

Diffeomorphism invariant observables in gravity

Kasia Rejzner

University of York

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- The aim of this program is to study some aspects of observables in QG that are accessible to perturbative methods and to learn more about the algebraic structure they define.
- The ultimate goal is to break away from the classical picture and have an intrinsically quantum formulation.

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where f has compact support and $j_x^k(g)$ is the k -th jet of g at x (i.e. f depends on derivatives of g up to order k).

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- A weaker notion: require all the functional derivatives $\frac{\delta^n F}{\delta g^n}(g_0)[h]$ to be local. **This is sufficient for perturbative renormalization in the sense of Epstein-Glaser.**

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- We realize the choice of a coordinate system by constructing four scalars X_g^μ , $\mu = 0, \dots, 3$ which will parametrize points of spacetime. The fields X_g^μ should transform under diffeomorphisms χ as

$$X_{\chi^*g}^\mu = X_g^\mu \circ \chi ,$$

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- Take another local field $A[g](x)$ (e.g. a metric scalar). Then

$$\mathcal{A}_g := A_g \circ \alpha_g$$

is invariant under diffeos.

Physical interpretation

Fields X_g^μ are configuration-dependent coordinates such that $[A[g] \circ X_g^{-1}](Y)$ corresponds to the value of the quantity $A[g]$ provided that the quantity X_g has the value $X_g = Y$.

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- By considering $\mathcal{A}[g] = A_g \circ X_g^{-1} \circ X_{g_0}$ we identify this observable with a functional

$$F_{\mathcal{A}}(g) = \int \mathcal{A}[g](x) f(x) = \int A[g](X_g^{-1}(Y)) f(X_{g_0}^{-1}(Y)),$$

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- If X_g^μ and $A[g]$ are all local fields themselves, then $F_{\mathcal{A}}$ is **non-local with local derivatives**.

Examples:

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- See also papers by Fröb et. al. [1703.01158], [1801.02632].

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- For a review see the book: *Perturbative algebraic quantum field theory. An introduction for mathematicians*, KR, Springer 2016.

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- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M . For the scalar field: $\mathcal{E}(M) \equiv C^\infty(M, \mathbb{R})$. For perturbative gravity $\mathcal{E}(M) = \Gamma((T^*M)^{\otimes 2})$.

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- The choice of action functional S specifies the **dynamics**. We use a modification of the Lagrangian formalism (fully covariant).

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- On $\mathcal{F}(M)$ we introduce first **classical** dynamics by means of a Poisson structure (**Peierls bracket**): $\{F, G\} = \left\langle \frac{\delta F}{\delta \varphi}, \Delta \frac{\delta G}{\delta \varphi} \right\rangle$,
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- Use the deformation quantization to construct the non-commutative algebra $\mathfrak{A}(M) = (\mathcal{F}(M)[[\hbar]], \star)$, such that

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- We work all the time on the **same vector space of functionals**, but we equip it with different algebraic structures (Poisson bracket, \star -product).

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- To introduce the **interaction**, we construct, for a given interaction term $V \in \mathcal{F}_{\text{loc}}(M)$, the **formal S-matrix**

$$\mathcal{S}(\lambda V) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\lambda}{\hbar} \right)^n V \cdot_{\mathcal{T}} \dots \cdot_{\mathcal{T}} V,$$

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- Let $F \in \mathcal{F}(M)$ be a **functional with local derivatives** (could be non-local). We define the interacting field corresponding to F by

$$F_{\text{int}} = -i\hbar \left. \frac{d}{dt} (\mathcal{S}(V)^{-1} \star \mathcal{S}(V + tF)) \right|_{t=0},$$

where the inverse of \mathcal{S} is the \star -inverse.

- In the algebraic approach, **states** are functionals $\omega : \mathfrak{A}(M) \rightarrow \mathbb{C}$ with $\omega(\mathbb{1}) = 1$ and $\omega(A^*A) \geq 0$. (Relation to Hilbert spaces via GNS theorem).

Correlation functions

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- **Wightman n -point functions** of the free theory are

$$W_n(f_1, \dots, f_n) = (\Phi(f_1) \star \dots \star \Phi(f_n))(0),$$

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- **Interacting correlation functions** are obtained as:

$$(\Phi_{\text{int}}(f_1) \star \dots \star \Phi_{\text{int}}(f_n))(0),$$

similarly for other observables in the theory.

What about gravity?

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- Formally, this product depends only on S , not on the splitting (Hawkins, KR [1612.09157]). Some obstructions could appear when renormalization is performed.



Thank you very much for your attention!