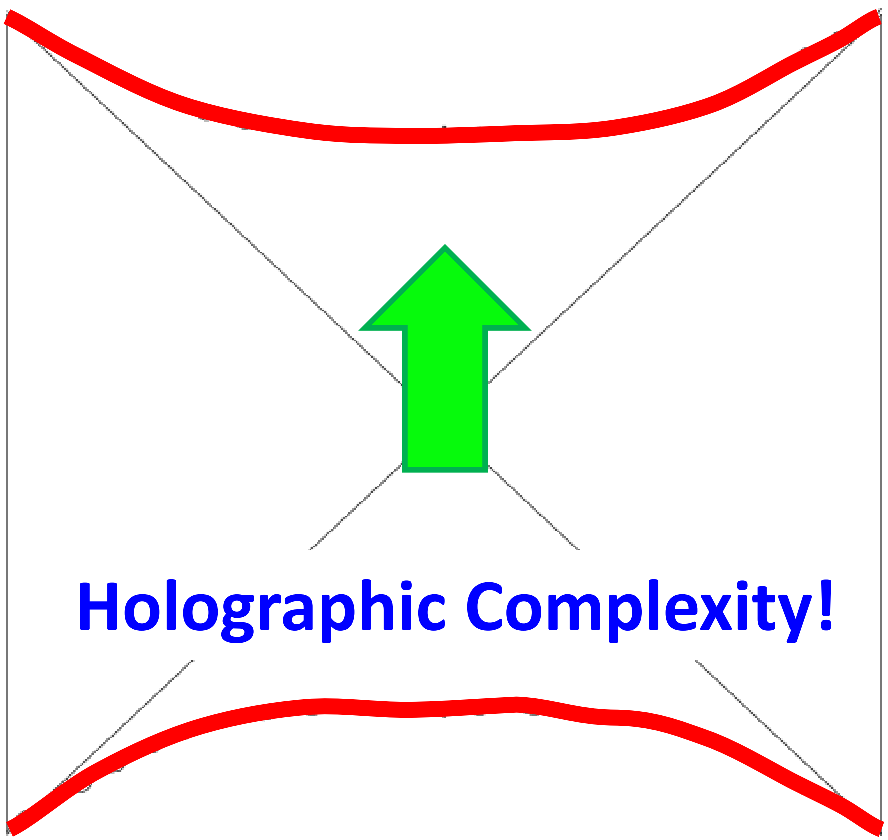

$$S(A) = \min_{\partial V \sim \Sigma} \frac{A_V}{4G_N}$$

- holographic EE is a fruitful forum for bulk-boundary dialogue:

**<u>Susskind</u>: Entanglement ^Entropy is not enough!**

# Susskind: Entanglement ^Entropy is not enough!

- "to understand the rich geometric structures that exist behind the horizon and which are predicted by general relativity."




- recall $S_{EE}$ only probes the eigenvalues of the density matrix

$$S_{EE} = -Tr\left[\rho_A \,\log \rho_A\right] = -\sum \lambda_i \,\log \lambda_i$$

- would like a new probe which is "sensitive to phases"

$$|\mathrm{TFD}\rangle \simeq \sum_\alpha e^{-E_\alpha/(2T)-iE_\alpha(t_L+t_R)} \times |E_\alpha\rangle_L |E_\alpha\rangle_R$$

# Complexity:

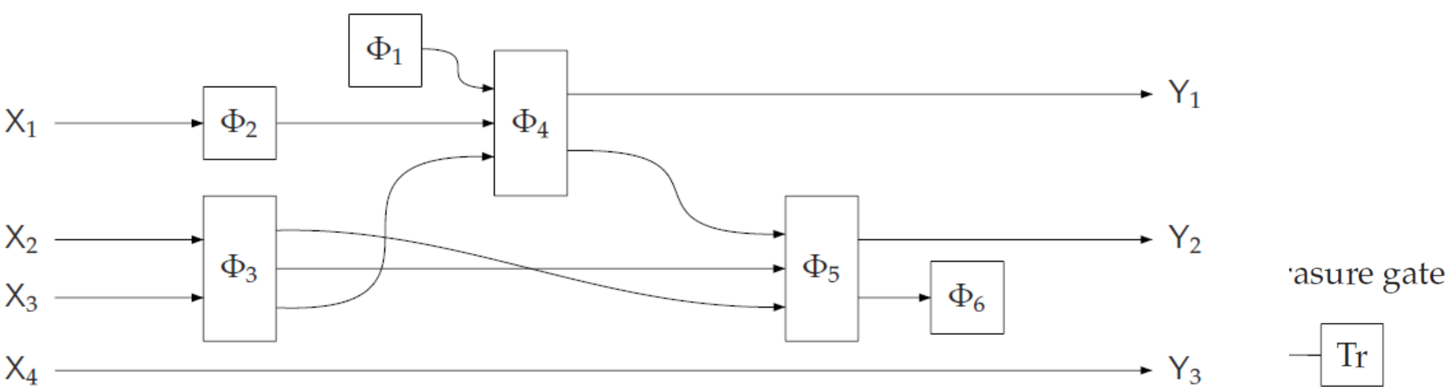
- computational complexity: how difficult is it to implement a task? eg, how difficult is it to prepare a particular state?

- quantum circuit model:

$$j\tilde{A}_T i = U j\tilde{A}_R i$$

unitary operator built from set of simple gates

simple reference state eg, $|00000\cdots0\rangle$



## Complexity:

- computational complexity: how difficult is it to implement a task? eg, how difficult is it to prepare a particular state?

- quantum circuit model:

$$|\psi_T\rangle = U\,|\psi_R\rangle$$

unitary operator built from set of simple gates

simple reference state eg, $|00000\cdots0\rangle$

tolerance: $\big|\,|\psi_T\rangle - U|\psi_R\rangle\,\big|^2 \le \varepsilon$

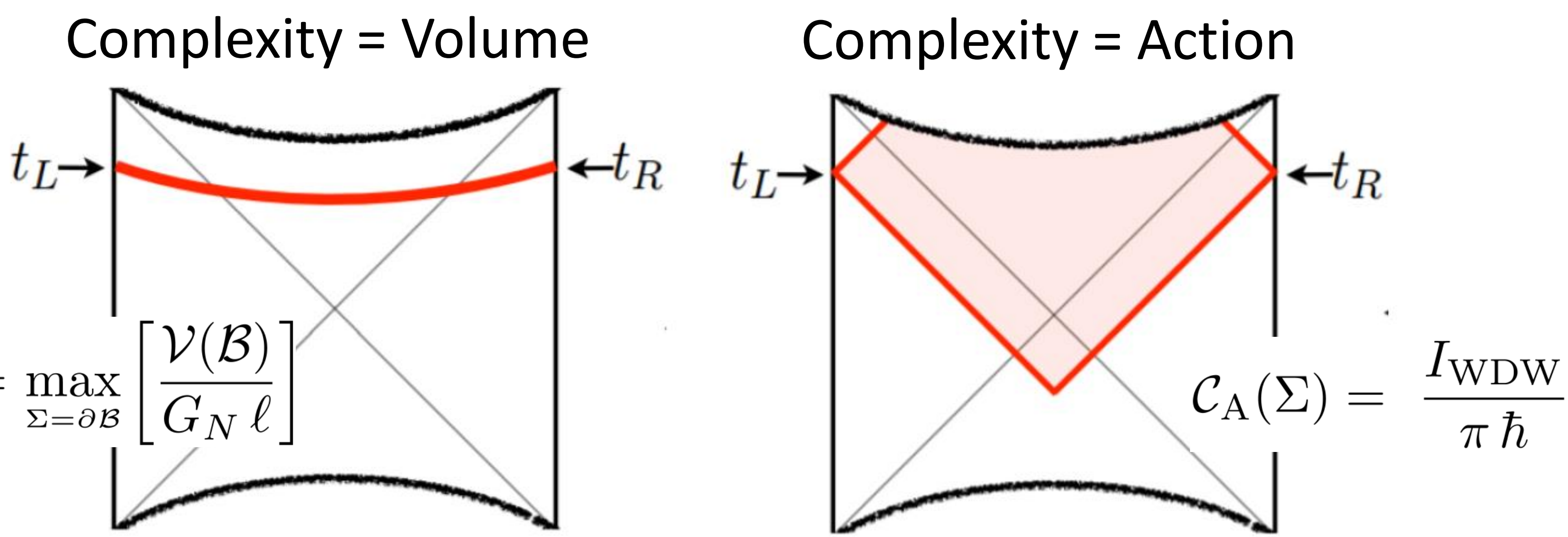
- **complexity** = minimum number of gates required to prepare the desired target state (ie, need to find optimal circuit)

- does the answer depend on the choices??

# Holographic Complexity: A Tale of Two Dualities

- complexity=volume: evaluate proper volume of extremal codim-one surface connecting Cauchy surfaces in boundary theory (cf holo EE)

(Stanford & Susskind)




$$\mathcal{C}_{\rm V}(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_N\,\ell}\right]$$

$$\mathcal{C}_{\rm A}(\Sigma) = \frac{I_{\rm WDW}}{\pi\,\hbar}$$

- complexity=action: evaluate gravitational action for Wheeler-DeWitt patch = domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT

(Brown, Roberts, Swingle, Susskind & Zhao)

- both of these gravitational "observables" probe the black hole interior (at arbitrarily late times on boundary)

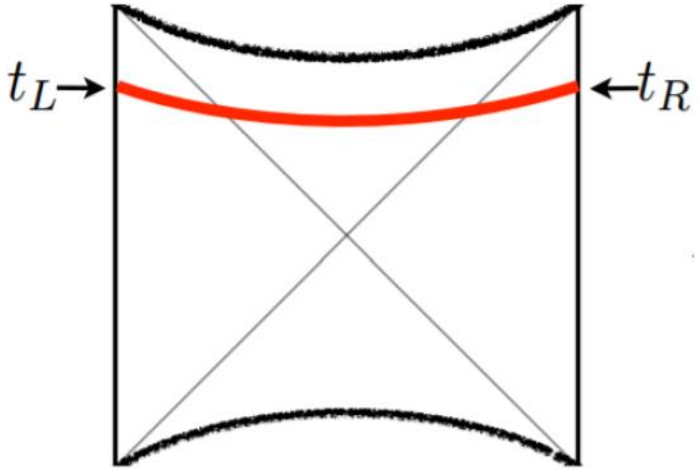
- complexity=volume2.0: evaluate spacetime volume of WDW patch

$$\mathcal{C}^0_{\rm V}(\Sigma) = \frac{V_{\rm WDW}}{G_N\,\ell^2}$$

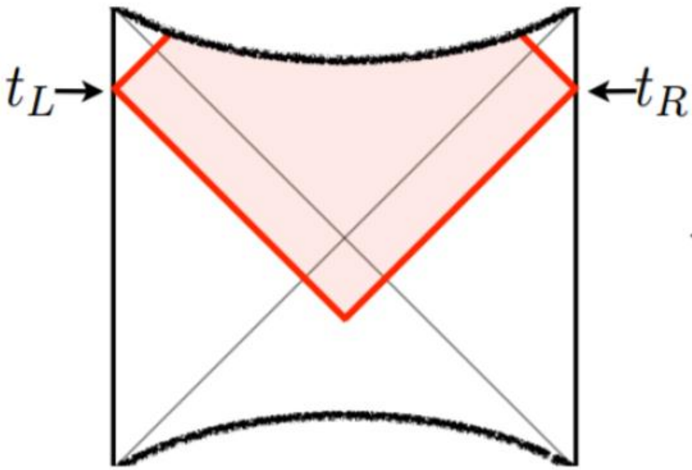
(Couch, Fischler & Nguyen)

# Holographic Complexity:

Complexity = Volume




Complexity = Action




**WHY COMPLEXITY??**

- connection of complexity=volume to AdS/MERA
- linear growth (at late times)

(d = boundary dimension)

$$\left.\frac{d\mathcal{C}_{\rm V}}{dt}\right|_{t\to\infty} = \frac{8\pi}{d-1}\,M \quad \text{(planar)} \qquad\qquad \left.\frac{d\mathcal{C}_{\rm A}}{dt}\right|_{t\to\infty} = \frac{2M}{\pi}$$

- "switchback" effect (out of time-order correlators)

# Complexity of Formation:

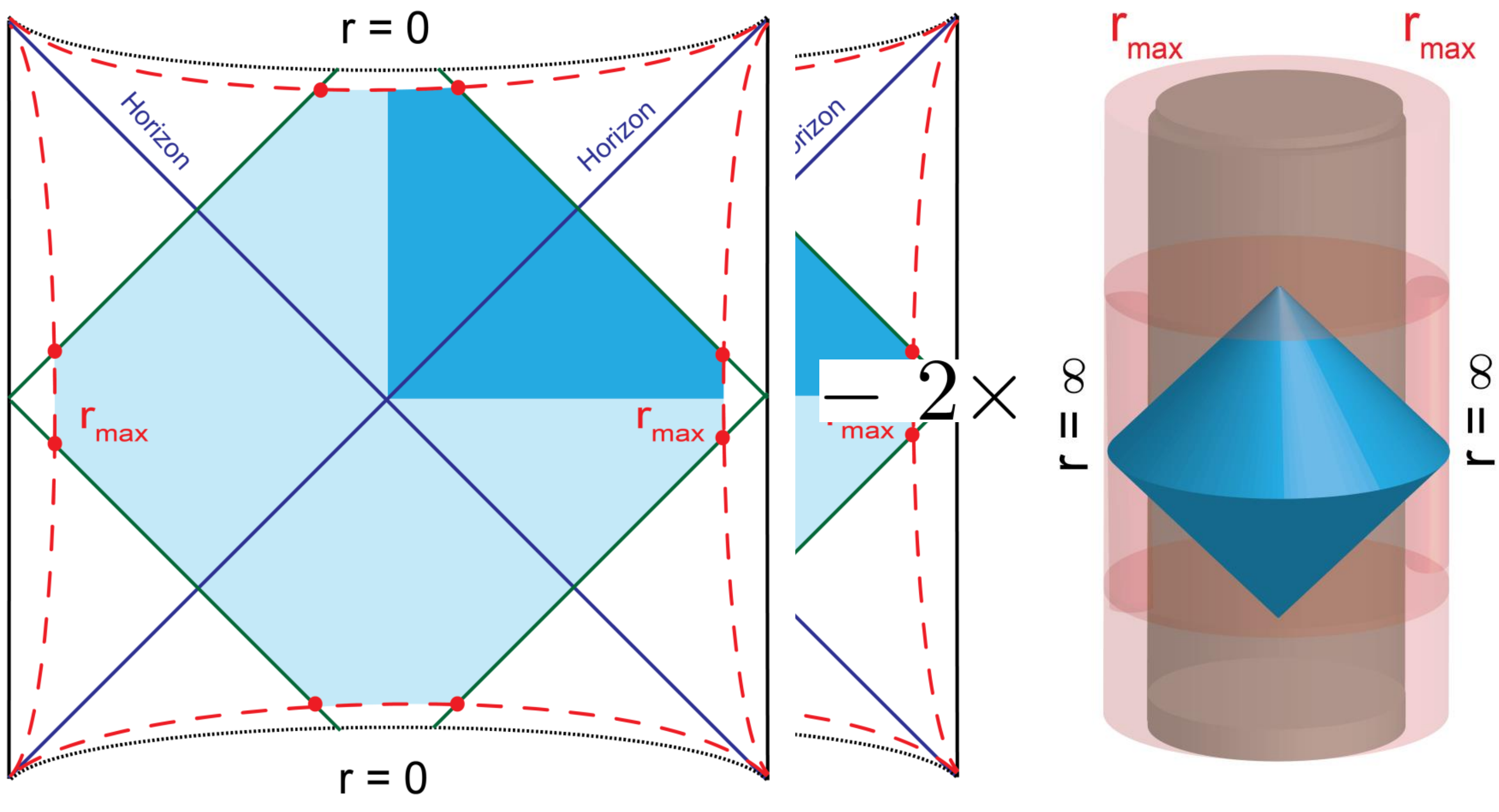
Shira Chapman, Hugo Marrochio & RCM

$$|\mathrm{TFD}\rangle = Z^{-1/2} \sum_\alpha e^{-E_\alpha/(2T)} |E_\alpha\rangle_L |E_\alpha\rangle_R$$

- additional complexity involved in forming thermofield double state compared to preparing two copies of vacuum state? (Brown et al)

$$\Delta\mathcal{C} = \mathcal{C}(|\mathrm{TFD}\rangle) - \mathcal{C}(|0\rangle \otimes |0\rangle)$$

$\Delta\mathcal{C}_A =$



# Complexity of Formation:

Shira Chapman, Hugo Marrochio & RCM

- additional complexity involved in forming thermofield double state compared to preparing two copies of vacuum state?

$$\Delta\mathcal{C} = \mathcal{C}(|\mathrm{TFD}\rangle) - \mathcal{C}(|0\rangle \otimes |0\rangle)$$

$$\Delta\mathcal{C}_A = \frac{d-2}{d\,\pi}\cot\left(\frac{\pi}{d}\right) S + \cdots \qquad \cdot \quad \mathrm{c}$$

$$\frac{d-2}{\pi^2} + \mathcal{O}(1/d)$$

thermal/ent. entropy

curvature corrections

**both complexity, but different microscopic rules?**

$$\boldsymbol{d = 2}$$

$$\Delta\mathcal{C}_V = 4\sqrt{\pi}\,\frac{(d-2)\,\Gamma(1+\frac{1}{d})}{(d-1)\,\Gamma(\frac{1}{2}+\frac{1}{d})} S + \cdots \qquad \cdot \quad \mathrm{c}$$

$$4 + \mathcal{O}(1/d)$$

$(d = \text{boundary dimension})$

# Questions?

- What is “holographic complexity”?
  - ➢ what is boundary dual of these gravitational observables?
  - ➢ QFT/path integral description of “complexity” in boundary CFT?

- is there a privileged role for (states on) null Cauchy surfaces?
  - ➢ provide distinguished reference states?

- is there a “renormalized holographic complexity”?
  - ➢ wha … of F)

- ambiguit …
  - ➢ conn… …dary?

**What does “complexity” mean in a quantum field theory?**

- more boundary terms: higher codim. intersections; “complex” joint contributions; boundary “counterterms”

- why is complexity of formation positive?

- $C_A$ contribution of spacetime singularity?
- subregion complexity?

# Complexity in Quantum Field Theory??

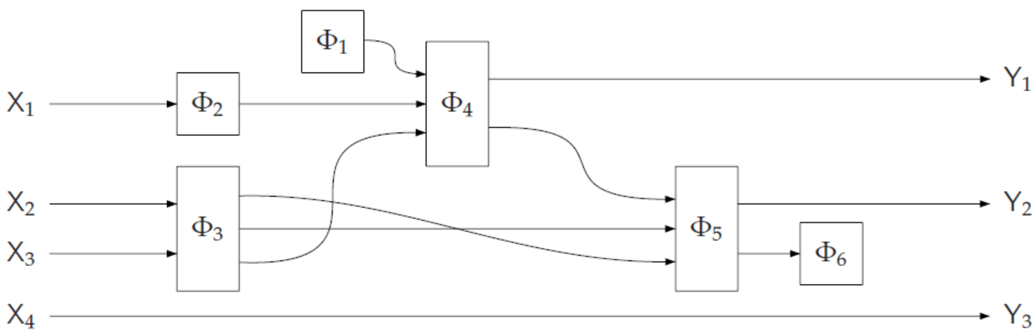
- computational complexity: how difficult is it to implement a task? eg, how difficult is it to prepare a particular state?

- quantum circuit model:

$$|\psi\rangle = U\,|\psi_0\rangle$$

unitary operator built from set of simple gates

simple reference state eg, $|00000\cdots 0\rangle$




tolerance:

$$\big|\,|\psi\rangle - |\psi\rangle_{\text{Target}}\,\big|^2 \le \varepsilon$$

- **complexity** = minimum number of gates required to prepare the desired target state (ie, need to find optimal circuit)

# Quantum Field Theory:

- **free** scalar field theory (in d spacetime dimensions)
- an infinite family of coupled harmonic oscillators

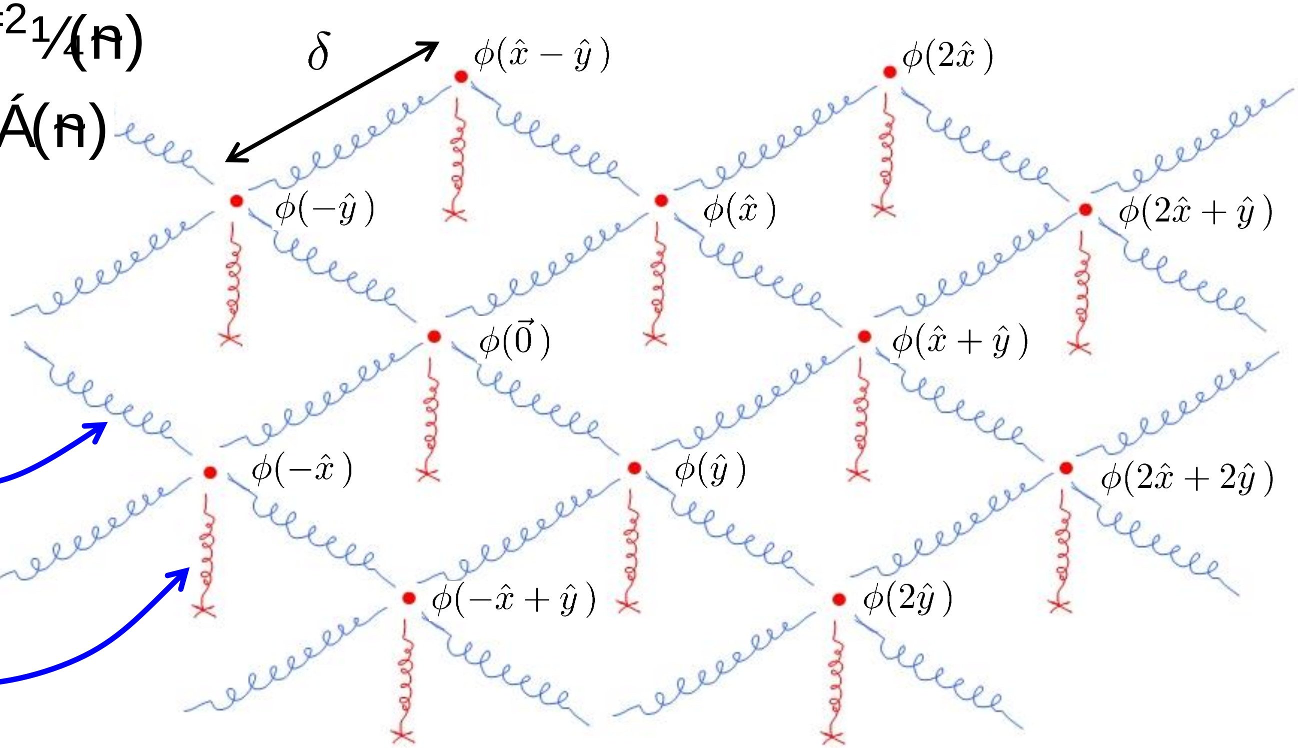
$$H = \frac{1}{2}\int d^{d-1}x\left[\pi(x)^2 + \vec{\nabla}\phi(x)^2 + m^2\phi(x)^2\right]$$

$$= \frac{1}{2}\sum_{\vec{n}}\left[\frac{p(\vec{n})^2}{\delta^{d-1}M} + M\delta^{d-1}\left(\sum_i \frac{1}{\delta^2}\left[x(\vec{n}) - x(\vec{n}-\hat{x}_i)\right]^2 + m^2 x(\vec{n})^2\right)\right]$$

$$p(\vec{n}) = \delta^{d/2}\pi(\vec{n})$$

$$x(\vec{n}) = \delta^{d/2}\phi(\vec{n})$$

$$M = 1/\delta$$




$\Omega^2 = 1/\delta^2$

$\omega^2 = m^2$

**Reference state:** $\psi_R(x_i) \simeq \exp\left(-\frac{1}{2}\,\omega_0^2 \sum x_i^2\right)$ **??????**

**Gates/Unitaries:** **??????**

- factorized Gaussian: all lattice sites disentangled
- natural operators: $x_i\,;\ p_j$ $\quad \left(\omega_0^2\, x_i + i p_i\right)\psi_R(x_j) = 0 \quad [x_i\,;p_j] = i\,\delta_{ij}$

→ $Q_{ij} = \exp[i\,\epsilon\, x_i\, p_j] \quad (i \neq j)$ "shift $x_j$ by $\epsilon\, x_i$" (entangling)

$Q_{ii} = \exp\left[i\,\frac{\epsilon}{2}(x_i\, p_i + p_i\, x_i)\right]$ "rescale $x_i$ to $e^{\epsilon} x_i$" (scaling)

$= e^{\epsilon/2} \exp[i\,\epsilon\, x_i\, p_i]$

infinitesimal parameter

**Target state:** **??????**

→ ground state; $\epsilon \ll 1$ thermofield double state

w/ Jefferson

w/ Chapman, Eisert, Heller, Jefferson, Marrochio & Pastawski

**How do we find optimal circuit??**

(see also: Chapman, Heller, Marrochio & Pastawski)

(see also: Yang, Kim, Niu, Yang & Zhang)

$\psi_T(x_i) = \cdots Q_{ij}^{\alpha_{11,2}} Q_{22}^{\alpha_{22,1}} Q_{21}^{\alpha_{21,1}} Q_{11}^{\alpha_{11,1}} \psi_R(x_i)$

- follow approach of **Mike Neilsen** (eg, Hamiltonian control theory)

(both Gaussian states!)

"circuit depth": $\mathcal{D}_1 = \sum |\alpha_{ij,n}|$

Nielsen [arXiv:0502070]; Neilsen et al [arXiv … & Dowling [arXiv:0701004]

"What is the minimal size quantum circuit required to exactly implement
a specified n-qubit unitary operation, U, without the use of ancilla qubits?"

## Nielsen approach:

- work with smooth functions on a smooth space (rather than discrete)

$\Delta s = \epsilon$ on/off

$$\psi_T(x_1,x_2) = U\,\psi_R(x_1,x_2) \quad \text{with} \quad U = P\exp\left(-i\int_0^1 ds\, Y^I(s)\, O_I\right)$$

right-to-left $s$: position label

where $O_{ij} = \frac{i}{2}(x_i p_j + p_j x_i)$

- consider trajectories:

$$U(s) = P\exp\left(-i\int_0^s d\tilde{s}\, Y^I(\tilde{s})\, M_I\right) \quad \text{where} \quad U(s=0) = 1; \quad U(s=1) = U_{fin}$$

velocity: $Y^I(s) = \mathrm{Tr}\left(i\,\partial_s U(s)\, U^{-1}(s)\, M_I\right)$

“What is the minimal size quantum circuit required to exactly implement a specified n-qubit unitary operation, U, without the use of ancilla qubits?”

## Nielsen approach:

- work with smooth functions on a smooth space (rather than discrete)
- consider trajectories:

$$U(s) = P\exp\left(\int_0^s ds\, Y^I(s)\, M_I\right) \quad \text{where} \quad U(s=0)=1;\ \ U(s=1)=U_{fin}$$

velocity: $Y^I(s) = \mathrm{Tr}\left(\partial_s U(s)\, U^{-1}(s)\, M_I\right)$

- analogy with motion of a particle determined by minimizing an action

minimizing the cost function/action: $D = \int_0^1 ds \sqrt{\sum_{I,J} g_{IJ}\, Y^I(s) Y^J(s)}$

$[\, F_1 \to F_2 \mapsto F_q \,]$

⟶ extremal path U(s) is geodesic in a Riemannian geometry

**Calculate, Calculate, Calculate, . . .**

## Ground state complexity:

$$\psi_R(x_{\vec k}) \simeq \exp\left[-\frac{1}{2}\omega_0^2 \sum |x_{\vec k}|^2\right] \longrightarrow \psi_T(x_{\vec k}) \simeq \exp\left[-\frac{1}{2}M \sum \omega_{\vec k}\, |x_{\vec k}|^2\right]$$

where for periodic square $N \times N \times N \times \cdots$ lattice:

$$\omega_{\vec k}^2 = m^2 + \frac{4}{\delta^2}\sum \sin^2\frac{\pi k_i}{N}\,; \qquad k_i = 0, 1, \cdots, N-1$$

QFT: $\rightarrow 0$

- reference state introduces scale
- $\omega_0 \sim {}^1/_R$: complexity is superextensive
- $\omega_0 = {}^1/_{\ell_0}$: complexity depends on new (unphysical?) scale
- $\omega_0 \sim {}^1/_\delta$: IR contributions depend on UV cutoff

Complexity = Action: $\alpha = L/\ell$

- normalizing null normals introduces scale
- $\ell \sim R$ : complexity is superextensive
- $\ell = \ell_0$: complexity depends on new (unphysical?) scale
- $\ell \sim \delta$: IR contributions depend on UV cutoff, eg, $dC_A/dt$

Dean Carmi, RCM & Pratik Rath

# Complexity of Formation:

$$\Delta\mathcal{C} = \mathcal{C}(|\mathrm{TFD}\rangle) - 2\,\mathcal{C}_{vac}$$

- thermofield double state: $|TFD\rangle \simeq \prod_{\vec{k}} \sum_{n_{\vec{k}}} e^{-n_{\vec{k}}\beta\omega_{\vec{k}}/2} |n_{\vec{k}}\rangle_L |n_{\vec{k}}\rangle_R$

$$\Delta\mathcal{C} = V \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \log\left(\frac{1+e^{-\beta\omega_{\vec{k}}/2}}{1-e^{-\beta\omega_{\vec{k}}/2}}\right)$$

$$\Delta\mathcal{C}_{m=0} = V T^{d-1} \frac{\Omega_{d-2}}{(2\pi)^{d-1}} \left(2^d - 1\right) \Gamma(d-1)\,\zeta(d) = \frac{2^d-1}{d} S_{m=0}$$

- compare to holography: $\Delta\mathcal{C} \propto S > 0$ ; UV finite; independent of $\omega_0$

**but** in $d=2$, $\Delta\mathcal{C}_{m=0} \propto RT$ vs $\Delta\mathcal{C}_{holo} \propto c$

- consider massive theory:

$$\Delta\mathcal{C}_m = \frac{2^d-1}{d} S_m \left(1 + \#\,\frac{m^2}{T^2} + \cdots\right)$$

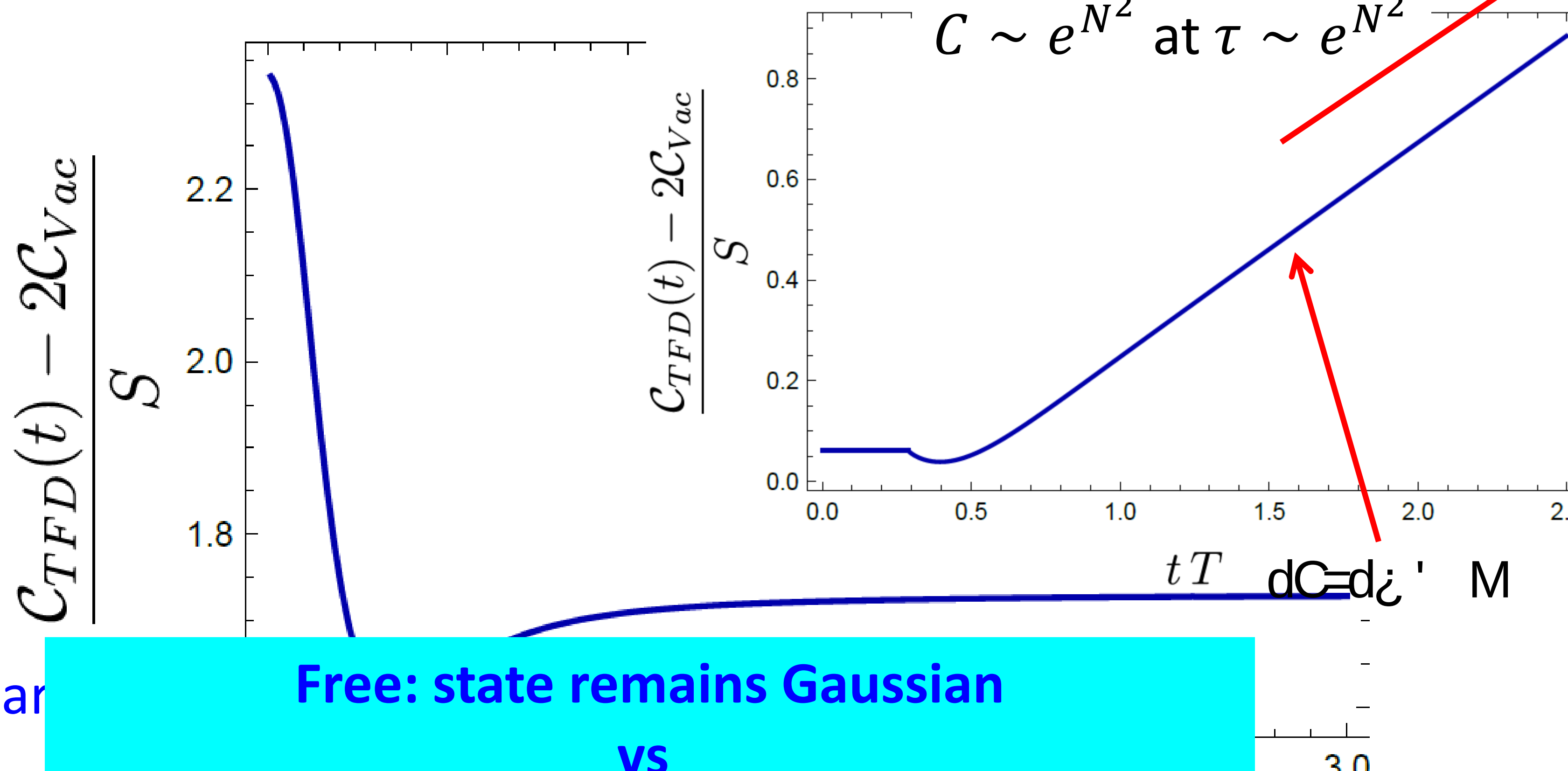
Chapman, Eisert, Hackel, Heller, Marrochio & RCM

# Complexity of Time-dependent TFD:

- time-dependent TFD (with $t_L = t_R = t/2$):

$$|TFD(t)\rangle \propto \prod_{\vec{k}} \sum_{n_{\vec{k}}} e^{-n_{\vec{k}}\beta\omega_{\vec{k}}/2}\, e^{-i(n_{\vec{k}}+\frac{1}{2})\omega_{\vec{k}} t}\, |n_{\vec{k}}\rangle_L |n_{\vec{k}}\rangle_R$$

saturates with $C \sim e^{N^2}$ at $\tau \sim e^{N^2}$




$dC/d\tau \propto M$

- compar

**Free: state remains Gaussian**

**vs**

**Holography: explore full Hilbert space (chaotic)**

$\omega_0$ and $\alpha$)

# Conclusions/Questions:

- complexity model for free scalar shows surprising similarities to holographic proposals for complexity of boundary CFT states
- possible extensions of QFT model:
  - ➢ complexity for excited QFT states? in interacting QFT's?
  - ➢ appropriate gate set? appropriate cost functions?
- geometry of "states" versus geometry of "unitaries"?

  Chapman, Heller, Marrochio & Pastawski (1707.08582)
- complexity for gauge theories? Hashimoto, Iizuka & Sugishita (1707.03840)
- concrete connection to "holographic complexity"?
  - ➢ QFT/path integral description of "complexity" in boundary CFT?
  - ➢ what is boundary dual of these gravitational observables?
- **need/want a better idea?**

build $\rho_A \to S_{EE} = -\Sigma\,\lambda_n \,\log \lambda_n$ ⟶ Replica Trick

build optimal $U \to \mathcal{C} = \#$ gates ⟶ **? ? ? ? ?**

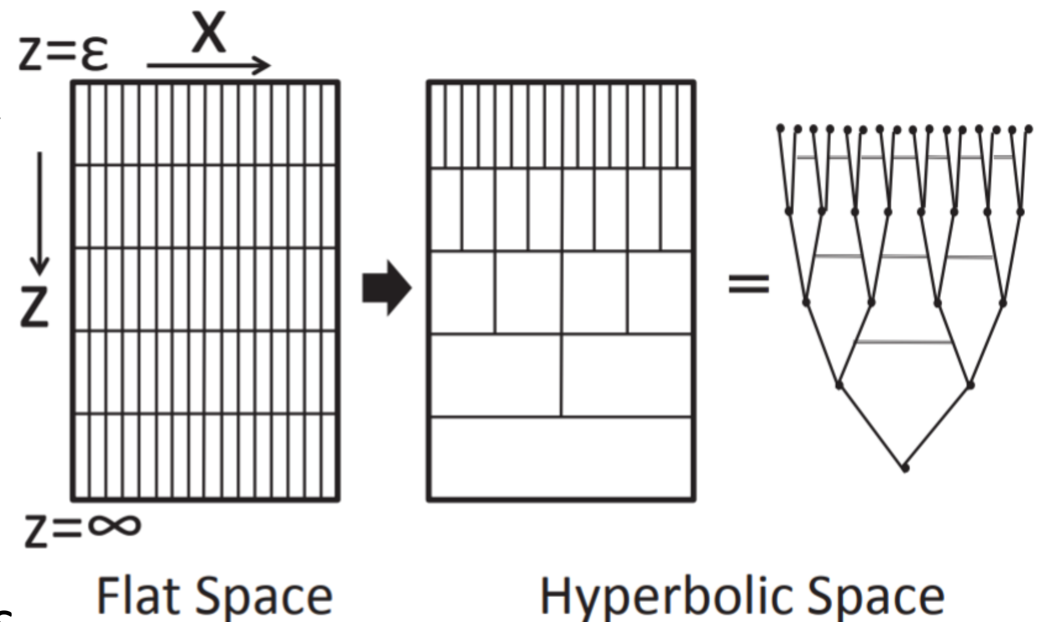
⟶ preliminary suggestions:

Caputa et al (1703.00456; 1706.07056); Czech (1706.00965)

# Path Integral Complexity(??):

- optimize resources/geometry for path integral construction of CFT states

  → cost functional = Liouville action for d=2 CFTs




- network elements ≠ unitaries

Bhattacharyya, Caputa, Das, Kundu, Miyaji, Takayanagi, Watanabe (1703.00456; 1706.07056; 1804.01999)

- similar optimization trading between "euclideons" and scaling unitaries

Czech (1706.00965); Sully & Vidal (unpublished)

- holographic spacetimes = "quantum circuits" based on path-integrals

Takayanagi (1808.09072)

see also: Milsted & Vidal (1807.02501, 1812.00529)

# Conclusions/Questions:

- complexity model for free scalar shows surprising similarities to holographic proposals for complexity of boundary CFT states
- possible extensions of QFT model:
  - complexity for excited QFT states? in interacting QFT's?
  - appropriate gate set? appropriate cost functions?
- geometry of "states" versus geometry of "unitaries"?

Chapman, Heller, Marrochio & Pastawski (1707.08582)

- complexit 707.03840)
- concrete c
  - QFT/pa ary CFT?
  - what is es?

**Lots to explore!**

- **need/want a better idea?**

$\text{build } \rho_A \rightarrow S_{EE} = -\Sigma\, \lambda_n \log \lambda_n$ ⟶ Replica Trick

$\text{build optimal } U \rightarrow \mathcal{C} = \#\text{ gates}$ ⟶ **? ? ? ? ?**

⟶ preliminary suggestions:

Caputa et al (1703.00456; 1706.07056); Czech (1706.00965)