QUANTUM EFFECTS FOR A SPHERICAL SHELL IN THE MILNE UNIVERSE

Tigran Petrosyan

Department of physics,
Yerevan State University, Armenia

The Modern Physics of Compact Stars and Relativistic Gravity,
17 September – 21 September, 2019, Yerevan, Armenia
CONTENT

- Introduction
- Problem formulation
- Mode functions
- Vacuum states
- Wightman function
- Vacuum expectation values
- Summary
THE CASIMIR EFFECT

- In a number of cosmological problems, additional boundary conditions are imposed on the operators of quantum fields.
- Boundary conditions modify the spectrum of quantum fluctuations of fields.
- Expectation values of physical observables are changed.

Casimir effect

- Shift depends on the bulk and boundary geometries and on the boundary conditions.
AIM OF THE WORK

- We consider the influence of the cosmological expansion on the local characteristics of the scalar vacuum
  - Background geometry is the Milne Universe
  - Boundary geometry consists of a spherical shell
  - Scalar field operator obeys the Robin boundary condition on the spherical shell

- The two-point Wightman function and the vacuum expectation values (VEVs) of the field squared and the energy-momentum tensor are evaluated
PROBLEM FORMULATION

- Scalar field
  - mass: $m$
  - curvature coupling parameter: $\xi$
- Background is the $(D+1)$-dimensional Milne Universe, which is described by flat spacetime and has constant negative spatial curvature
- Scale factor is a linear function of proper time coordinate

Line element: $ds^2 = dt^2 - t^2 (dr^2 + \sinh^2 r d\Omega^2_{D-1})$

Spacetime curvature: $R = 0$
PROBLEM FORMULATION

- Boundary geometry consists of a sphere of radius $r_o$ with Robin boundary conditions (BC) on it

$$ (A - \delta_{(j)} B \partial_r) \varphi(x) \bigg|_{r=r_0} = 0, $$

$$ \delta_{(i)} = 1, \delta_{(e)} = -1 $$

- Special cases are Neumann and Dirichlet BCs

$$ A = 0, B = 1 \quad A = 1, B = 0 $$

- Equation of motion

$$ (\nabla_\mu \nabla^\mu + m^2 + \xi R) \varphi(x) = 0 $$

  - Minimal coupling $\rightarrow \xi = 0$
  - Conformal coupling $\rightarrow \xi = \frac{D-1}{4D}$
MODE FUNCTIONS

\[ \varphi_{\sigma}(t, r, \theta, \phi) = f(t) \frac{Z^{-\mu}_{iz-1/2}(u)}{\sinh \frac{D/2-1}{r}} Y(m_p, \vartheta, \phi) \]

where

\( \sigma \) - the set of quantum numbers specifying the modes

\( f(t) \) - expressed in terms of Bessel functions or Hankel functions

\[ f(t) \equiv \frac{X_{iz}(mt)}{t^{(D-1)/2}} = \frac{d_1 J_{-iz}(mt) + d_2 J_{iz}(mt)}{t^{(D-1)/2}} = \frac{e^{-z\pi/2} w_1 H^{(1)}_{iz}(mt) + e^{z\pi/2} w_2 H^{(2)}_{iz}(mt)}{t^{(D-1)/2}} \]

\( Z^{-\mu}_{iz-1/2}(u) \) - expressed in terms of associated Legendre functions of the first and second kinds

\[ Z^{-\mu}_{iz-1/2}(u) = b_1 P^{-\mu}_{iz-1/2}(u) + b_2 Q^{-\mu}_{iz-1/2}(u), u = \cosh r \]

\( Y(m_p, \vartheta, \phi) \) - hyperspherical harmonics of degree \( l \)
MODE FUNCTIONS

We can find the relations between the coefficients by using the normalization condition and the BC.

\[(A - \delta_{(j)} B \partial_r ) \varphi(x) \bigg|_{r=r_0} = 0 \quad \implies \quad \frac{b_2}{b_1}\]

\[-i \int d^D x \sqrt{g} \varphi_\sigma(x) \tilde{\partial}_t \varphi_\sigma^*(x) = \delta_{\sigma' \sigma} \quad \implies \quad b_1\]

\[f^*(t)f'(t) - f(t)f^*(t) = -it^{-D} \quad \implies \quad \begin{cases} |d_1|^2 - |d_2|^2 = \frac{\pi}{2 \sinh(\pi \zeta)} \\ |w_2|^2 - |w_1|^2 = \frac{\pi}{4} \end{cases}\]
VACUUM STATES

• Adiabatic vacuum

\[ w_1 = 0 \quad \Rightarrow \quad X_{iz}(mt) = \frac{\sqrt{\pi}}{2} e^{z_{\pi/2}} H^{(2)}_{iz}(mt) \]

• Conformal vacuum

\[ d_2 = 0 \quad \Rightarrow \quad X_{iz}(mt) = d_1 J_{-iz}(mt), \left|d_1\right|^2 = \frac{\pi}{2 \sinh(\pi z)} \]
WIGHTMAN FUNCTION

\[ W(x, x') = \sum_{\sigma} \varphi_{\sigma}(x) \varphi_{\sigma}^*(x') \quad W(x, x') = W_0(x, x') + W_b(x, x') \]

- **Boundary-free part**

\[
W_0(x, x') = -\frac{\pi^{-1}(tt')^{(1-D)/2}}{(\sinh r \sinh r')^{D/2-1}} \sum_{l=0}^{\infty} \frac{2l + n}{nS_D} C_l^{n/2} (\cos \theta)
\]

\[
\int_0^\infty dx x \sinh(\pi x) |\Gamma(\mu + i x + 1/2)|^2 X_{ix}(mt)X_{ix}(mt')^* P^{-\mu}_{ix-1/2}(u)P^{-\mu}_{ix-1/2}(u')
\]

- **Interior region of the sphere, conformal vacuum**

\[
W_b(x, x') = -\frac{(tt')^{(1-D)/2}}{2(\sinh r \sinh r')^{D/2-1}} \sum_{l=0}^{\infty} \frac{2l + n}{nS_D} e^{-i\mu\pi} C_l^{n/2} (\cos \theta)
\]

\[
\int_0^\infty dx \frac{\bar{Q}_{x-1/2}(u_0)}{\bar{P}_{x-1/2}(u_0)} xP^{-\mu}_{x-1/2}(u)P^{-\mu}_{x-1/2}(u') \frac{J_x(mt)J_{-x}(mt') + J_{-x}(mt)J_x(mt')}{\sin(\pi x)}
\]

- **Exterior region of the sphere, conformal vacuum**

\[
W_b(x, x') = -\frac{(tt')^{(1-D)/2}}{2(\sinh r \sinh r')^{D/2-1}} \sum_{l=0}^{\infty} \frac{2l + n}{nS_D} e^{-i\mu\pi} C_l^{n/2} (\cos \theta)
\]

\[
\int_0^\infty dx \frac{\bar{P}_{x-1/2}(u_0)}{\bar{Q}_{x-1/2}(u_0)} xQ^{-\mu}_{x-1/2}(u)Q^{-\mu}_{x-1/2}(u') \frac{J_x(mt)J_{-x}(mt') + J_{-x}(mt)J_x(mt')}{\sin(\pi x)}
\]
WIGHTMAN FUNCTION

- Interior region ➡ Summation over $z_k$

$$P_{iz_k - 1/2}^{-\mu}(u_0) = 0, u_0 = \cosh r_0$$

Summation formula

$$\sum_{k=1}^{\infty} T_\mu(z_k, u) h(z_k) = \frac{e^{-i\mu\pi}}{2} \int_0^\infty dx \sinh(\pi x) h(x)$$

$$- \frac{1}{2\pi} \int_0^\infty dx \frac{Q_{x-1/2}^{-\mu}(u_0)}{P_{x-1/2}^{-\mu}(u_0)} \cos[\pi(x - \mu)] \sum_j h(xe^{j\pi/2})$$

where

$$h(z) = z^{\Gamma(\mu + iz + 1/2)} \Gamma(\mu - iz + 1/2) P_{iz-1/2}^{-\mu}(u) P_{iz-1/2}^{-\mu}(u') X_{iz}(mt) X_{iz}(mt')^*$$

$$T_\mu(z, u) = \frac{Q_{iz-1/2}^{-\mu}(u)}{\partial_z P_{iz-1/2}^{-\mu}(u)} \cos[\pi(\mu - iz)]$$

- Exterior region ➡ Integration by $z$
VEV OF THE FIELD SQUARED

\[ \langle \varphi^2 \rangle = \langle \varphi^2 \rangle_0 + \langle \varphi^2 \rangle_b \]

- **Interior region**

\[ \langle \varphi^2 \rangle_b = -\frac{t^{1-D}}{S_D} \sum_{l=0}^{\infty} e^{-i\mu \pi} D_1 \int_0^\infty dx \int_0^\infty \frac{Q_{x-1/2}^\mu (u_0)}{P_{x-1/2}^\mu (u_0)} [p_{x-1/2}^{-\mu} (u)]^2 \frac{J_x (m t) J_{-x} (m t)}{\sin(\pi x)} \]

- **Exterior region**

\[ F(x) = x \frac{Q_{x-1/2}^\mu (u_0)}{P_{x-1/2}^\mu (u_0)} [p_{x-1/2}^{-\mu} (u)]^2 J_x (m t) J_{-x} (m t) \]

\[ \overline{P}_{x-1/2}^{-\mu} (u_0) \leftrightarrow \overline{Q}_{x-1/2}^{\mu} (u_0) \]

\[ p_{x-1/2}^{-\mu} (u) \leftrightarrow q_{x-1/2}^{\mu} (u) \]
VEV of EMT

\[ \langle T_{ik} \rangle = \lim_{x' \to x} \partial_i \partial_k W(x, x') + \left( (\xi - 1/4) g_{ik} \nabla_p \nabla^p - \xi \nabla_i \nabla_k \right) \langle \phi^2 \rangle \]

\[ \langle T_{ik} \rangle = \langle T_{ik} \rangle_0 + \langle T_{ik} \rangle_b \]

\[ \begin{pmatrix}
\varepsilon & \langle T_1^0 \rangle_b & 0 & 0 & 0 \\
\langle T_0^1 \rangle_b & -p & 0 & 0 & 0 \\
0 & 0 & -p_\perp & 0 & 0 \\
0 & 0 & 0 & \ddots & \ddots \\
0 & 0 & 0 & \cdots & -p_\perp \\
\end{pmatrix} \]
ENERGY FLUX AND ENERGY DENSITY

- Interior region

\[
\langle T^1_0 \rangle_b = \frac{1}{4S_D t^{D+2}} \sum_{l=0}^{\infty} e^{-i\mu \pi} D_l \int_0^\infty dx x \frac{\overline{Q}_{x-1/2}^{\mu} (u_0)}{\overline{P}_{x-1/2}^{-\mu} (u_0)} \sqrt{u^2 - 1} \partial_u [p_{x-1/2}^{-\mu} (u)]^2 \\
\times [4D(\xi - \xi_D) + (1 - 4\xi)t\partial_t] \frac{J_x(mt)J_{-x}(mt)}{\sin(\pi x)}
\]

Energy flux

Energy density

\[
\langle T^0_0 \rangle_b = \frac{1}{S_D t^{D+1}} \sum_{l=0}^{\infty} e^{-i\mu \pi} D_l \int_0^\infty dx \frac{x}{\sin(\pi x)} \frac{\overline{Q}_{x-1/2}^{\mu} (u_0)}{\overline{P}_{x-1/2}^{-\mu} (u_0)} \hat{F}_0(t,u) [p_{x-1/2}^{-\mu} (u)]^2 J_x(mt)J_{-x}(mt)
\]

\[
\hat{F}_0(t,u) = -\frac{1}{4} t^2 \partial_t^2 - \left( D\xi - \frac{D-2}{4} \right) t \partial_t - m^2 t^2 \\
+ (\xi - \frac{1}{4}) \left[ (u^2 - 1) \partial_u^2 + Du \partial_u \right] + x^2 + D(D-1)(\xi - \xi_D)
\]
VACUUM STRESSES

- **Interior region**

\[
\langle T^1_1 \rangle_b = \frac{1}{S_D t^{D+1}} \sum_{l=0}^{\infty} e^{-i\mu \pi} D_l \int_0^\infty dx \frac{x}{\sin(\pi x)} \frac{-\mu}{P_{x-1/2}(u_0)} \hat{F}_1(t,u)[p_{x-1/2}(u)]^2 J_x(mt)J_{-x}(mt),
\]

- **Normal stress**

- **Azimuthal stress**

\[
\langle T^2_2 \rangle_b = \frac{1}{S_D t^{D+1}} \sum_{l=0}^{\infty} e^{-i\mu \pi} D_l \int_0^\infty dx \frac{x}{\sin(\pi x)} \frac{-\mu}{P_{x-1/2}(u_0)} \hat{F}_2(t,u)[p_{x-1/2}(u)]^2 J_x(mt)J_{-x}(mt),
\]

\[
\hat{F}_1(t,u) = \left( \frac{1}{4} - \xi \right) t^2 \partial_t^2 + \left( (D-1)\xi - \frac{D-2}{4} \right) t \partial_t + \frac{1}{4} \left( u^2 - 1 \right) \partial_u^2
\]
\[
+ \left( (D-1)\xi + \frac{D}{4} \right) u \partial_u - x^2 + \frac{1}{4} + \frac{nD}{4} - \frac{l(l+n)}{u^2-1} - \xi(D-1),
\]

\[
\hat{F}_2(t,u) = \left( \frac{1}{4} - \xi \right) t^2 \partial_t^2 + \left( (D-1)\xi - \frac{D-2}{4} \right) t \partial_t - \left( \frac{1}{4} - \xi \right) \left( u^2 - 1 \right) \partial_u^2 - \left( \frac{D}{4} - (D-1)\xi \right) u \partial_u - \xi(D-1),
\]
CONSERVATION EQUATION

- Interior region

\[ \langle T_2^2 \rangle_b = \langle T_3^3 \rangle_b = \cdots = \langle T_D^D \rangle_b \]

Covariant conservation equation

\[ \nabla_k \langle T_i^k \rangle_b = 0 \]

\[ \frac{\partial_0 \left( t^D \langle T_0^0 \rangle_b \right)}{t^D} + \frac{\partial_1 \left( \sinh^{D-1} r \langle T_1^1 \rangle_b \right)}{\sinh^{D-1} r} - \frac{1}{t} \langle T_1^1 \rangle_b - \frac{D-2}{t} \langle T_2^2 \rangle_b = 0, \]

\[ \frac{\partial_0 \left( t^D \langle T_1^1 \rangle_b \right)}{t^D} + \frac{\partial_1 \left( \sinh^{D-1} r \langle T_1^1 \rangle_b \right)}{\sinh^{D-1} r} - (D-2) \frac{\cosh r}{\sinh r} \langle T_2^2 \rangle_b = 0. \]

- Exterior region

\[ \overline{P}_{x-1/2}^{-\mu} (u_0) \leftrightarrow \overline{Q}_{x-1/2}^{\mu} (u_0) \]

\[ p_{x-1/2}^{-\mu} (u) \leftrightarrow q_{x-1/2}^{\mu} (u) \]
NUMERICAL RESULTS

Inside

Outside

$D = 3, r_0 = 1, 1.5, 2, mt = 1, \xi = 0$

Energy flux

Dirichlet BC

Neumann BC

$D = 3, r_0 = 1, m = 1, \xi = 0$
NUMERICAL RESULTS

- Inside the sphere:
  - \( \langle T_0^i \rangle_b < 0 \)
  - \( \langle T_0^i \rangle_b > 0 \)

- Outside the sphere:
  - \( \langle T_0^i \rangle_b < 0 \)
  - \( \langle T_0^i \rangle_b > 0 \)

Energy flux

- Dirichlet BC
- Neumann BC

Energy density

- Inside the sphere
- Outside the sphere

\( D = 3, r_0 = 1, 1.5, 2, mt = 1, \xi = 0 \)

Dirichlet BC
Neumann BC
NUMERICAL RESULTS

Field squared

\[ D = 3, r_0 = 1, 1.5, 2, mt = 1, \xi = 0 \]

Outside the sphere

\[ \| \varphi \|^2 = 0, 1, 2, 5 \]

Neumann BC

Dirichlet BC

Robin BC

\[ 0, 2, 1, 2 \]

\[ 1, 3 \]

\[ \| \varphi \|^2 = 0 \]

\[ \| \varphi \|^2 = 0 \]

\[ \frac{\langle T_1 \rangle}{b} \]

\[ \frac{\langle T_2 \rangle}{b} \]

\[ \frac{\langle \varphi^2 \rangle}{b} \]

Outside the sphere

\[ D = 3, mt = 1 \]

\[ r_0 = 2, r = 1, 2.5, \xi = 0 \]

Robin BC
SUMMARY

- The Wightman function and the vacuum expectation values (VEVs) of the field squared and of the energy-momentum tensor (EMT) are investigated in the geometry of a spherical shell on background of the Milne Universe. The field obeys the Robin boundary condition on the sphere.
- The boundary-induced contribution is explicitly extracted and the renormalization is done only for the boundary-free contribution.
- Rapidly convergent integral representations are provided for the boundary-induced parts.
- Adiabatic and conformal vacuum states are considered, and the corresponding mode functions are found.
- There is a nonzero energy flux along the radial direction (the off-diagonal component of the VEV of the EMT).
- Depending on the coefficients in the Robin BC, sphere-induced part in the energy density can be either positive or negative.
THANK YOU!