QUANTUM EFFECTS FOR A SPHERICAL SHELL IN THE MILNE UNIVERSE

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CONTENT

THE CASIMIR EFFECT

1948

- In a number of cosmological problems, additional boundary conditions are imposed on the operators of quantum fields
- Boundary conditions modify the spectrum of quantum fluctuations of fields
- Expectation values of physical observables are changed

→ Casimir effect

- Shift depends on the bulk and boundary geometries and on the boundary conditions

AIM OF THE WORK

- We consider the influence of the cosmological expansion on the local characteristics of the scalar vacuum
  - Background geometry is the Milne Universe
  - Boundary geometry consists of a spherical shell
  - Scalar field operator obeys the Robin boundary condition on the spherical shell
- The two-point Wightman function and the vacuum expectation values (VEVs) of the field squared and the energy-momentum tensor are evaluated

PROBLEM FORMULATION

- Scalar field

  mass - $\boldsymbol{m}$

  curvature coupling parameter - ξ
- Background is the ***(D+1)*** -dimensional Milne Universe, which is described by flat spacetime and has constant negative spatial curvature
- Scale factor is a linear function of proper time coordinate

Line element ⟶ $ds^2 = dt^2 - t^2(dr^2 + \sinh^2 r d\Omega_{D-1}^2)$

Spacetime curvature ⟶ $R = 0$

PROBLEM FORMULATION

- Boundary geometry consists of a sphere of radius ***r₀*** with Robin boundary conditions (BC) on it

$$(A-\delta_{(j)}B\partial_r)\varphi(x)\Big|_{r=r_0}=0,$$

$$\delta_{(i)}=1,\delta_{(e)}=-1$$

- Special cases are Neumann and Dirichlet BCs

$$A=0,B=1 \qquad A=1,B=0$$

- Equation of motion

$$(\nabla_\mu\nabla^\mu+m^2+\xi R)\varphi(x)=0$$

Minimal coupling $\longrightarrow \xi=0$

Conformal coupling $\longrightarrow \xi=\frac{D-1}{4D}$

MODE FUNCTIONS

$$\varphi_\sigma(t,r,\theta,\phi) = f(t)\frac{Z_{iz-1/2}^{-\mu}(u)}{\sinh^{D/2-1} r} Y(m_p;\vartheta,\phi)$$

where

σ - the set of quantum numbers specifying the modes

$f(t)$ - expressed in terms of Bessel functions or Hankel functions

$$f(t) \equiv \frac{X_{iz}(mt)}{t^{(D-1)/2}} = \frac{d_1 J_{-iz}(mt) + d_2 J_{iz}(mt)}{t^{(D-1)/2}} = \frac{e^{-z\pi/2} w_1 H_{iz}^{(1)}(mt) + e^{z\pi/2} w_2 H_{iz}^{(2)}(mt)}{t^{(D-1)/2}}$$

$Z_{iz-1/2}^{-\mu}(u)$ - expressed in terms of associated Legendre functions of the first and second kinds

$$Z_{iz-1/2}^{-\mu}(u) = b_1 P_{iz-1/2}^{-\mu}(u) + b_2 Q_{iz-1/2}^{-\mu}(u), u = \cosh r$$

$Y(m_p;\vartheta,\phi)$ - hyperspherical harmonics of degree ***l***

MODE FUNCTIONS

We can find the relations between the coefficients by using **the normalization condition and the BC**

$$(A-\delta_{(j)}B\partial_r)\varphi(x)\Big|_{r=r_0}=0 \qquad \Longrightarrow \qquad \frac{b_2}{b_1}$$

$$-i\int d^Dx\sqrt{|g|}\varphi_\sigma(x)\overleftrightarrow{\partial}_t\varphi^*_{\sigma'}(x)=\delta_{\sigma\sigma'} \qquad \Longrightarrow \qquad b_1$$

$$f^*(t)f'(t)-f(t)f^{*\prime}(t)=-it^{-D} \qquad \Longrightarrow \qquad \begin{cases} |d_1|^2-|d_2|^2=\dfrac{\pi}{2\sinh(\pi z)} \\ |w_2|^2-|w_1|^2=\dfrac{\pi}{4} \end{cases}$$

VACUUM STATES

- **Adiabatic vacuum**

$$w_1 = 0 \longrightarrow X_{iz}(mt) = \frac{\sqrt{\pi}}{2} e^{z\pi/2} H_{iz}^{(2)}(mt)$$

- **Conformal vacuum**

$$d_2 = 0 \longrightarrow X_{iz}(mt) = d_1 J_{-iz}(mt), |d_1|^2 = \frac{\pi}{2\sinh(\pi z)}$$

WIGHTMAN FUNCTION

$$W(x,x') = \sum_{\sigma} \varphi_{\sigma}(x)\varphi_{\sigma}^{*}(x') \qquad W(x,x') = W_0(x,x') + W_b(x,x')$$

- Boundary-free part

$$W_0(x,x') = \frac{\pi^{-1}(tt')^{(1-D)/2}}{(\sinh r \sinh r')^{D/2-1}} \sum_{l=0}^{\infty} \frac{2l+n}{nS_D} C_l^{n/2}(\cos\theta)$$

$$\int_0^{\infty} dx\, x \sinh(\pi x) \left|\Gamma(\mu + ix + 1/2)\right|^2 X_{ix}(mt)[X_{ix}(mt')]^* P_{ix-1/2}^{-\mu}(u) P_{ix-1/2}^{-\mu}(u')$$

- Interior region of the sphere, conformal vacuum

$$W_b(x,x') = -\frac{(tt')^{(1-D)/2}}{2(\sinh r \sinh r')^{D/2-1}} \sum_{l=0}^{\infty} \frac{2l+n}{nS_D} e^{-i\mu\pi} C_l^{n/2}(\cos\theta)$$

$$\int_0^{\infty} dx \frac{\overline{Q}_{x-1/2}^{\mu}(u_0)}{\overline{P}_{x-1/2}^{-\mu}(u_0)} x P_{x-1/2}^{-\mu}(u) P_{x-1/2}^{-\mu}(u') \frac{J_x(mt)J_{-x}(mt') + J_{-x}(mt)J_x(mt')}{\sin(\pi x)}$$

- Exterior region of the sphere, conformal vacuum

$$W_b(x,x') = -\frac{(tt')^{(1-D)/2}}{2(\sinh r \sinh r')^{D/2-1}} \sum_{l=0}^{\infty} \frac{2l+n}{nS_D} e^{-i\mu\pi} C_l^{n/2}(\cos\theta)$$

$$\int_0^{\infty} dx \frac{\overline{P}_{x-1/2}^{-\mu}(u_0)}{\overline{Q}_{x-1/2}^{\mu}(u_0)} x Q_{x-1/2}^{\mu}(u) Q_{x-1/2}^{\mu}(u') \frac{J_x(mt)J_{-x}(mt') + J_{-x}(mt)J_x(mt')}{\sin(\pi x)}$$

WIGHTMAN FUNCTION

- Interior region → Summation over z_k

$$\bar{P}^{-\mu}_{iz_k-1/2}(u_0)=0, u_0=\cosh r_0$$

Summation formula

$$\sum_{k=1}^{\infty} T_\mu(z_k,u)h(z_k)=\frac{e^{-i\mu\pi}}{2}\int_0^\infty dx\sinh(\pi x)h(x)$$

$$-\frac{1}{2\pi}\int_0^\infty dx\frac{\bar{Q}^{\mu}_{x-1/2}(u_0)}{\bar{P}^{-\mu}_{x-1/2}(u_0)}\cos[\pi(x-\mu)]\sum_{j=\pm}h(xe^{j\pi i/2})$$

where

$$h(z)=z\Gamma(\mu+iz+1/2)\Gamma(\mu-iz+1/2)P^{-\mu}_{iz-1/2}(u)P^{-\mu}_{iz-1/2}(u')X_{iz}(mt)\left|X_{iz}(mt')\right|^*$$

$$T_\mu(z,u)=\frac{\bar{Q}^{-\mu}_{iz-1/2}(u)}{\partial_z\bar{P}^{-\mu}_{iz-1/2}(u)}\cos[\pi(\mu-iz)]$$

- Exterior region → Integration by z

VEV OF THE FIELD SQUARED

$$\langle \varphi^2 \rangle = \langle \varphi^2 \rangle_0 + \langle \varphi^2 \rangle_b$$

- Interior region

$$\langle \varphi^2 \rangle_b = -\frac{t^{1-D}}{S_D} \sum_{l=0}^{\infty} e^{-i\mu\pi} D_l \int_0^{\infty} dx\, x \frac{\bar{Q}^{\mu}_{x-1/2}(u_0)}{\bar{P}^{-\mu}_{x-1/2}(u_0)} [p^{-\mu}_{x-1/2}(u)]^2 \frac{J_x(mt) J_{-x}(mt)}{\sin(\pi x)}$$

$$\langle \varphi^2 \rangle_b = -\frac{2t^{1-D}}{\pi S_D} \sum_{l=0}^{\infty} e^{-i\mu\pi} D_l \sum_{k=0}^{\infty}{}' (-1)^k \int_0^{\infty} dx \frac{xF(x) - kF(k)}{x^2 - k^2}$$

$$F(x) = x \frac{\bar{Q}^{\mu}_{x-1/2}(u_0)}{\bar{P}^{-\mu}_{x-1/2}(u_0)} [p^{-\mu}_{x-1/2}(u)]^2 J_x(mt) J_{-x}(mt)$$

- Exterior region

$$\bar{P}^{-\mu}_{x-1/2}(u_0) \leftrightarrow \bar{Q}^{\mu}_{x-1/2}(u_0)$$

$$p^{-\mu}_{x-1/2}(u) \leftrightarrow q^{\mu}_{x-1/2}(u)$$

ENERGY-MOMENTUM TENSOR (EMT)

VEV of EMT

$$\langle T_{ik}\rangle = \lim_{x'\to x}\partial_{i'}\partial_k W(x,x') + \left[(\xi - 1/4)g_{ik}\nabla_p\nabla^p - \xi\nabla_i\nabla_k\right]\langle\varphi^2\rangle$$

$$\langle T_{ik}\rangle = \langle T_{ik}\rangle_0 + \langle T_{ik}\rangle_b$$

$$\langle T_k^i\rangle_b = \begin{pmatrix} \varepsilon & \langle T_1^0\rangle_b & 0 & 0 & 0 \\ \langle T_0^1\rangle_b & -p & 0 & 0 & 0 \\ 0 & 0 & -p_\perp & 0 & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -p_\perp \end{pmatrix}$$

ENERGY FLUX AND ENERGY DENSITY

- Interior region

$$\left\langle T_0^1 \right\rangle_b = \frac{1}{4S_D t^{D+2}} \sum_{l=0}^{\infty} e^{-i\mu\pi} D_l \int_0^{\infty} dx x \frac{\overline{Q}_{x-1/2}^{\mu}(u_0)}{\overline{P}_{x-1/2}^{-\mu}(u_0)} \sqrt{u^2-1}\partial_u [p_{x-1/2}^{-\mu}(u)]^2$$

$$\times\left[4D(\xi-\xi_D)+(1-4\xi)t\partial_t\right]\frac{J_x(mt)J_{-x}(mt)}{\sin(\pi x)}$$

Energy flux

Energy density

$$\left\langle T_0^0 \right\rangle_b = \frac{1}{S_D t^{D+1}} \sum_{l=0}^{\infty} e^{-i\mu\pi} D_l \int_0^{\infty} dx \frac{x}{\sin(\pi x)} \frac{\overline{Q}_{x-1/2}^{\mu}(u_0)}{\overline{P}_{x-1/2}^{-\mu}(u_0)} \hat{F}_0(t,u)[p_{x-1/2}^{-\mu}(u)]^2 J_x(mt)J_{-x}(mt)$$

$$\hat{F}_0(t,u) = -\tfrac{1}{4}t^2\partial_t^2 - \left(D\xi - \frac{D-2}{4}\right)t\partial_t - m^2t^2$$

$$+\left(\xi-\tfrac{1}{4}\right)\left[\left(u^2-1\right)\partial_u^2 + Du\partial_u\right] + x^2 + D(D-1)(\xi-\xi_D)$$

VACUUM STRESSES

- Interior region

$$\left\langle T_1^1 \right\rangle_b = \frac{1}{S_D t^{D+1}} \sum_{l=0}^{\infty} e^{-i\mu\pi} D_l \int_0^{\infty} dx \frac{x}{\sin(\pi x)} \frac{\bar{Q}_{x-1/2}^{\mu}(u_0)}{\bar{P}_{x-1/2}^{-\mu}(u_0)} \hat{F}_1(t,u)[p_{x-1/2}^{-\mu}(u)]^2 J_x(mt) J_{-x}(mt),$$

Normal stress

Azimuthal stress

$$\left\langle T_2^2 \right\rangle_b = \frac{1}{S_D t^{D+1}} \sum_{l=0}^{\infty} e^{-i\mu\pi} D_l \int_0^{\infty} dx \frac{x}{\sin(\pi x)} \frac{\bar{Q}_{x-1/2}^{\mu}(u_0)}{\bar{P}_{x-1/2}^{-\mu}(u_0)} \hat{F}_2(t,u)[p_{x-1/2}^{-\mu}(u)]^2 J_x(mt) J_{-x}(mt),$$

$$\hat{F}_1(t,u) = \left(\frac{1}{4} - \xi\right) t^2 \partial_t^2 + \left((D-1)\xi - \frac{D-2}{4}\right) t\partial_t + \frac{1}{4}\left(u^2 - 1\right)\partial_u^2$$
$$+ \left((D-1)\xi + \frac{D}{4}\right) u\partial_u - x^2 + \frac{1}{4} + \frac{nD}{4} - \frac{l(l+n)}{u^2 - 1} - \xi(D-1),$$

$$\hat{F}_2(t,u) = \left(\frac{1}{4} - \xi\right) t^2 \partial_t^2 + \left((D-1)\xi - \frac{D-2}{4}\right) t\partial_t - \left(\frac{1}{4} - \xi\right)\left(u^2 - 1\right)\partial_u^2 - \left(\frac{D}{4} - (D-1)\xi\right) u\partial_u - \xi(D-1),$$

CONSERVATION EQUATION

- Interior region

$$\langle T_2^2\rangle_b = \langle T_3^3\rangle_b = \cdots = \langle T_D^D\rangle_b$$

Covariant conservation equation

$$\nabla_k \langle T_i^k\rangle_b = 0$$

$$\frac{\partial_0\left(t^D\langle T_0^0\rangle_b\right)}{t^D} + \frac{\partial_1\left(\sinh^{D-1} r\langle T_0^1\rangle_b\right)}{\sinh^{D-1} r} - \frac{1}{t}\langle T_1^1\rangle_b - \frac{D-2}{t}\langle T_2^2\rangle_b = 0,$$

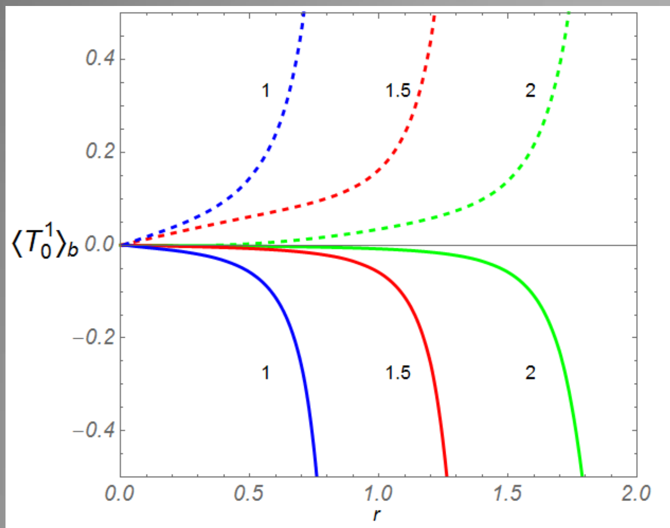
$$\frac{\partial_0\left(t^D\langle T_1^0\rangle_b\right)}{t^D} + \frac{\partial_1\left(\sinh^{D-1} r\langle T_1^1\rangle_b\right)}{\sinh^{D-1} r} - (D-2)\frac{\cosh r}{\sinh r}\langle T_2^2\rangle_b = 0.$$

- Exterior region

$$\overline{P}^{-\mu}_{x-1/2}(u_0) \leftrightarrow \overline{Q}^{\mu}_{x-1/2}(u_0)$$

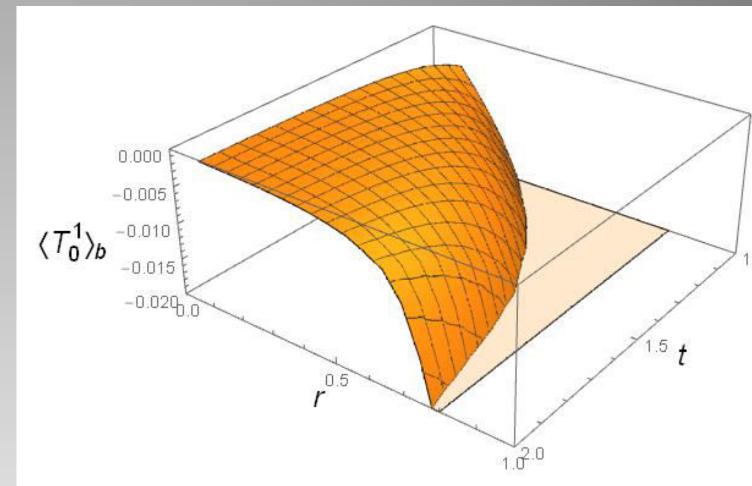
$$p^{-\mu}_{x-1/2}(u) \leftrightarrow q^{\mu}_{x-1/2}(u)$$

NUMERICAL RESULTS



Energy flux

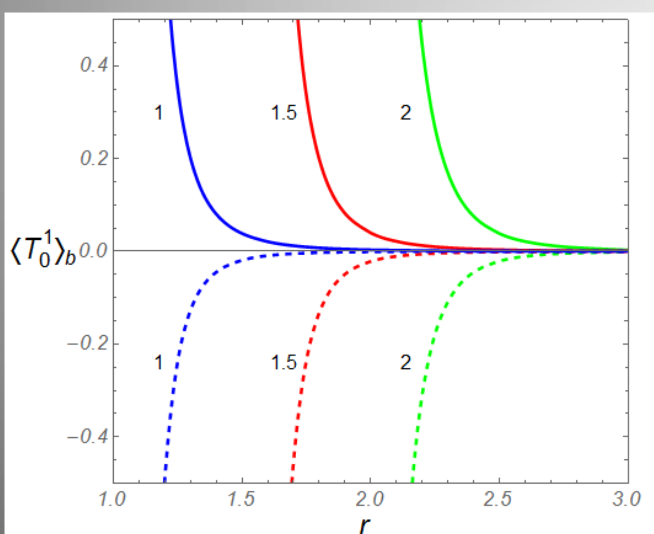
Inside



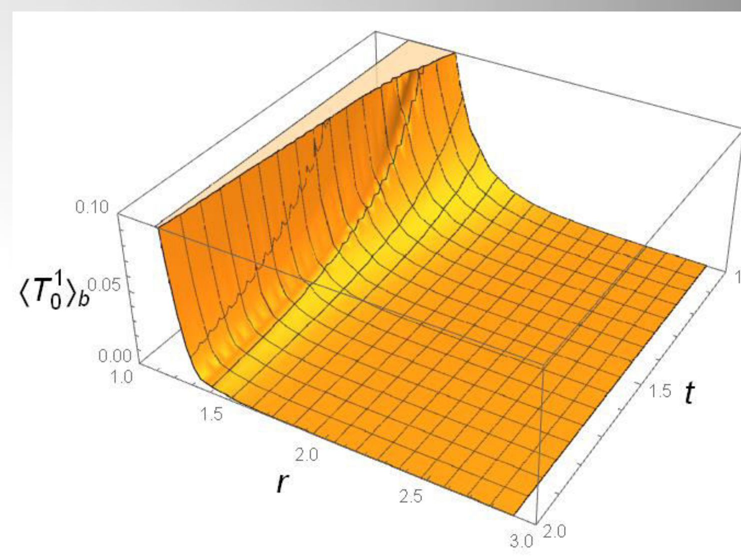
$D = 3, r_0 = 1, m = 1, \xi = 0$
Dirichlet BC

$D = 3, r_0 = 1, 1.5, 2, mt = 1, \xi = 0$

Dirichlet BC (solid line)
Neumann BC (dashed line)



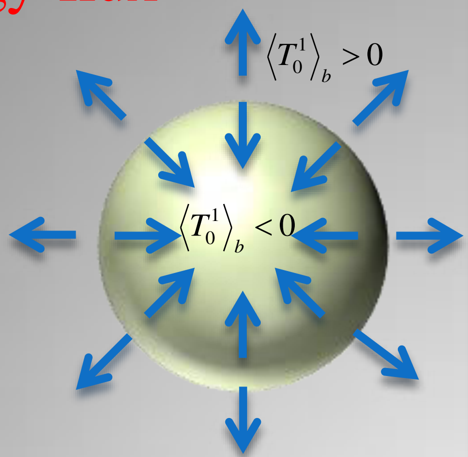
Outside



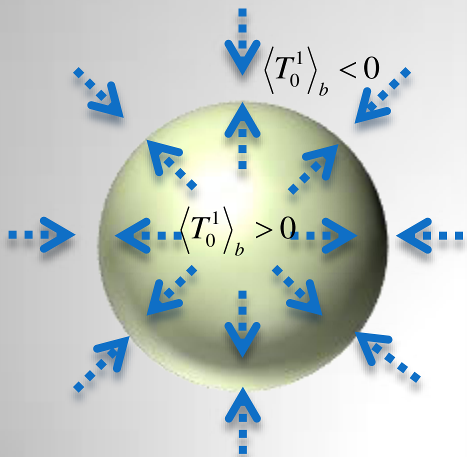
NUMERICAL RESULTS

Energy flux

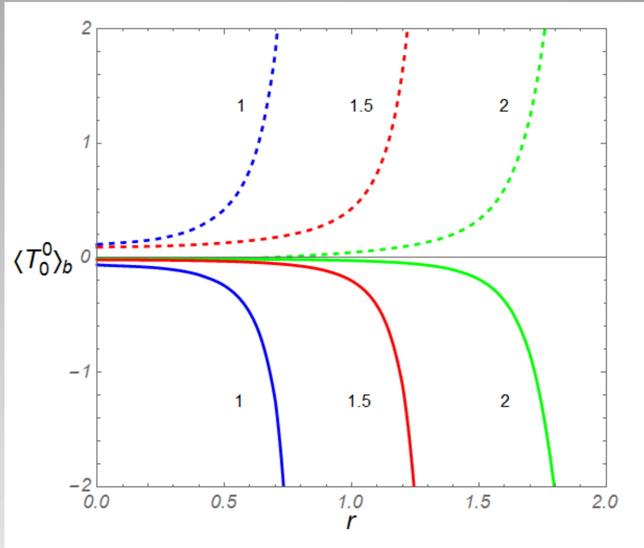
Dirichlet BC



Neumann BC



Energy density

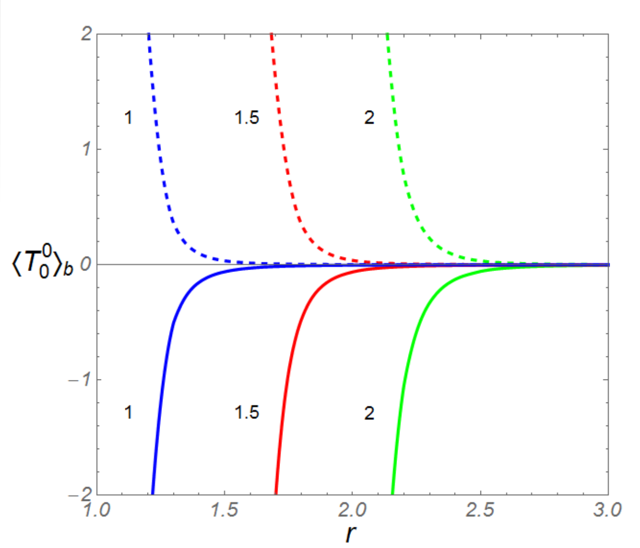


Inside the sphere

$D = 3, r_0 = 1, 1.5, 2, mt = 1, \xi = 0$

Dirichlet BC

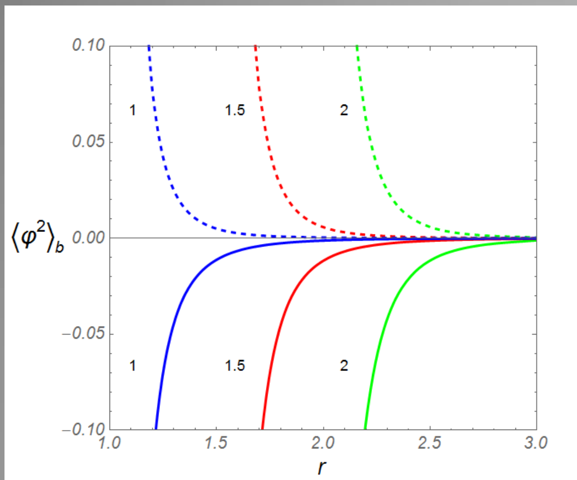
Neumann BC

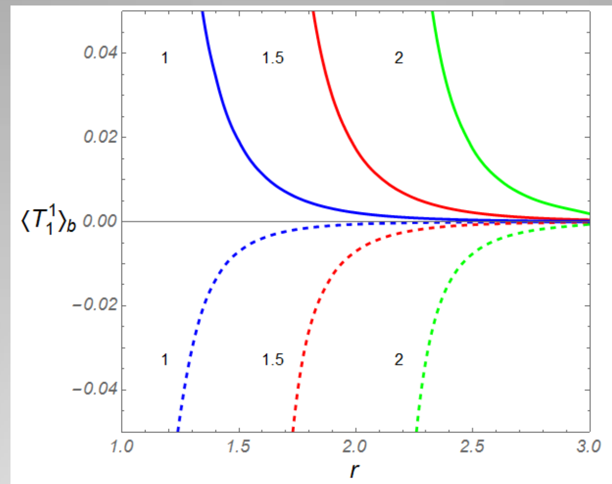


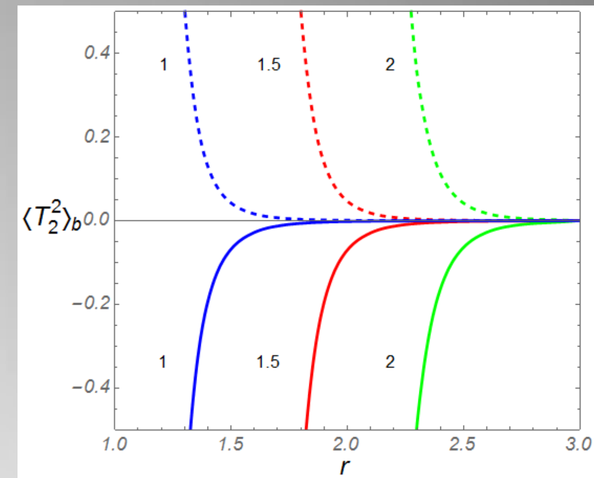
Outside the sphere

NUMERICAL RESULTS

Field squared





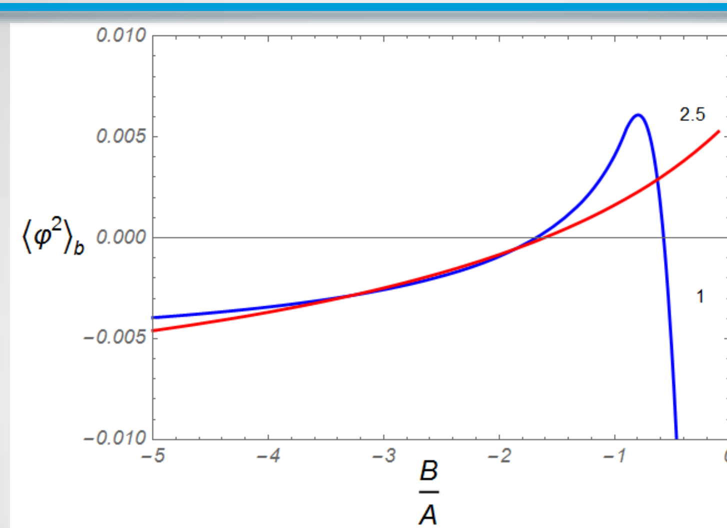


$D = 3, r_0 = 1, 1.5, 2, mt = 1, \xi = 0$

Dirichlet BC

Neumann BC

Outside the sphere



$D = 3, mt = 1$

$r_0 = 2, r = 1, 2.5, \xi = 0$

Robin BC

SUMMARY

- The Wightman function and the vacuum expectation values (VEVs) of the field squared and of the energy-momentum tensor (EMT) are investigated in the geometry of a spherical shell on background of the Milne Universe. The field obeys the Robin boundary condition on the sphere.
- The boundary-induced contribution is explicitly extracted and the renormalization is done only for the boundary-free contribution
- Rapidly convergent integral representations are provided for the boundary-induced parts
- Adiabatic and conformal vacuum states are considered, and the corresponding mode functions are found
- There is a nonzero energy flux along the radial direction (the off-diagonal component of the VEV of the EMT)
- Depending on the coefficients in the Robin BC, sphere-induced part in the energy density can be either positive or negative

THANK YOU!