Vacuum polarization around a cosmic string

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Outline

- Global and local cosmic strings
- Gravitational field of a cosmic string
- Vacuum polarization in a boundary-free cosmic string spacetime
- Vacuum polarization by a cosmic string in de Sitter spacetime
- Vacuum polarization in a cosmic string spacetime with a cylindrical boundary
- Induced fermionic current in a (2+1)-dimensional conical space with a circular boundary and magnetic flux

Cosmic strings

- Cosmic strings generically arise within the framework of grand unified theories and could be produced in the early universe as a result of symmetry breaking phase transitions
- Observational data on the cosmic microwave background radiation have ruled out cosmic strings as the primary source for primordial density perturbations
- Cosmic strings are candidates for the generation of a number of interesting physical effects:
  - *Gravitational lensing*
  - *Generation of gravitational waves*
  - *High energy cosmic rays*
  - *Gamma ray bursts*

Global strings

- Simplest theory exhibiting string solutions is that of a complex scalar field described by the Lagrangian

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi - V(\varphi), \; V(\varphi) = \frac{1}{2}\lambda(|\varphi|^2 - \eta^2)^2$$

- Global *U*(1) symmetry: $\varphi \to e^{i\alpha}\varphi \quad \alpha = \text{const}$

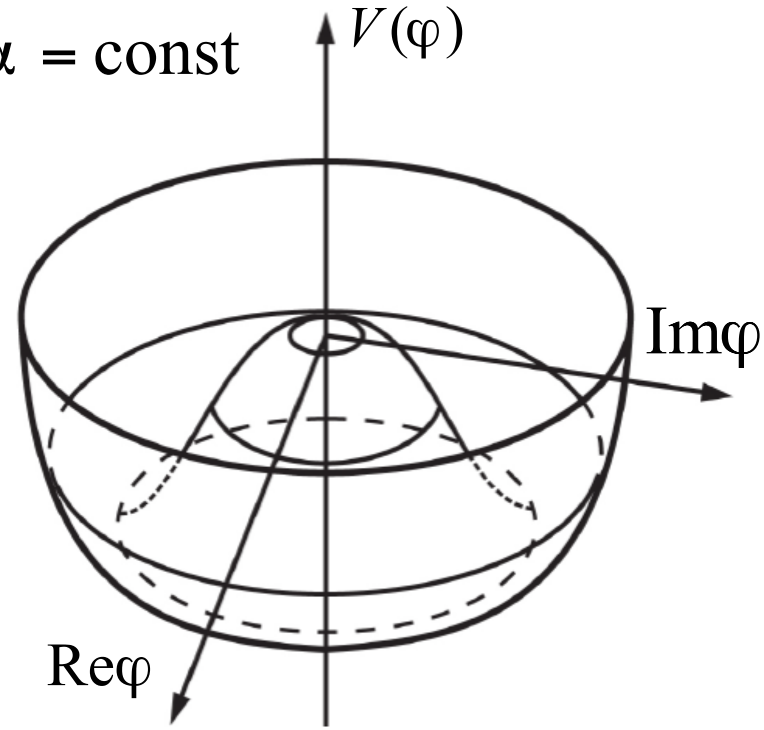
- Field equation:

$$[\Box + \lambda(|\varphi|^2 - \eta^2)]\varphi = 0$$

- Ground state or vacuum solution:

$$\varphi = \eta e^{i\alpha_0} \qquad \alpha_0 = \text{const}$$

- Is not invariant under the *U*(1) symmetry transformation:
Symmetry is broken by the vacuum



Global strings or Vortices

- Cylindrically symmetric ansatz:

$$\varphi = \eta f(m_s\rho)e^{in\phi}, \quad m_s^2 = 2\lambda\eta^2$$

Mass of the scalar particle in the symmetry breaking vacuum

$\{\rho, \phi, z\}$ Cylindrical coordinates $\quad n = \text{integer}$

- From the field equation: $f'' + \frac{1}{\xi}f' - \frac{n^2}{\xi^2}f - \frac{1}{2}(f^2-1)f = 0, \ \xi = m_s\rho$
- Field approaches its ground state $|\varphi| = \eta$ at infinity: $f \to 1$
- At large ξ: $\delta f \sim n^2/\xi^2 \quad (\delta f = f - 1)$
- Energy density: $\mathcal{E} = |\dot{\varphi}|^2 + |\nabla\varphi|^2 + V(\varphi) \propto \xi^{-2}$
- Energy per unit length is infinite:
  Energy inside a cylinder of radius $R \gg m_s^{-1} \Rightarrow 2\pi n^2\eta^2 \ln(m_s R)$
- These solutions are known as global strings or vortices
- Closely related to the vortices in superfluid ^4He (scalar field $\Rightarrow$ wavefunction of the condensed ^4He atoms)

Local or gauge strings

- Local internal symmetry requires the introduction of a vector field
- Lagrangian: $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\varphi|^2 - V(\varphi)$$

$$D_\mu = \partial_\mu + ieA_\mu \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Local U(1) invariance: $\varphi \to e^{i\Lambda(x)}\varphi,\ A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\Lambda(x)$
- Field equations:

$$\begin{aligned}[D^2 + \lambda(|\varphi|^2 - \eta^2)]\varphi &= 0 \\ \partial_\nu F^{\mu\nu} + ie(\varphi^* D^\mu\varphi - \varphi D^\mu\varphi^*) &= 0\end{aligned}$$

- Properties of a vortex solution are quite different from those of the global vortex ➡ *Energy per unit length is finite*

Local or gauge strings (continued)

- Radial gauge $A_\rho = 0$
- General cylindrically symmetric ansatz

$$\varphi = \eta f(m_v \rho) e^{in\phi}, \;\; A^i = \frac{n}{e\rho}\Phi^i a(m_v\rho)$$

$$m_v \;=\; \sqrt{2}e\eta$$ Mass gained by the vector field

- Asymptotic behavior

$$f \;\simeq\; \begin{cases} f_0 \xi^{|n|}, & \xi \to 0 \\ 1 - f_1 \xi^{-1/2}\exp(-\sqrt{\beta}\xi), & \xi \to \infty \end{cases}$$

$$a \;\simeq\; \begin{cases} a_0 \xi^2, & \xi \to 0 \\ 1 - a_1 \xi^{1/2}\exp(-\xi), & \xi \to \infty \end{cases} \quad .$$

Notations

$$\boxed{\begin{aligned} &\xi = m_v \rho \\ &\beta = \lambda/e^2 = (m_s/m_v)^2 \end{aligned}}$$

- Local string also contains a tube of quantized magnetic flux

$$\int d^2x\, \mathbf{B}\cdot\hat{z} = \int_{L_\infty} dx^i A^i = \frac{2\pi n}{e}$$

- Energy per unit length of the local string: $\mu = \int d\rho d\phi\, \rho \mathcal{E}(\rho) = \pi\eta^2 \epsilon(\beta)$

String gravity

- Gravitating string is described by the combined system of Einstein, scalar and gauge field equations
- No exact solutions have been found to date
- For most cosmological applications the problem can be made tractable assuming that the string thickness is much smaller than all other relevant dimensions

String gravity: Simplified model

- String is approximated as a line of zero width with a distributional δ-function energy-momentum tensor

$$T^{\mu\nu} = \mu \mathrm{diag}(1,0,0,-1)\delta(x)\delta(y)$$

- Line element

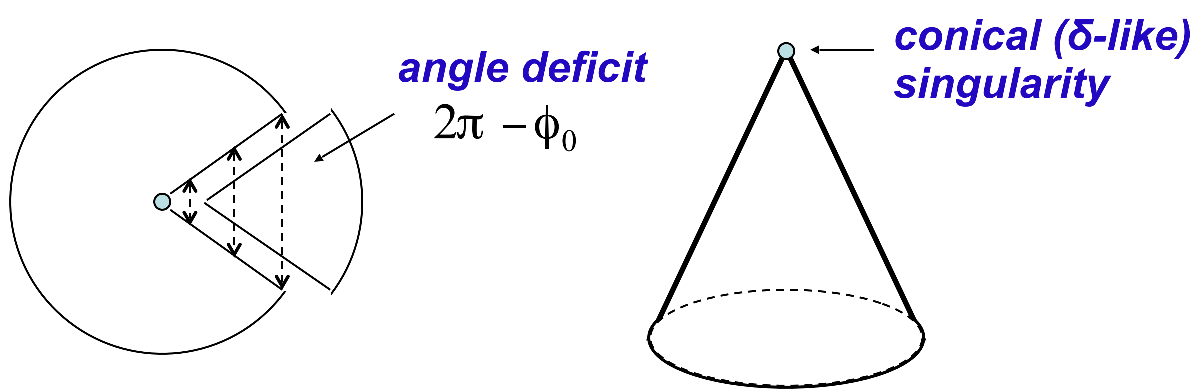
$$ds^2 = dt^2 - d\rho^2 - (1-4G\mu)^2\rho^2 d\phi^2 - dz^2$$

- New angular coordinate $\tilde{\phi} = (1-4G\mu)\phi$ $\quad 0 \le \tilde{\phi} \le 2\pi - \delta, \ \delta = 8\pi G\mu$

- Spacetime is flat everywhere except $\rho = 0$

- Cone in the plane transverse to the string, with an angle deficit δ

Cosmic string



Vacuum polarization by a cosmic string

- Non-trivial topology of the cosmic string spacetime results in the distortion of the zero-point vacuum fluctuations of quantized fields

- Non-zero vacuum expectation values (VEVs) for physical observables (field squared, energy-momentum tensor,...)

VEVs of the field squared and EMT

- Among the most important quantities characterizing the vacuum properties are the vacuum expectation values (VEVs) of the field squared and the energy-momentum tensor

- Though the corresponding operators are local, due to the global nature of the vacuum, the VEVs carry an important information about the global properties of the bulk

- In addition to describing the physical structure of the quantum field at a given point, the VEV of the energy-momentum tensor acts as the source of gravity in the quasiclassical Einstein equations and plays an important role in modelling a self-consistent dynamics involving the gravitational field

Vacuum polarization by a cosmic string: Scalar field

- Field equation

Curvature coupling parameter

$$(\nabla_l \nabla^l + m^2 + \xi R)\varphi = 0$$

Ricci scalar

- For a massless scalar field the renormalized VEV of the field squared has the form (units $\hbar = c = 1$)

$$\langle \varphi^2 \rangle = \frac{q^2 - 1}{48\pi^2 r^2},\ q = 2\pi/\phi_0,\ 0 \leqslant \phi \leqslant \phi_0$$

- Energy-momentum tensor

$$\begin{aligned} \langle T_0^0 \rangle &= \langle T_3^3 \rangle = -\frac{q^2 - 1}{96\pi^2 r^4}\left(8\xi + \frac{q^2 - 19}{15}\right), \\ \langle T_1^1 \rangle &= -\langle T_2^2 \rangle /3 = -\frac{q^2 - 1}{96\pi^2 r^4}\left(\frac{q^2 + 11}{15} - 4\xi\right) \end{aligned}$$

Scalar field: Special cases

- Minimally coupled scalar field ($\xi = 0$)

$$\begin{aligned}\langle T_0^0 \rangle &= \langle T_3^3 \rangle = -\frac{(q^2-1)(q^2-19)}{1440\pi^2 r^4}, \\ \langle T_1^1 \rangle &= -\langle T_2^2 \rangle /3 = -\frac{(q^2-1)(q^2+11)}{1440\pi^2 r^4}\end{aligned}$$

- Conformally coupled scalar field ($\xi = 1/6$)

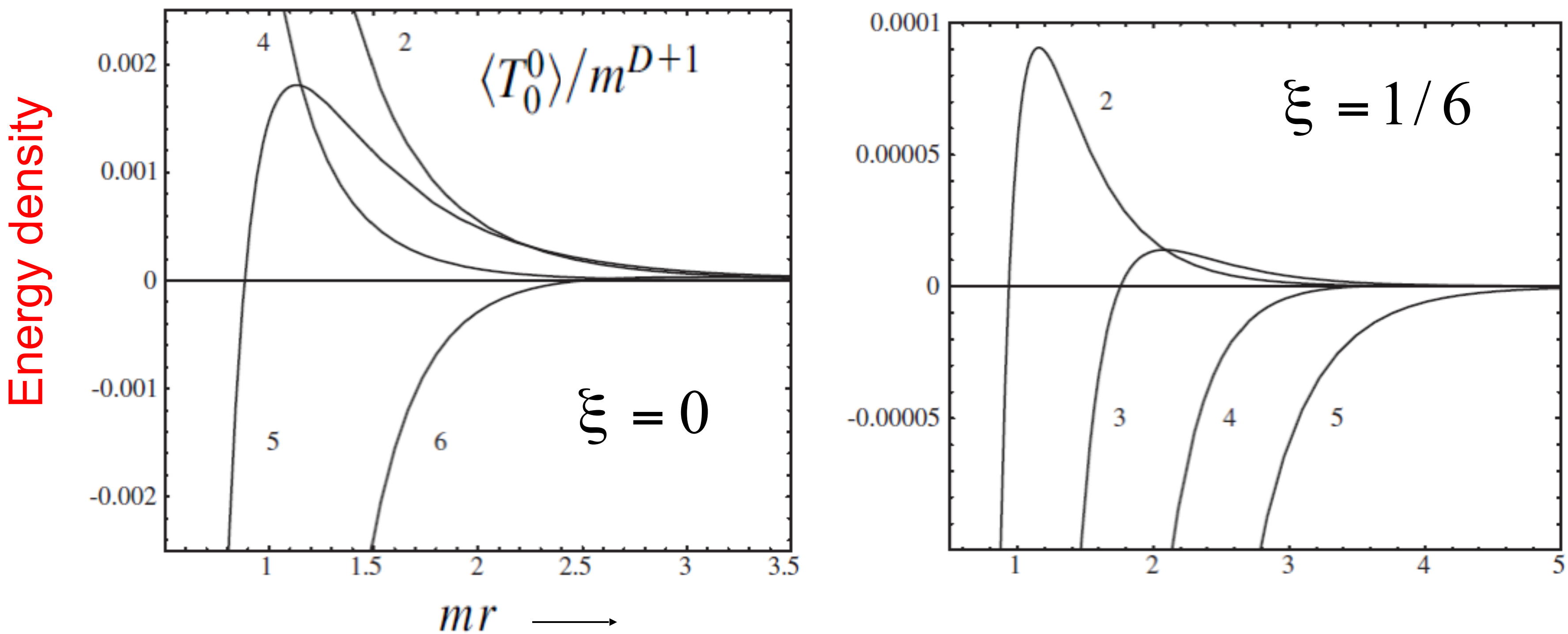
$$\langle T_0^0 \rangle = \langle T_1^1 \rangle = -\langle T_2^2 \rangle /3 = \langle T_3^3 \rangle = -\frac{q^4-1}{1440\pi^2 r^4}$$

- For a conformally coupled field the VEV is traceless

- For integer values of the parameter $q = 2\pi/\phi_0$

$$\langle \varphi^2 \rangle = \frac{m^{D-1}}{(2\pi)^{\frac{D+1}{2}}} \sum_{l=1}^{q-1} \frac{K_{(D-1)/2}(2mry_l)}{(2mry_l)^{(D-1)/2}}, \quad y_l = \sin(\pi l/q)$$

- At large distances $mr \gg 1$: $\langle \varphi^2 \rangle \approx \frac{m^{D/2-1}}{(4\pi r)^{D/2}} \frac{e^{-2mr\sin(\pi/q)}}{\sin^{D/2}(\pi/q)}$



Higher spin fields

- Massless fermionic field

$$\left\langle T_0^0 \right\rangle = \left\langle T_1^1 \right\rangle = -\left\langle T_2^2 \right\rangle/3 = \left\langle T_3^3 \right\rangle = -\frac{(q^2-1)(7q^2+17)}{2880\pi^2 r^4}$$

- Electromagnetic field: Field squared

$$\langle E^2 \rangle = \langle B^2 \rangle = -\frac{(q^2-1)(q^2+11)}{180\pi r^4}$$

- Electromagnetic field: Energy-momentum tensor

$$\left\langle T_0^0 \right\rangle = \left\langle T_1^1 \right\rangle = -\left\langle T_2^2 \right\rangle/3 = \left\langle T_3^3 \right\rangle = -\frac{(q^2-1)(q^2+11)}{720\pi^2 r^4}$$

Cosmic string in de Sitter spacetime

- De Sitter (dS) spacetime has been most studied in quantum field theory during the past two decades
- The main reason resides in the fact that it is maximally symmetric and several physical problems can be exactly solvable on that background
- In great number of inflationary models, approximated dS spacetime is employed to solve relevant problems in standard cosmology
- Quantum fluctuations during an inflationary epoch play an important role in the generation of cosmic structures from inflation
- More recently astronomical observations of high redshift supernovae, galaxy clusters and cosmic microwave background indicate that at the present epoch, the Universe is accelerating and can be well approximated by a world with a positive cosmological constant

Geometry and the field

- Locally dS spacetime with planar angle deficit

$$ds^2 = dt^2 - e^{2t/\alpha}\left(dr^2 + r^2 d\phi^2 + \sum_{i=1}^{N} dz_i^2\right), \phi \in [0,\ 2\pi/q]$$

$D = 3 + N \Longrightarrow$ Spacetime dimension

$$\Lambda = \frac{(D-1)(D-2)}{2\alpha^2}, \quad R = \frac{D(D-1)}{\alpha^2}$$

Cosmological constant
Ricci scalar

- Conformal time $\tau = -\alpha e^{-t/\alpha}, \ -\infty < \tau < 0$

- Scalar field with curvature coupling parameter ξ

$$\left(\nabla_l \nabla^l + m^2 + \xi R\right) \Phi(x) = 0$$

- Field is prepared in the Bunch-Davies vacuum state

VEV of the field squared

- VEV of the field squared is decomposed as

$$\langle \Phi^2 \rangle = \langle \Phi^2 \rangle_{\rm dS} + \langle \Phi^2 \rangle_{\rm s}$$

String induced part

VEV in dS in the absence of cosmic string

- From the maximal symmetry of dS and BD vacuum

$\langle \Phi^2 \rangle_{\rm dS}$ does not depend on spacetime point

- String-induced contribution (topological part)

$\eta = \alpha e^{-t/\alpha}$

$$\langle \Phi^2 \rangle_{\rm s} = \frac{8\alpha^{2-D}}{(2\pi)^{(D+3)/2}} \int_0^\infty dv\, g(q,v) \int_0^\infty dz \frac{e^{-z}}{z} K_{iv}(z) F\left(z\eta^2/r^2\right)$$

$$g(q,v) = \sinh(\pi v) \left(\frac{1}{e^{2\pi v/q} - 1} - \frac{1}{e^{2\pi v} - 1} \right)$$

$$F(x) = x^{\frac{D-1}{2}} e^x K_\nu(x)$$

MacDonald function

- For a conformally coupled massless field

$$\langle \Phi^2 \rangle_{\rm s} = \frac{8(4\pi)^{-(D+1)/2}}{\Gamma((D-1)/2)} \left(\frac{\eta}{\alpha r}\right)^{D-2} \int_0^\infty dv\, g(q,v)\, \Gamma(D/2-1+iv)\Gamma(D/2-1-iv)$$

For even *D* the integral is evaluated by using $\Gamma(1+iv)\Gamma(1-iv) = \pi v / \sinh(\pi v)$

VEV of the field squared: Asymptotics

- Points near the cosmic string: $r/\eta \ll 1 \ (\alpha r/\eta \ll \alpha)$

$$\langle \Phi^2 \rangle_s \approx \left(\frac{\eta}{\alpha r}\right)^{D-2} \int_0^\infty dv\, g(q,v) h(v)$$

$$h(v) = \frac{8(4\pi)^{-(D+1)/2}}{\Gamma((D-1)/2)} \Gamma(D/2-1+iv)\Gamma(D/2-1-iv)$$

At small distances from the string the leading order term of the string induced part in the VEV of the field squared coincides with the corresponding result for a massless conformally coupled field and the effects of gravity are weak

- Large distances from the string: $r/\eta \gg 1$

$$\langle \Phi^2 \rangle_s \approx \frac{\alpha^{2-D}\Gamma(\nu)}{\pi^{D/2+1}\Gamma(D/2-\nu)} \left(\frac{\eta}{2r}\right)^{D-1-2\nu} \int_0^\infty dv\, g(q,v) \Gamma\left(\frac{D-1}{2}-\nu+iv\right) \Gamma\left(\frac{D-1}{2}-\nu-iv\right)$$

ν is real

$$\langle \Phi^2 \rangle_s \approx -\frac{4B(q,|\nu|)}{(4\pi)^{D/2}\alpha^{D-2}} \left(\frac{\eta}{r}\right)^{D-1} \sin[2|\nu| \ln(\eta/2r) + \psi_0]$$

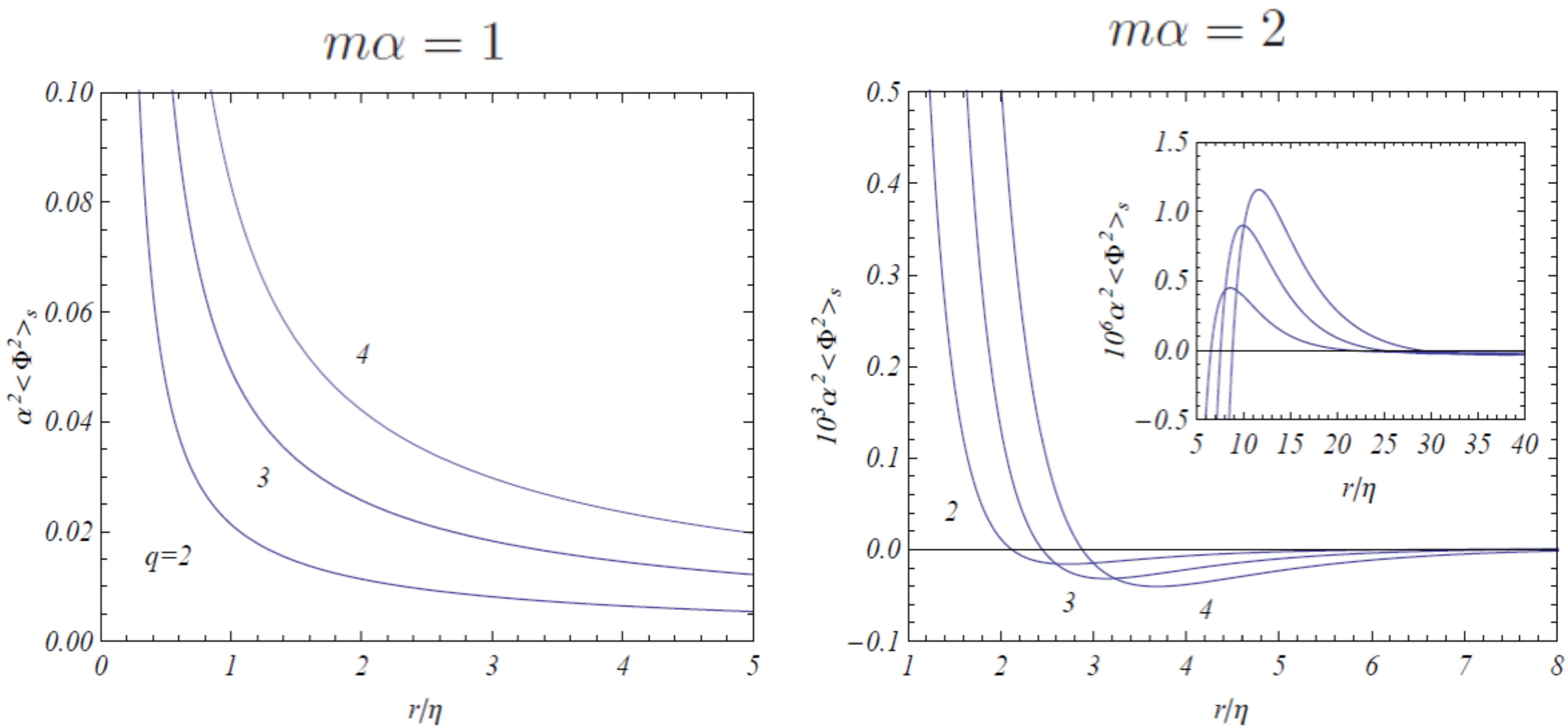
ν is imaginary

$$\nu = \sqrt{(D-1)^2/4 - \xi D(D-1) - m^2\alpha^2}$$

Behavior is essentially different from that in Minkowski bulk (exponential suppression for a massive field)

VEV of the field squared: Numeric

D=4 minimally coupled field



VEV of the energy-momentum tensor

- Vacuum energy-momentum tensor is decomposed as

$$\langle T_{\mu\nu}\rangle = \langle T_{\mu\nu}\rangle_{\mathrm{dS}} + \langle T_{\mu\nu}\rangle_{\mathrm{s}}$$

String induced part

VEV in dS in the absence of cosmic string

- Diagonal components

$$\langle T_{\mu}^{\mu}\rangle_{\mathrm{s}} = A - \frac{8\alpha^{-D}}{(2\pi)^{(D+3)/2}} \int_0^\infty dv\, g(q,v) \int_0^\infty dz\, e^{-yz} K_{iv}(yz) g^{(\mu)}(z)$$

$$A = \frac{8\alpha^{-D}}{(2\pi)^{(D+3)/2}} \int_0^\infty dv\, g(q,v) \int_0^\infty dz e^{-yz} K_{iv}(yz) \qquad F(x) = x^{\frac{D-1}{2}} e^x K_\nu(x)$$

$$\times \left\{ (4\xi - 1)\left[\left(1-\frac{1}{y}\right)[zF'(z)]' + \frac{1-D}{2}F'(z)\right] - \xi\frac{D-1}{z}F(z) \right\}$$

- Nonzero off-diagonal component

$$\langle T_1^0\rangle_{\mathrm{s}} = -\frac{8\alpha^{-D}\eta/r}{(2\pi)^{(D+3)/2}} \int_0^\infty dv\, g(q,v) \int_0^\infty dz\, K_{iv}(zy) e^{-zy} \left\{(4\xi-1)[zF'(z)]' + 2\xi F'(z)\right\}$$

Energy flux along the radial direction

Energy-momentum tensor: Asymptotics

- Near the cosmic string: $r/\eta \ll 1 \; (\alpha r/\eta \ll \alpha)$

$$\langle T_0^0\rangle_{\mathrm{s}} \approx \langle T_3^3\rangle_{\mathrm{s}} \approx -\left(\frac{\eta}{\alpha r}\right)^D \int_0^\infty dv\, g(q,v)h(v)\left[(D-2)^2(\xi-\xi_c)+\frac{v^2}{D-1}\right],$$

$$\langle T_1^1\rangle_{\mathrm{s}} \approx \frac{\langle T_2^2\rangle_{\mathrm{s}}}{1-D} \approx \left(\frac{\eta}{\alpha r}\right)^D \int_0^\infty dv\, g(q,v)h(v)\left[(D-2)(\xi-\xi_c)-\frac{v^2}{D-1}\right],$$

$$\langle T_1^0\rangle_{\mathrm{s}} \approx -(D-2)(\xi-\xi_c)\frac{D-1}{\alpha}\left(\frac{\eta}{\alpha r}\right)^{D-1} \int_0^\infty dv\, g(q,v)h(v)$$

- Large distances from the $r/\eta \gg 1$ ν is real

$$\langle T_0^0\rangle_{\mathrm{s}} \approx -[(D-1-2\nu)(\xi-1/4)+\xi]\frac{(D-1)\alpha^{-D}\Gamma(\nu)}{\pi^{D/2+1}\Gamma(D/2-\nu)}\left(\frac{\eta}{2r}\right)^{D-1-2\nu}$$
$$\times\int_0^\infty dv\, g(q,v)\Gamma\left(\frac{D-1}{2}-\nu+iv\right)\Gamma\left(\frac{D-1}{2}-\nu-iv\right).$$

$$\langle T_\mu^\mu\rangle_{\mathrm{s}} \approx \frac{2\nu}{D-1}\langle T_0^0\rangle_{\mathrm{s}},\ \mu=1,2,\dots,D-1,\ \langle T_1^0\rangle_{\mathrm{s}} \approx \frac{\eta}{r}\frac{D-1-2\nu}{D-1}\langle T_0^0\rangle_{\mathrm{s}}$$

- Large distances from the string, ν is imaginary

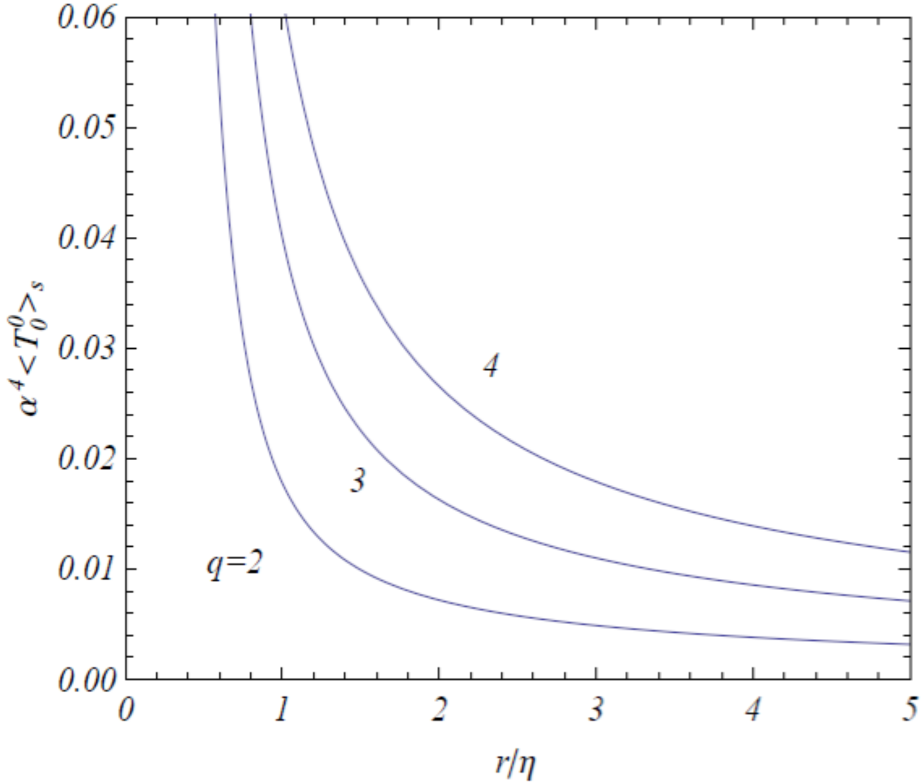
Diagonal components $\Rightarrow$ $(\eta/r)^{D-1}\sin[2|\nu|\ln(\eta/2r)+\psi_1]$

Off-diagonal component $(\eta/r)^D$

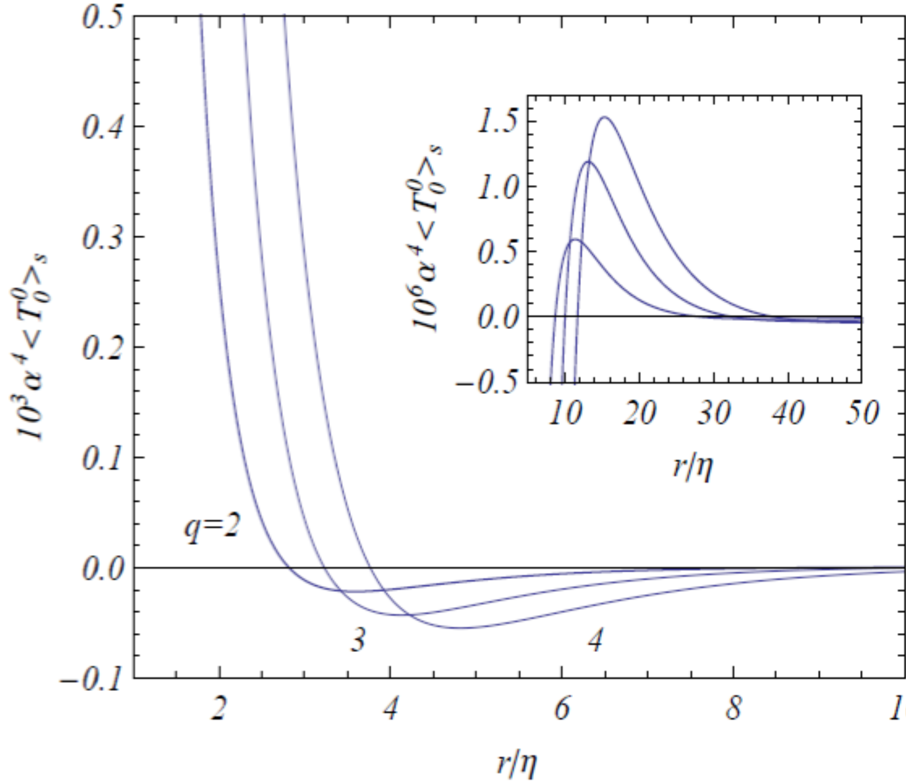
Energy density: Numeric

D=4 minimally coupled field

$m\alpha = 1$



$m\alpha = 2$



Cosmic string in de Sitter spacetime (D=4): Fermionic field

- Field equation $i\gamma^{\mu}\nabla_{\mu}\psi - m\psi = 0\,,\ \nabla_{\mu} = \partial_{\mu} + \Gamma_{\mu}$
- String-induced contribution in fermionic condensate

$$\langle\bar{\psi}\psi\rangle_{\rm s} = \langle 0|\bar{\psi}\psi|0\rangle - \langle 0|\bar{\psi}\psi|0\rangle_{\rm dS}$$

$$\langle\bar{\psi}\psi\rangle_{\rm s} = \frac{4\sqrt{2}}{\pi^{7/2}\alpha^{3}}\int_{0}^{\infty} du\, g(q,u)\int_{0}^{\infty} dy\, y^{3/2} e^{y(1-r^2/\eta^2)} \times {\rm Im}[K_{1/2-iu}(yr^2/\eta^2)]\,{\rm Im}\left[K_{1/2-im\alpha}(y)\right]$$

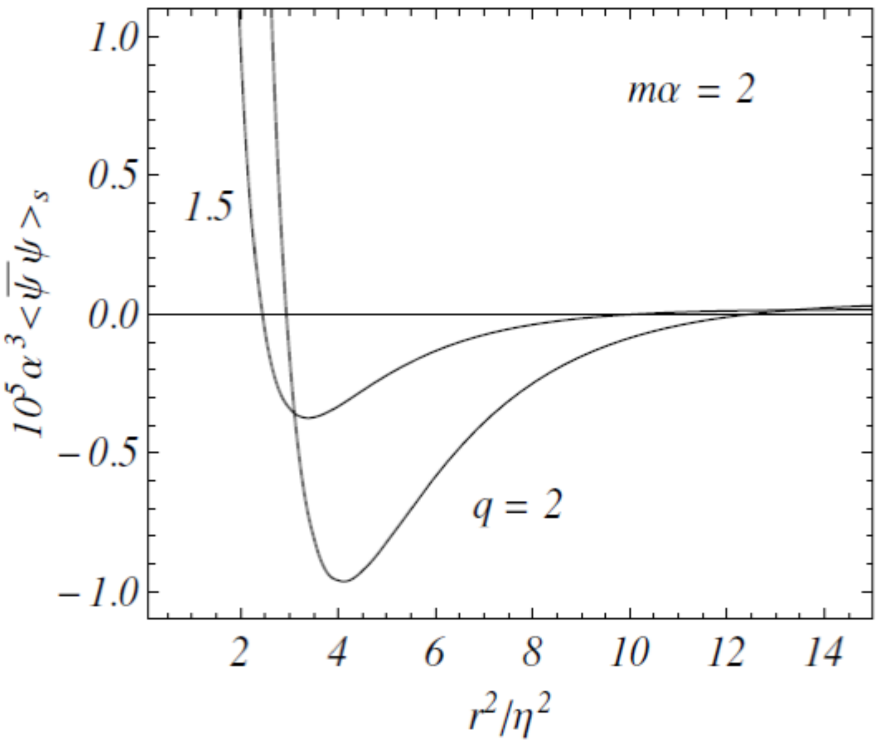
$$g(q,u) = \cosh(u\pi)\left(\frac{1}{e^{2\pi u/q}+1} - \frac{1}{e^{2\pi u}+1}\right)$$

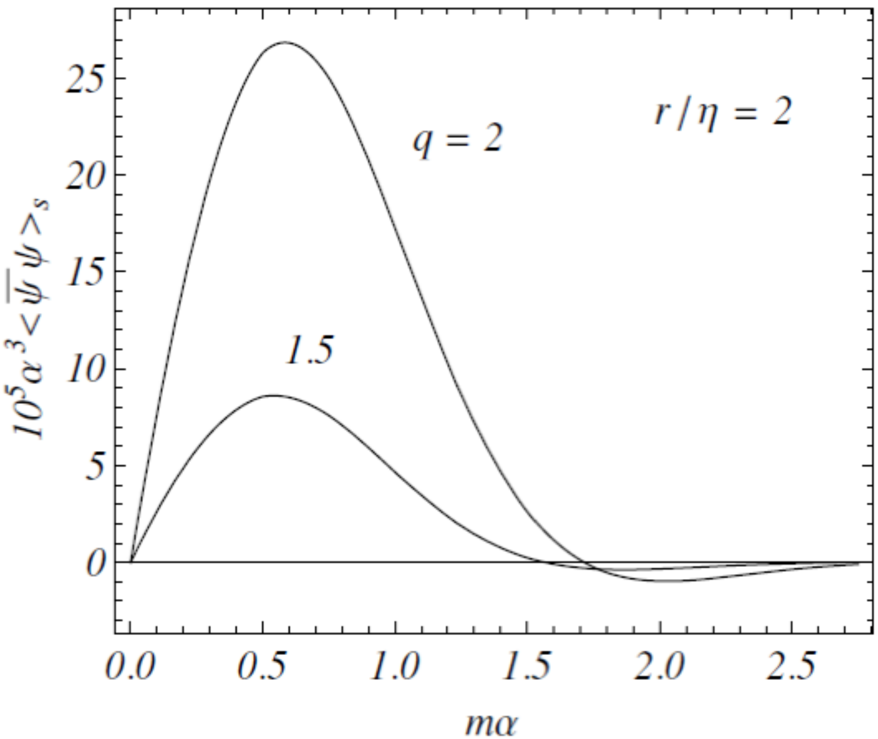
- Near the cosmic string: $\langle\bar{\psi}\psi\rangle_{\rm s} \approx \dfrac{m(q^2-1)}{24\pi^2(\alpha r/\eta)^2},\ r/\eta \ll 1$

- Large distances from the string:

$$\langle\bar{\psi}\psi\rangle_{\rm s} \approx \frac{\alpha A(q,m\alpha)}{\pi^3(\alpha r/\eta)^4}\sin\left[2m\alpha\ln(2r/\eta) - \varphi_0\right],\ r/\eta \gg 1$$

String-induced fermionic condensate





Energy-momentum tensor

- String-induced contribution in the energy-momentum tensor

$$\langle T_\mu^\nu \rangle_{\rm s} = \langle 0|T_\mu^\nu|0\rangle - \langle 0|T_\mu^\nu|0\rangle_{\rm dS}$$

- Energy-momentum tensor is diagonal

$$\langle T_\mu^\mu \rangle_{\rm s} = \frac{4\alpha^{-4}}{2^{1/2}\pi^{7/2}} \int_0^\infty dx\, g(q,x) \int_0^\infty dy\, y^{3/2} e^{y(1-r^2/\eta^2)} \times \mathrm{Im}[K_{1/2-ix}(yr^2/\eta^2)]\mathrm{Re}\left[K_{1/2-im\alpha}(y)\right] \quad \mu = 0,1,3$$

$$\langle T_2^2 \rangle_{\rm s} = \frac{8\alpha^{-4}}{2^{1/2}\pi^{7/2}} \int_0^\infty dx\, x g(q,x) \int_0^\infty dy\, y^{3/2} e^{y(1-r^2/\eta^2)} \times \mathrm{Re}\left[K_{1/2-ix}(yr^2/\eta^2)\right] \mathrm{Re}\left[K_{1/2-im\alpha}(y)\right]$$

- Massless field: $\langle T_\mu^\nu \rangle_{\rm s} = -\dfrac{(q^2-1)(7q^2+17)}{2880\pi^2(\alpha r/\eta)^4}\mathrm{diag}(1,1,-3,1)$
- Near the string the leading term coincides with
- Large distances from the string:

$$\langle T_\mu^\mu \rangle_{\rm s} \approx -\frac{A(q,m\alpha)}{2\pi^3(\alpha r/\eta)^4}\cos\left[2m\alpha\ln(2r/\eta) - \varphi_0\right],\ \mu = 0,1,3,$$

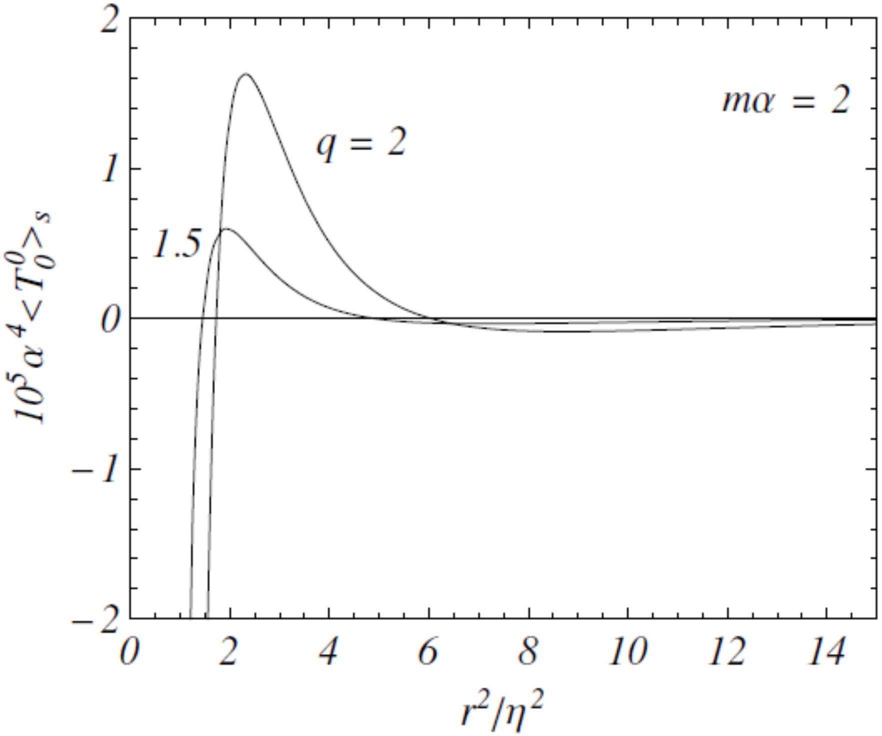
$$\langle T_2^2 \rangle_{\rm s} \approx \frac{A(q,m\alpha)}{\pi^3(\alpha r/\eta)^4}\sqrt{9/4+m^2\alpha^2}\sin[2m\alpha\ln(2r/\eta) - \varphi_0 + \varphi_1]$$

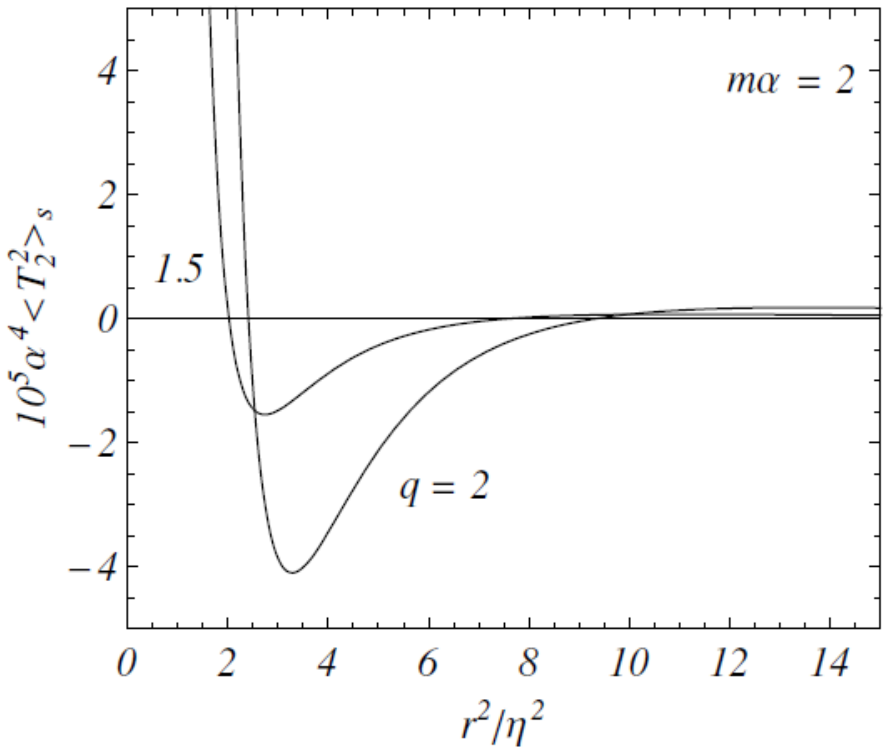
Energy-momentum tensor: Numeric

Energy density

Azimuthal stress

$$m\alpha = 2$$





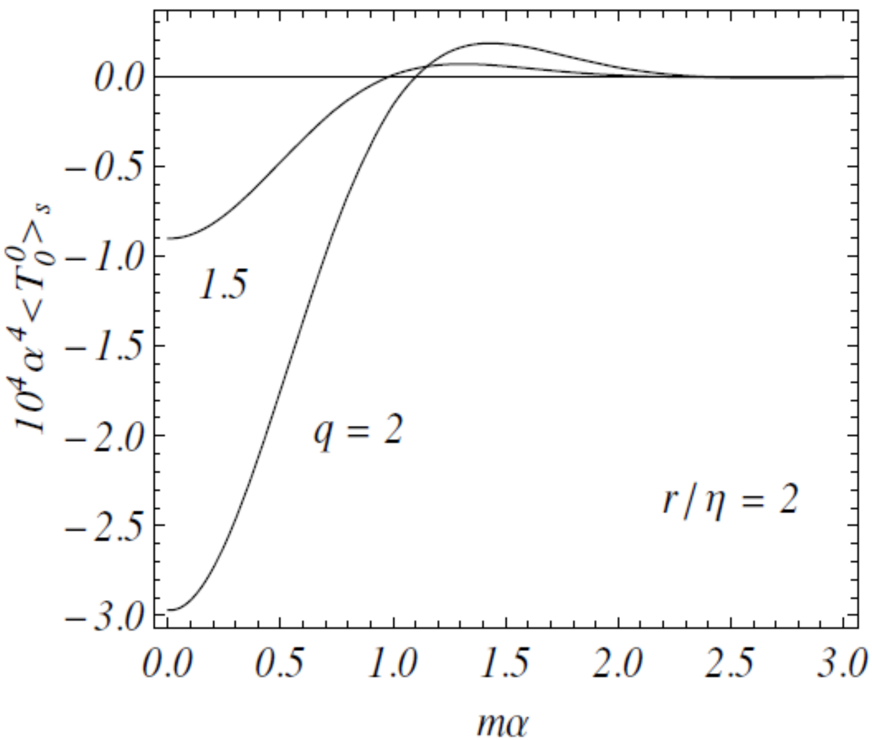
r^2/η^2

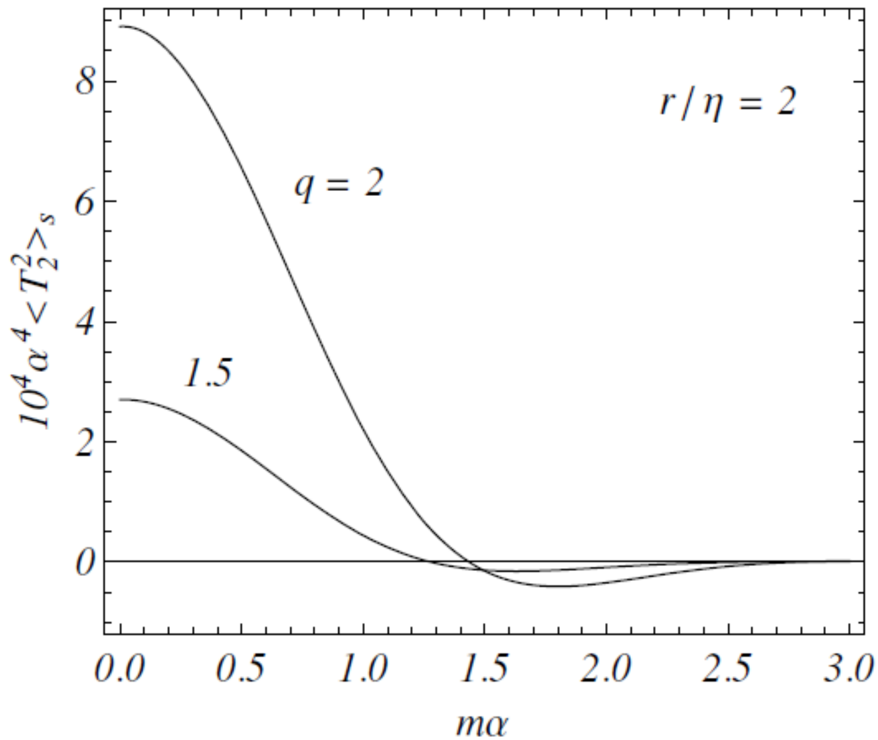
Energy-momentum tensor: Numeric

Energy density

Azimuthal stress

$r/\eta = 2$





$m\alpha$

Boundaries

- In many problems the physical model is considered on background of manifolds with boundaries on which the dynamical variables satisfy some prescribed boundary conditions

- In the presence of boundaries, new degrees of freedom appear which can essentially influence the dynamics of the model

- Branches where the boundary effects play an important role

  - *Surface and finite-size effects in condensed matter*
  - *Phase transitions and critical phenomena*
  - *Bag models of hadrons in QCD*
  - *String and M theories, braneworld models*
  - *Horizons and the thermodynamics of black holes*
  - *AdS/Conformal Field Theory (CFT) correspondence*
  - *Holographic principle*
  - *….*

Boundaries and QFT vacuum

- In Quantum Field Theory the influence of boundaries on the vacuum state of a quantized field leads to interesting physical consequences

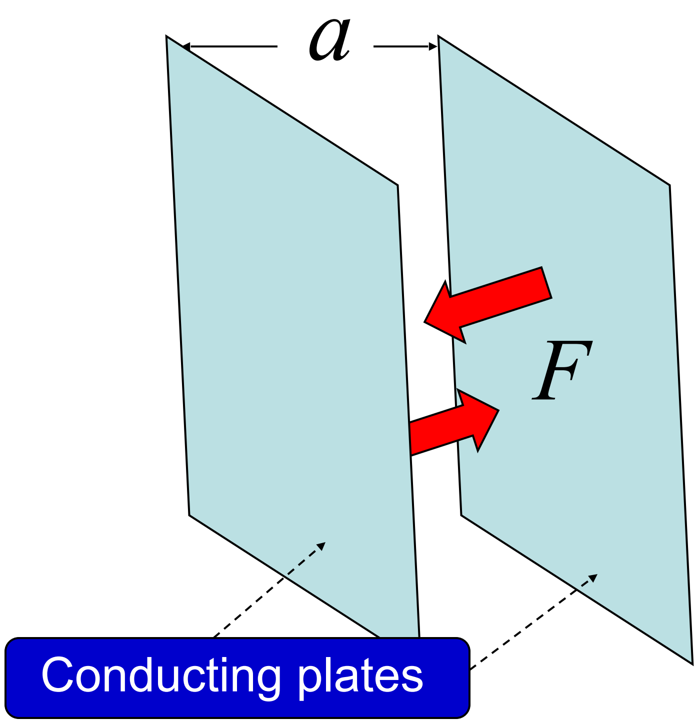
- Imposition of boundary conditions leads to the modification of the zero-point fluctuations spectrum and results in the shifts in the VEVs of physical quantities (energy density, vacuum stresses)

- Vacuum forces arise acting on constraining boundaries

  Casimir effect

- Features of the resulting vacuum forces depend on
  - *Nature of the quantum field*
  - *Bulk ans boundary geometries*
  - *Boundary conditions imposed on the field*

Casimir effect for parallel conducting plates



Two conducting neutral parallel plates in the vacuum attract by the force per unit surface (Casimir, 1948)

$$\boxed{F = \frac{\pi^2 \hbar c}{240 a^4}}$$

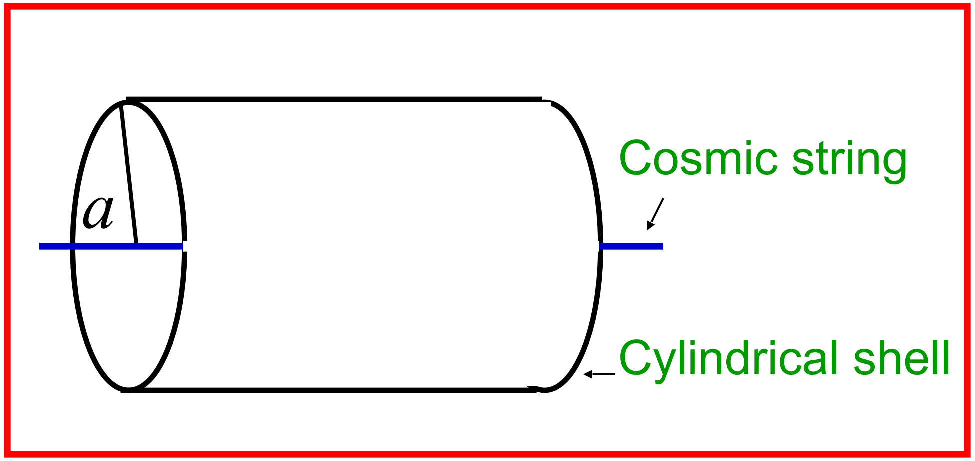
Combined effects from topology and boundaries

- An interesting topic in the investigations of the Casimir effect is its dependence on the geometry of the background spacetime

- Relevant information is encoded in the vacuum fluctuations spectrum and analytic solutions can be found for highly symmetric geometries only

- Two sources of vacuum polarization in the geometry of a cosmic string:
  - *Non-trivial topology due to the cosmic string*
  - *Boundaries*

Cylindrical boundary in cosmic string spacetime: Scalar field



- Line element

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - \sum_{i=1}^{N} dz_i{}^2$$

$$0 \leq \phi \leq \phi_0$$

- Massive scalar field: $(\nabla^i \nabla_i + m^2 + \xi R)\varphi(x) = 0$ Curvature coupling parameter

- Special cases: Minimal coupling $\xi = 0$

  Conformal coupling $\xi = \xi_D \equiv (D-1)/4D$

- Robin boundary condition on the cylindrical shell

$$\left(A + B\frac{\partial}{\partial r}\right)\varphi = 0, \qquad r = a$$

- Special cases of BC: Dirichlet (*B=0*) and Neumann (*A=0*)

Motivation

- From the point of view of the physics in the region outside the string, the geometry considered can be viewed as a simplified model for the nontrivial core
- This model presents a framework in which the influence of the finite core effects on physical processes in the vicinity of the cosmic string can be investigated
- It enables to specify conditions under which the idealized model with the core of zero thickness can be used
- Corresponding results may shed light upon features of finite core effects in more realistic models, including those used for string-like defects in crystals and superfluid helium
- In addition, the problem considered is of interest as an example with combined topological and boundary-induced quantum effects in which the vacuum characteristics such as energy density and stresses can be found in closed analytic form

Wightman function

- Among the most important local characteristics of the vacuum state are the vacuum expectation values (VEVs) of the field squared, $\langle\varphi^2\rangle$, and the energy-momentum tensor $\langle T_{ik}\rangle$
- These VEVs are obtained from the positive frequency Wightman function in the coincidence limit of the arguments
- In addition, WF determines the response of a particle detector in an arbitrary state of motion
- WF is presented as the mode sum

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \sum_{\alpha}\varphi_{\alpha}(x)\varphi_{\alpha}^{*}(x')$$

$\{\varphi_{\alpha}(x),\, \varphi_{\alpha}^{*}(x)\}$ Complete set of solutions to classical field equation obeying the boundary condition on the cylindrical shell

Mode functions and mode sum for WF

- Mode functions: $\varphi_\alpha(x) = \beta_\alpha J_{q|n|}(\gamma r)\exp(iqn\phi + i\mathbf{k}\mathbf{r}_\parallel - i\omega t)$

$$n = 0, \pm 1, \pm 2, \ldots, \mathbf{k} = (k_1, \ldots, k_N), -\infty < k_j < \infty,$$

$$\omega = \sqrt{\gamma^2 + k^2 + m^2},\ q = 2\pi/\phi_0,$$

- Eigenvalues for γ : $\gamma = \lambda_{n,j}/a, \qquad j = 1, 2, \ldots,$

$\lambda_{n,j}$ ➡ positive zeros of the function $\bar{J}_{q|n|}(z)$: $\bar{J}_{q|n|}(\lambda_{n,j}) = 0$

Notation: $\bar{f}(z) = Af(z) + (B/a)zf'(z)$

- Mode sum for the WF:

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = 2\int d^N\mathbf{k}\, e^{i\mathbf{k}(\mathbf{r}_\parallel - \mathbf{r}'_\parallel)} \sum_{n=0}^{\infty}{}' \cos[qn(\phi - \phi')]$$

$$\times \sum_{j=1}^{\infty} \beta_\alpha^2 J_{qn}(\gamma r) J_{qn}(\gamma r') e^{-i\omega(t-t')}|_{\gamma=\lambda_{n,j}/a}$$

- Is not convenient for the evaluation ov the VEVs:
  - No explicit expressions for $\lambda_{n,j}$
  - Summands in the series over *j* are strongly oscillating functions for large values of *j*

Generalized Abel-Plana formula

- Generalized Abel-Plana summation formula

$$\sum_{j=1}^{\infty} T_{qn}(\lambda_{n,j}) f(\lambda_{n,j}) = \frac{1}{2}\int_0^{\infty} dz f(z) - \frac{1}{2\pi}\int_0^{\infty} dz \frac{\bar{K}_{qn}(z)}{\bar{I}_{qn}(z)}$$

$$\times \left[e^{-qn\pi i} f(ze^{(\pi i)/2}) + e^{qn\pi i} f(ze^{-\lfloor(\pi i)/2\rfloor})\right]$$

- *Advantages*:
  - Explicit form for the normal modes is not necessary
  - First integral corresponds to the physical quantity in the situation without boundaries
  - Boundary-induced part is presented in terms of exponentially convergent integrals convenient for numerical evaluations
  - For the local characteristics of the vacuum (energy density, vacuum stresses) at the points away from the boundary the renormalization is needed for the boundary free part only

Decomposed form of WF

- After the application of GAPF, WF is presented in the decomposed from:

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \underbrace{\langle 0_s|\varphi(x)\varphi(x')|0_s\rangle}_{\text{WF for the geometry without cylindrical boundary}} + \underbrace{\langle \varphi(x)\varphi(x')\rangle_a}_{\text{Induced by the cylindrical boundary}}$$

- Boundary induced part

$$\langle \varphi(x)\varphi(x')\rangle_a = -\frac{2}{\pi\phi_0}\int \frac{d^N\mathbf{k}}{(2\pi)^N} e^{i\mathbf{k}(\mathbf{r}_\parallel - \mathbf{r}'_\parallel)} \int_{\sqrt{k^2+m^2}}^{\infty} dz \, \frac{z\cosh[(t-t')\sqrt{z^2-k^2-m^2}]}{\sqrt{z^2-k^2-m^2}}$$
$$\times \sum_{n=0}^{\infty}{}' \cos[qn(\phi-\phi')] I_{qn}(zr) I_{qn}(zr') \frac{\bar{K}_{qn}(za)}{\bar{I}_{qn}(za)}.$$

- For points away from boundary the boundary induced part is finite in the coincidence limit

WF outside a cylindrical shell

- Mode functions are obtained from the interior ones by

$$J_{qn}(\gamma r) \rightarrow g_{qn}(\gamma r, \gamma a) \equiv J_{qn}(\gamma r)\bar{Y}_{qn}(\gamma a) - \bar{J}_{qn}(\gamma a)Y_{qn}(\gamma r)$$

- Decomposed form for WF

$$\langle 0|\varphi(x)\varphi(x')|0\rangle = \langle 0_s|\varphi(x)\varphi(x')|0_s\rangle + \langle \varphi(x)\varphi(x')\rangle_a$$

- Boundary induced part

$$\langle \varphi(x)\varphi(x')\rangle_a = -\frac{2}{\pi\phi_0}\int \frac{d^N\mathbf{k}}{(2\pi)^N} e^{i\mathbf{k}(\mathbf{r}_\parallel - \mathbf{r}'_\parallel)} \int_k^\infty dz \frac{z\cosh[(t-t')\sqrt{z^2-k^2}]}{\sqrt{z^2-k^2}}$$

$$\times \sum_{n=0}^{\infty}{}' \cos[qn(\phi-\phi')]K_{qn}(zr)K_{qn}(zr')\frac{\bar{I}_{qn}(za)}{\bar{K}_{qn}(za)}$$

VEV of the field squared

- VEV of the field squared: $\langle 0|\varphi^2|0\rangle = \langle 0_s|\varphi^2|0_s\rangle + \langle \varphi^2\rangle_a$

$$\langle \varphi^2\rangle_a = -\frac{A_D}{\phi_0}\sum_{n=0}^{\infty}{}' \int_m^\infty dz z(z^2-m^2)^{(D-3)/2}\,\frac{\bar{K}_{qn}(za)}{\bar{I}_{qn}(za)} I_{qn}^2(zr), \quad r<a$$

$$A_D = \frac{2^{3-D}\pi^{(1-D)/2}}{\Gamma(\frac{D-1}{2})}$$

$$I_{qn} \leftrightarrow K_{qn}, \quad r>a$$

- Boundary induced part is negative for Dirichlet scalar and is positive for Neumann scalar

- Near the string, $r \ll a$,

$$\langle \varphi^2\rangle_a \approx -\frac{A_D}{2a^{D-1}\phi_0}\int_{ma}^\infty dz z(z^2-m^2a^2)^{(D-3)/2}\frac{\bar{K}_0(z)}{\bar{I}_0(z)}$$

Total VEV is dominated by the boundary-free part and is positive

Field squared in asymptotic regions

- Diverges on the cylindrical shell
- Near the shell: $\langle\varphi^2\rangle_a \approx -\dfrac{(2\delta_{B0}-1)\Gamma(\frac{D-1}{2})}{(4\pi)^{(D+1)/2}(a-r)^{D-1}}$

Total VEV is dominated by the boundary part

- Large distances from the cylindrical case, $r \gg a$:

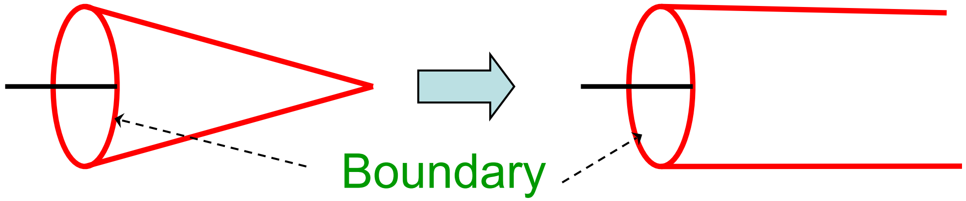
$$\langle\varphi^2\rangle_a \approx -\frac{\pi^{1-D/2}}{2^D\phi_0 r^{D-1}\ln(r/a)}\frac{\Gamma^2(\frac{D-1}{2})}{\Gamma(D/2)} \qquad A \neq 0$$

$$\langle\varphi^2\rangle_a \approx \frac{\pi^{1-D/2}(D-1)}{\phi_0 a^{D-1}}\frac{\Gamma^2(\frac{D+1}{2})}{\Gamma(D/2)}\left(\frac{a}{2r}\right)^{D+1} \qquad A = 0$$

Neumann BC

Total VEV is dominated by the boundary-free part $(\sim 1/r^{D-1})$

- Limiting case: $\phi_0 \to 0,\ r,\ a \to \infty,\ a-r$ and $a\phi_0 \equiv a_0$ are fixed



VEV of the energy-momentum tensor

- VEV of EMT $\langle 0|T_{ik}(x)|0\rangle = \lim_{x'\to x}\nabla_i\nabla'_k\langle 0|\varphi(x)\varphi(x')|0\rangle$

$$+\left[\left(\xi-\frac{1}{4}\right)g_{ik}\nabla^l\nabla_l-\xi\nabla_i\nabla_k\right]\langle 0|\varphi^2(x)|0\rangle$$

- Decomposed form: $\langle 0|T_{ik}|0\rangle = \langle 0_s|T_{ik}|0_s\rangle + \langle T_{ik}\rangle_a$
- Off-diagonal components vanish
- Diagonal components (interior region)

$$\langle T_i^i\rangle_a = \frac{A_D}{\phi_0}\sum_{n=0}^{\infty}{}' \int_m^{\infty} dz z^3(z^2-m^2)^{(D-3)/2}\frac{\bar{K}_{qn}(za)}{\bar{I}_{qn}(za)}F_{qn}^{(i)}[I_{qn}(zr)]$$

$$F_{qn}^{(1)}[f(y)] = \frac{1}{2}f'^2(y) + \frac{2\xi}{y}f(y)f'(y) - \frac{1}{2}\left(1+\frac{q^2n^2}{y^2}\right)f^2(y)$$

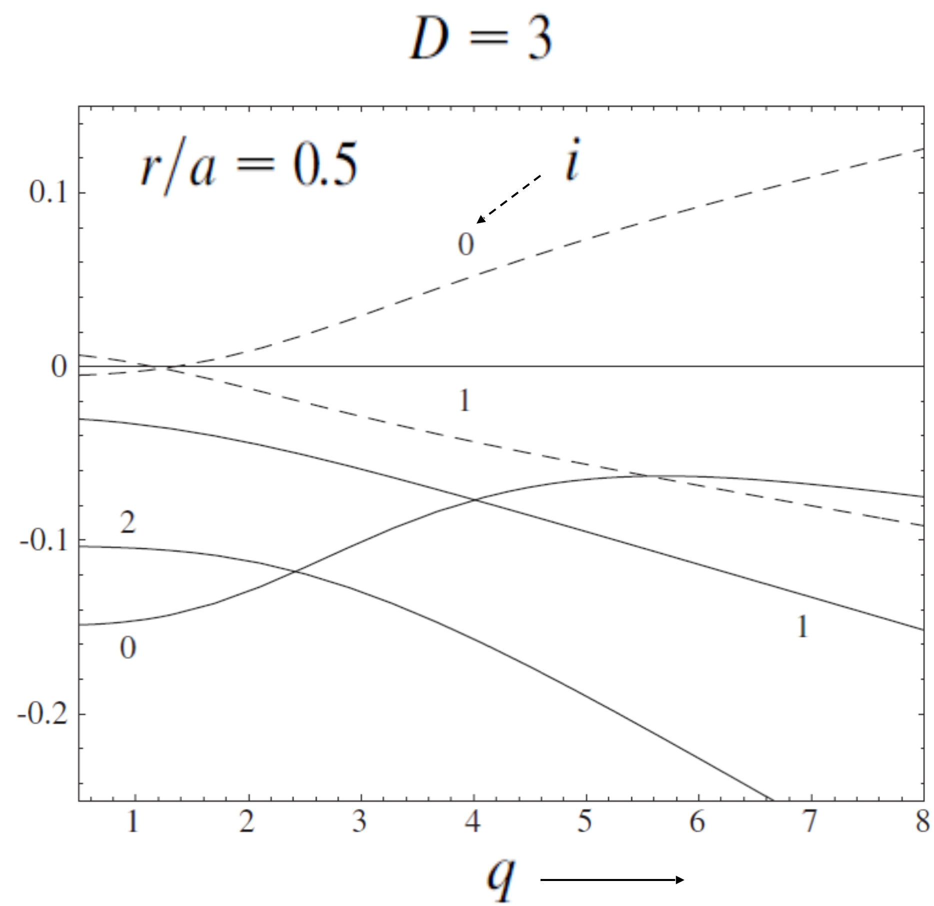
$$F_{qn}^{(i)}[f(y)] = F_{qn}^{(0)}[f(y)], \qquad i = 3,\ldots,D$$

- In the exterior region ➡ interchange $I_{qn} \leftrightarrow K_{qn}, \quad r > a$

Large deficit angle

- For $\phi_0 << 2\pi$ $(q >> 1)$ the contribution of the terms with $n \neq 0$ is suppressed by the factor $q^{(D-1)/2}(r/a)^{2q}$
- Main contribution comes from $n = 0$ term with the linear dependence on q
- Boundary-free part in the VEV of the energy-momentum tensor behaves as q^{D+1} and the total energy-momentum tensor is dominated by this part

$a^{D+1}\langle T_i^i \rangle_a$



Conformal

Minimal

VEV near the boundary

- Boundary part diverges on the cylindrical shell
- Inside the shell the leading terms in the asymptotic expansion

$$\langle T_i^i \rangle_a \approx \frac{D(\xi - \xi_D)(2\delta_{B0} - 1)}{2^D \pi^{(D+1)/2}(a-r)^{D+1}} \Gamma\left(\frac{D+1}{2}\right), \quad i = 0, 2, \ldots, D$$

$$\langle T_1^1 \rangle_a \sim (a-r)^{-D}$$

- In the exterior region: $a - r \rightarrow r - a$
- Leading divergence does not depend on the angle deficit and coincides with the corresponding one for a cylindrical surface in the Minkowski bulk
- Near the boundary the total VEV is dominated by the boundary-induced part

Large distances

- For a massless scalar field with $A \neq 0$

$$\langle T_0^0 \rangle_a \approx -(D-1)\langle T_1^1 \rangle_a \approx \frac{D-1}{D}\langle T_2^2 \rangle_a$$

$$\approx \frac{\Gamma^2(\frac{D+1}{2})}{(4\pi)^{D/2-1}\Gamma(D/2)} \frac{\xi - \xi_D}{\phi_0 r^{D+1}\ln(r/a)}$$

- For Neumann BC $\sim 1/r^{D+3}$

- For a massive field

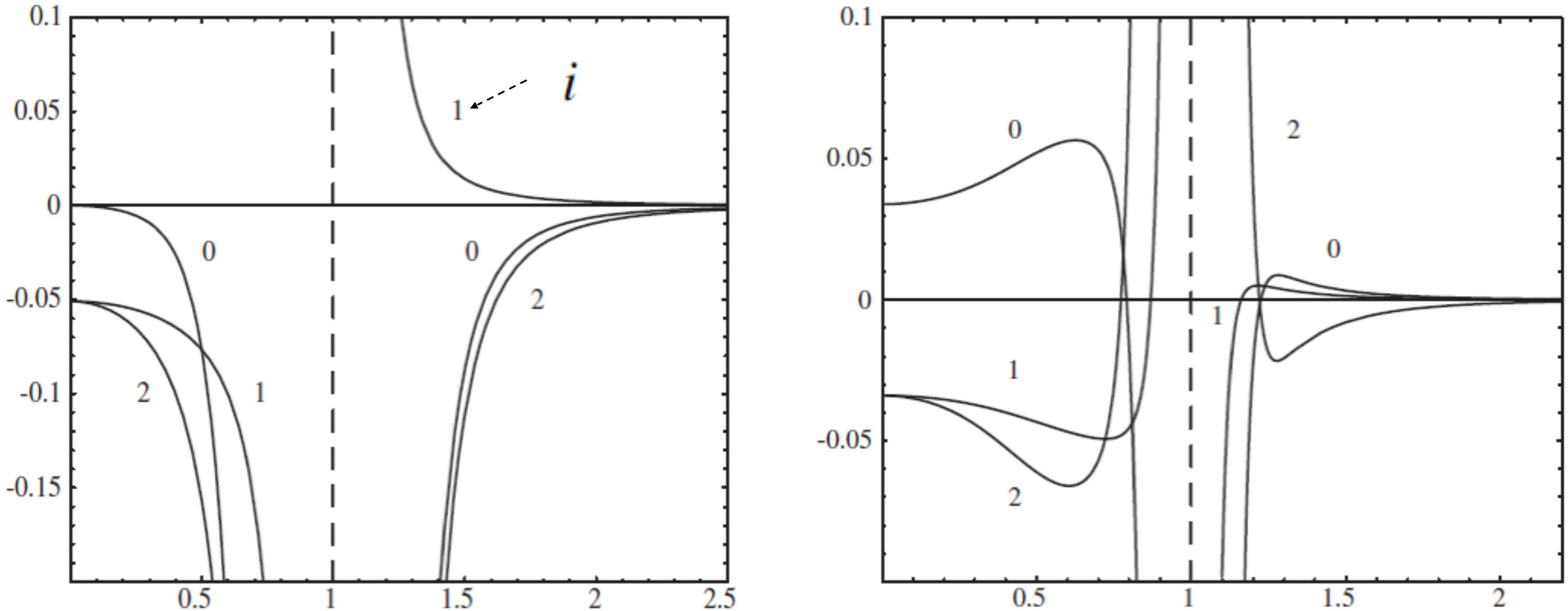
$$\langle T_0^0 \rangle_a \approx -2mr\langle T_1^1 \rangle_a \approx \langle T_2^2 \rangle_a$$

$$\approx \frac{(4\xi - 1)e^{-2mr}}{2^{D-1}\pi^{(D-3)/2}} \left(\frac{m}{r}\right)^{(D+1)/2} {\sum_{n=0}^{\infty}}' \frac{\bar{I}_{qn}(ma)}{\bar{K}_{qn}(ma)}$$

Boundary-induced part in VEV

$D=3$ massless scalar fields with a Dirichlet boundary condition

Minimal coupling $\quad a^{D+1}\langle T_i^i \rangle_a \quad$ Conformal coupling



$r/a \longrightarrow$

String with finite thickness

- From the point of view of the physics in the exterior region, the cylindrical surface can be considered as a simple model of cosmic string core

- In general, the string core is modeled by a cylindrically symmetric potential whose support lies in $r \leq a$

- Model of core: $$ds^2 = dt^2 - P^2(r/a)dr^2 - r^2 d\phi^2 - \sum_{i=1}^{N} dz_i^2$$

Smooth monotonic function

$$\lim_{x \to 0} P(x) = 1/q, \qquad P(x) = 1 \quad \text{for } x > 1$$

Mode functions

- Mode functions inside the core

$$\varphi_\alpha(x) = f_n(r/a, \gamma a)\exp(iqn\phi + i\mathbf{k}\mathbf{r}_\parallel - i\omega t)$$

- Radial equation

$$\left[\frac{1}{xP(x)}\frac{d}{dx}\frac{x}{P(x)}\frac{d}{dx} + \gamma^2 a^2 - \frac{q^2 n^2}{x^2} - \frac{2\xi}{x}\frac{P'(x)}{P^3(x)}\right] f_n(x, \gamma a) = 0$$

- Solution regular at $r = 0$ ➡ $R_n(x, \gamma a)$

- Radial part of the mode functions

$$\begin{array}{ll} R_n(r/a, \gamma a) & \text{for } r < a \\ A_n J_{q|n|}(\gamma r) + B_n Y_{q|n|}(\gamma r) & \text{for } r > a \end{array}$$

$$A_n = \frac{\pi}{2}[\gamma a Y'_{q|n|}(\gamma a) R_n(1, \gamma a) - Y_{q|n|}(\gamma a) R'_n(1, \gamma a)]$$

$$B_n = -\frac{\pi}{2}[\gamma a J'_{q|n|}(\gamma a) R_n(1, \gamma a) - J_{q|n|}(\gamma a) R'_n(1, \gamma a)]$$

Mode functions

- Mode functions in the region $r > a$ coincide with the corresponding functions outside a cylindrical boundary with the Robin coefficients

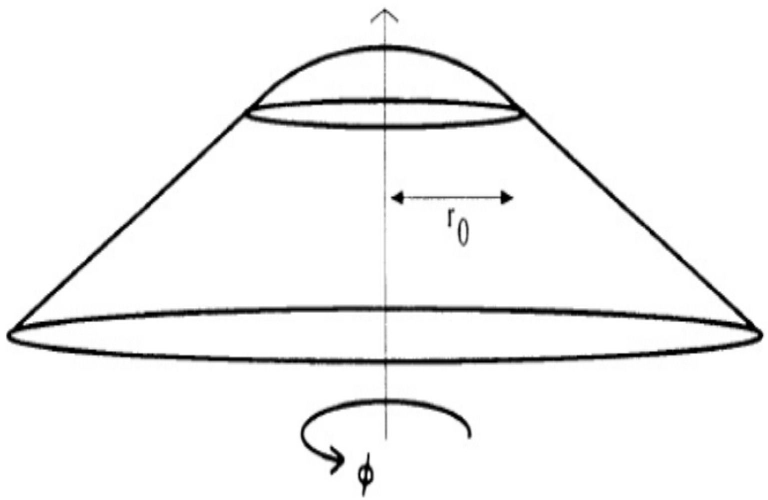
$$A/B \to -R_n'(1, \gamma a)/R_n(1, \gamma a)$$

- Part in the Wightman function and in the VEVs of the field squared and the energy-momentum tensor induced by the nontrivial structure of the string core is given by previous formulas in the exterior region, where we should substitute

$$\frac{A}{B} = -\frac{R_n'(1, zae^{\pi i/2})}{R_n(1, zae^{\pi i/2})}$$

Special models for string core

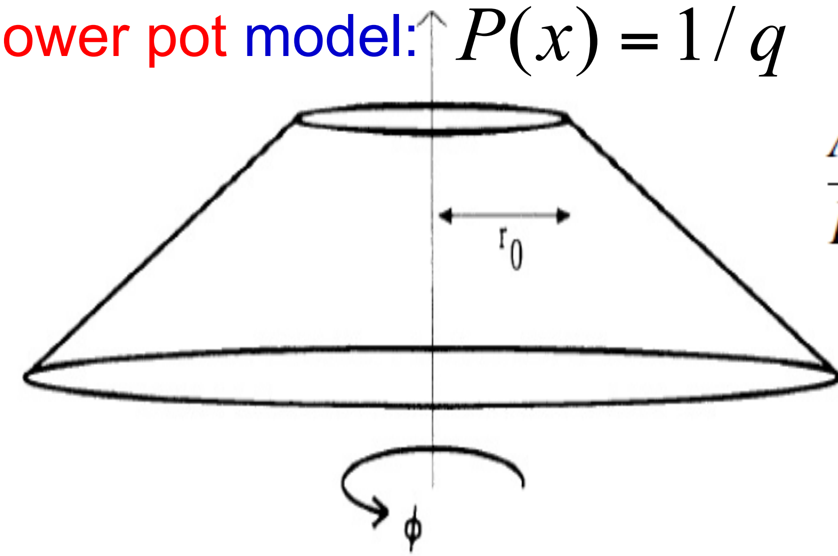
- Ball-point pen model: $P(x) = [x^2(1-q^2)+q^2]^{-1/2}$



$$\frac{A}{B} = -\sqrt{q^2-1}\,\frac{P_\nu^{|n|\prime}(1/q)}{P_\nu^{|n|}(1/q)},$$

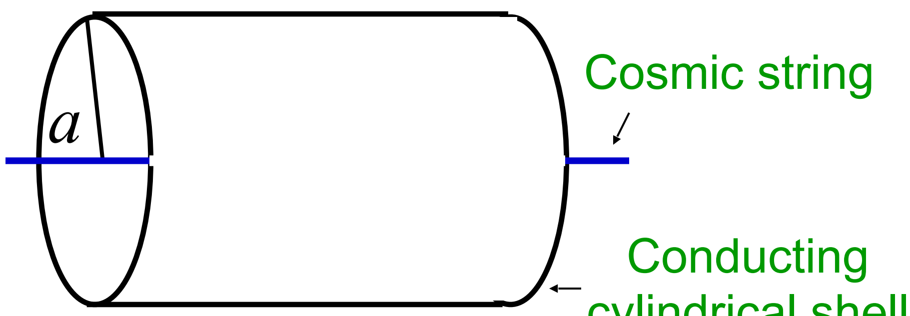
$$\nu(\nu+1) = -2\xi - \frac{z^2a^2}{q^2-1}$$

- Flower pot model: $P(x) = 1/q$



$$\frac{A}{B} = -za\,\frac{I'_{|n|}(za/q)}{I_{|n|}(za/q)} - 2\xi(q-1)$$

Electromagnetic field



- VEV for a quantity bilinear in the field

$$\langle 0|F\{A_i, A_k\}|0\rangle = \sum_{\alpha} F\{A_{\alpha i}, A^*_{\alpha k}\}$$

Complete set of solutions of the classical field equations

- Mode functions

TM modes

$$\mathbf{A}_\alpha = \beta_\alpha \begin{cases} (1/i\omega)(\gamma^2 \mathbf{e}_3 + ik\nabla_t) J_{q|m|}(\gamma r)\exp[i(qm\phi + kz - \omega t)], & \lambda = 0, \\ -\mathbf{e}_3 \times \nabla_t \{J_{q|m|}(\gamma r)\exp[i(qm\phi + kz - \omega t)]\}, & \lambda = 1, \end{cases}$$

TE modes

$$\omega^2 = \gamma^2 + k^2, \qquad q = 2\pi/\phi_0, \quad m = 0, \pm 1, \pm 2, \dots$$

- From boundary conditions $\mathbf{n} \times \mathbf{E} = 0$ and $\mathbf{n} \cdot \mathbf{B} = 0$

$$J^{(\lambda)}_{q|m|}(\gamma a) = 0, \quad \lambda = 0, 1,$$

- Eigenvalues:

$$\gamma a = j^{(\lambda)}_{m,n},\ n = 1, 2, \dots$$

VEVs of the electric and magnetic field squared

- Decomposed form of the VEVs: $\langle 0|F^2|0\rangle = \langle 0_s|F^2|0_s\rangle + \langle F^2\rangle_{\mathrm{b}}$, $F = E, B$
- Boundary induced parts inside the shell ($\eta_{E\lambda} = \lambda$, $\eta_{B\lambda} = 1-\lambda$)

$$\langle F^2\rangle_{\mathrm{b}} = \frac{4q}{\pi^2}\sum_{m=0}^{\infty}{}' \int_0^{\infty} dk \sum_{\lambda=0,1}\int_k^{\infty} dx\, x^3 \frac{K_{qm}^{(\lambda)}(xa)}{I_{qm}^{(\lambda)}(xa)} \frac{G_{qm}^{(\eta_{F\lambda})}[k, I_{qm}(xr)]}{\sqrt{x^2-k^2}}$$

$$G_{\nu}^{(j)}[k, f(x)] = \begin{cases} (k^2r^2/x^2)[f'^2(x) + \nu^2 f^2(x)/x^2] + f^2(x), & j=0, \\ (k^2r^2/x^2 - 1)[f'^2(x) + \nu^2 f^2(x)/x^2], & j=1. \end{cases}$$

- Exterior region ➡ Replacements $I_{qm} \rightleftarrows K_{qm}$

- Boundary-induced parts for the electric and magnetic fields are different

  Presence of the shell breaks the electric-magnetic symmetry

Asymptotic of the VEVs

- Near the boundary: $\langle E^2 \rangle_b \approx -\langle B^2 \rangle_b \approx \dfrac{3}{4\pi (a-r)^4}$
- In reality the expectation values will attain a limiting value on the conductor surface, which will depend on the molecular details of the conductor
- Near the string: $\langle E^2 \rangle_b \approx \dfrac{q}{\pi a^4} \displaystyle\int_0^\infty dx\, x^3 \frac{K_0(x)}{I_0(x)} \approx 0.320 \frac{q}{a^4}$

$$\langle B^2 \rangle_b \approx -\frac{q}{\pi a^4} \int_0^\infty dx\, x^3 \frac{K_1(x)}{I_1(x)} \approx -0.742 \frac{q}{a^4}$$

- Large distances from the cylinder

$$\langle E^2 \rangle_b \approx \frac{q}{2\pi r^4 \ln(r/a)} \int_0^\infty dx\, x^3 \left[2K_0^2(x) + K_1^2(x)\right] = \frac{2q}{3\pi r^4 \ln(r/a)},$$

$$\langle B^2 \rangle_b \approx -\frac{q}{2\pi r^4 \ln(r/a)} \int_0^\infty dx\, x^3 K_1^2(x) = -\frac{q}{3\pi r^4 \ln(r/a)}.$$

VEV of the energy-momentum tensor

- Decomposed form: $\langle 0|T_i^k|0\rangle = \langle 0_s|T_i^k|0_s\rangle + \langle T_i^k\rangle_{\rm b}$

- Part induced by a cylindrical shell

$$\langle T_i^k\rangle_{\rm b} = \frac{q\delta_i^k}{4\pi^2}\sum_{m=0}^{\infty}{}' \sum_{\lambda=0,1}\int_0^{\infty} dx\, x^3 \frac{K_{qm}^{(\lambda)}(xa)}{I_{qm}^{(\lambda)}(xa)} F_{qm}^{(i)}[I_{qm}(xr)]$$

$$F_\nu^{(0)}[f(y)] = F_\nu^{(3)}[f(y)] = f^2(y),$$

$$F_\nu^{(i)}[f(y)] = -(-1)^i f'^2(y) - [1-(-1)^i \nu^2/y^2] f^2(y), \quad i=1,2$$

- Exterior region ➡ Replacements $I_{qm} \rightleftarrows K_{qm}$

- Boundary-induced parts in the vacuum energy density and axial stress are negative
  Radial and azimuthal stresses are positive

Asymptotic

- Near the boundary:

$$\langle T_0^0 \rangle_{\mathrm{b}} \approx -\frac{1}{2}\langle T_2^2 \rangle_{\mathrm{b}} \approx -\frac{(a-r)^{-3}}{60\pi^2 a}, \qquad \langle T_1^1 \rangle_{\mathrm{b}} \approx \frac{(a-r)^{-2}}{60\pi^2 a^2}$$

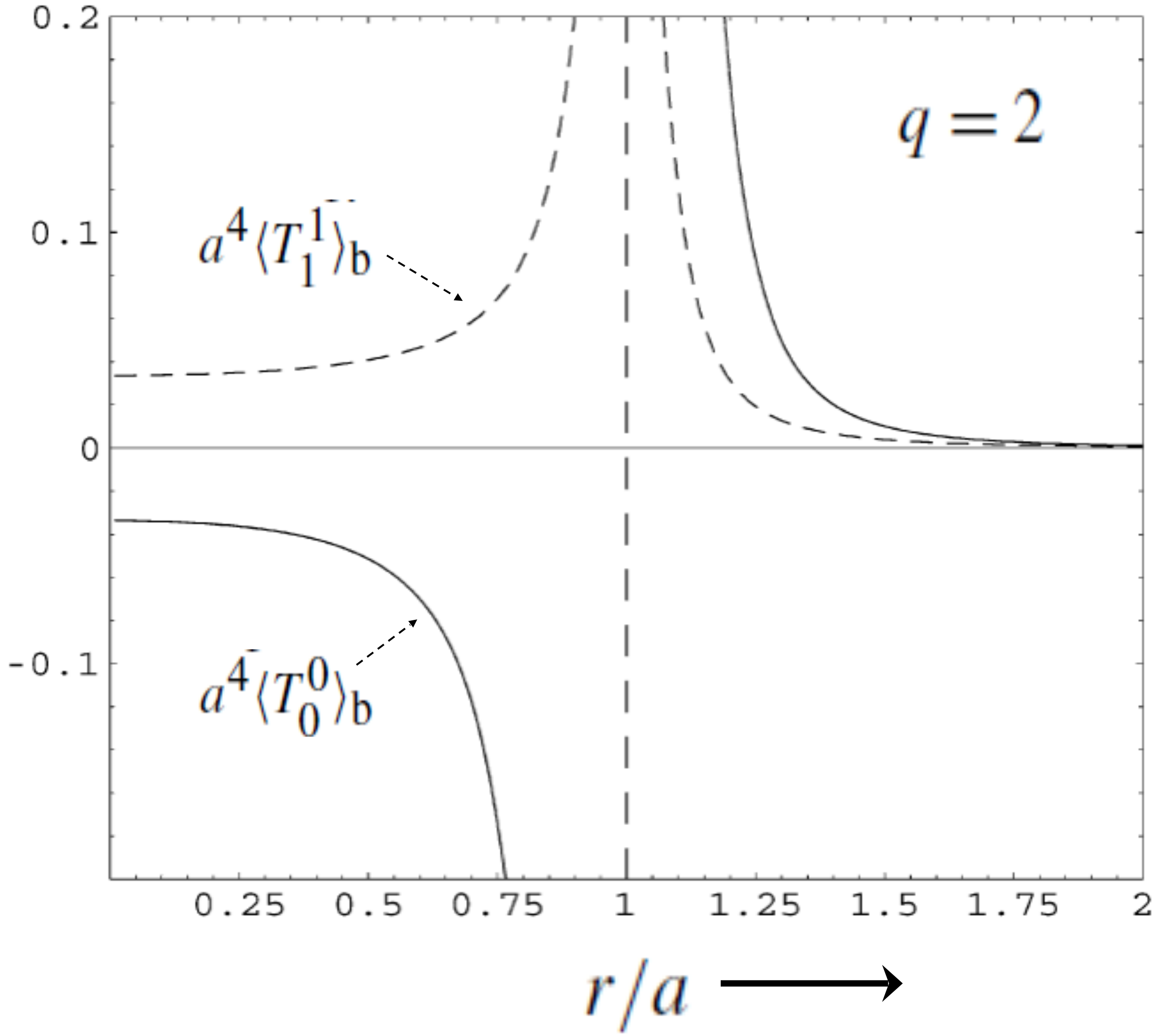
- Near the string:

$$\langle T_0^0 \rangle_{\mathrm{b}} \approx -\langle T_1^1 \rangle_{\mathrm{b}} \approx -\langle T_2^2 \rangle_{\mathrm{b}} \approx \frac{q}{8\pi^2 a^4} \int_0^\infty dx\, x^3 \left[\frac{K_0(x)}{I_0(x)} - \frac{K_1(x)}{I_1(x)} \right] = -0.0168 \frac{q}{a^4}$$

- Large distances from the cylinder

$$\langle T_0^0 \rangle_{\mathrm{b}} \approx \langle T_1^1 \rangle_{\mathrm{b}} \approx -\frac{1}{3}\langle T_2^2 \rangle_{\mathrm{b}} \approx \frac{q}{24\pi^2 r^4 \ln(r/a)}$$

Energy density and radial stress



Fermionic field

- Massive fermionic field

$$i\gamma^\mu \nabla_\mu \psi - m\psi = 0,\ \nabla_\mu = \partial_\mu + \Gamma_\mu \quad \gamma^\mu = e^\mu_{(a)} \gamma^{(a)}$$

- MIT bag boundary condition on a cylindrical shell

$$(1 + i\gamma^\mu n_\mu)\psi = 0, \qquad r = a,$$

Outward-pointing normal
to the boundary

- Eigenmodes inside the shell are expressed in terms of the zeros of the function

$$xJ'_{\beta_1}(x) + (\mu - s\sqrt{x^2 + \mu^2} - \epsilon_j \beta_1) J_{\beta_1}(x),\ \mu = ma$$

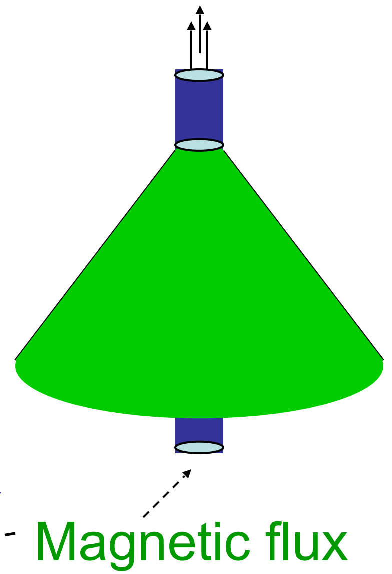
$$\beta_1 = |qj - 1/2|\,,\ \ s = \pm 1\,,$$

Projection of total momentum

- Fermionic condensate $\langle 0|\bar{\psi}\psi|0\rangle$

- VEV of the energy-momentum tensor $\langle 0|T_{\mu\nu}|0\rangle$

Fermionic current in the presence of magnetic flux



- (2+1) -dimensional conical spacetime

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 \, , \, 0 \leqslant \phi \leqslant \phi_0$$

- Field equation

$$i\gamma^\mu (\nabla_\mu + ieA_\mu)\psi - m\psi = 0 \; , \; \nabla_\mu = \partial_\mu + \Gamma_\mu ,$$

- Infinitely thin magnetic flux located at the cone apex

$$A_\mu = (0, 0, A) \text{ for } r > 0 , \; A = -\Phi / \phi_0$$

- Non-trivial topology of the background spacetime leads to Aharonov-Bohm-like effects on physical observables

- MIT bag boundary condition on the circle with radius a

$$\left(1 + i n_\mu \gamma^\mu\right) \psi \Big|_{r=a} = 0$$

- VEV of the fermionic current $j^\mu(x) = e\bar{\psi}\gamma^\mu\psi$

Graphitic cones

- The results obtained can be applied for the evaluation of the VEV of the fermionic current in graphitic cones

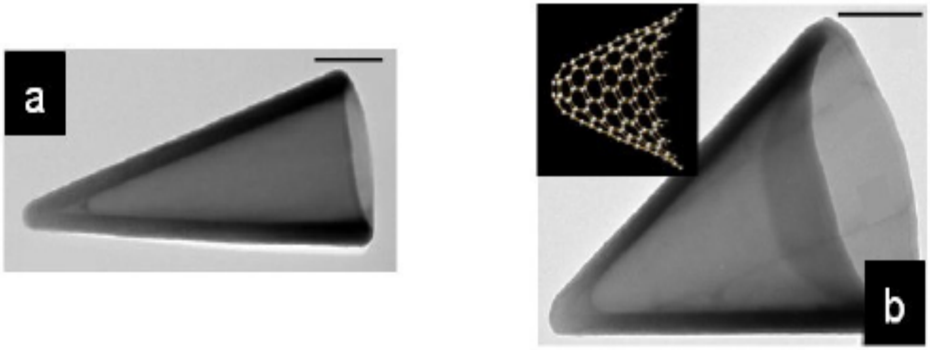
- Graphitic cones are obtained from the graphene sheet if one or more sectors are excised

- Opening angle of the cone

Number of sectors removed

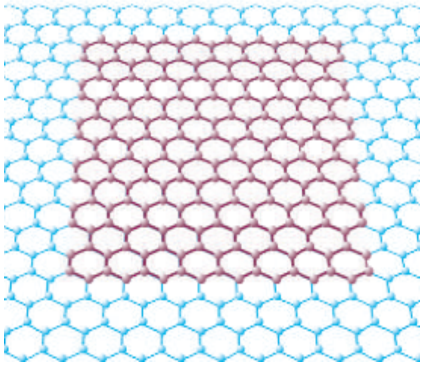
$$\phi_0 = 2\pi(1 - N_c/6),\ N_c = 1, 2, \ldots, 5$$

- All these angles have been observed in experiments

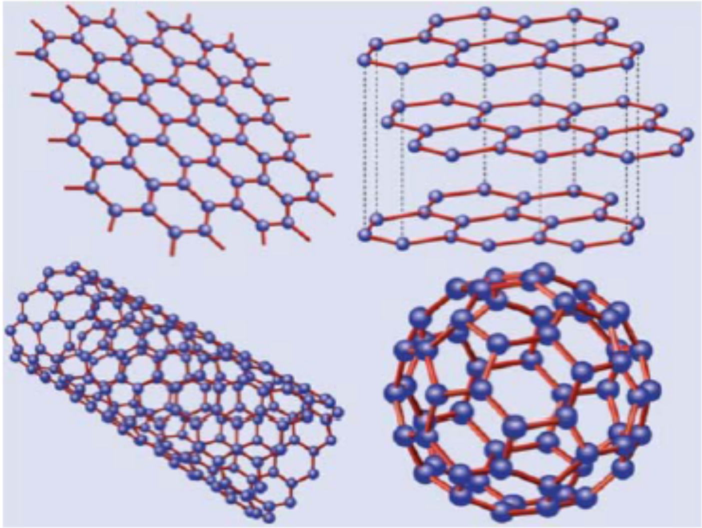


Graphene and its allotropes

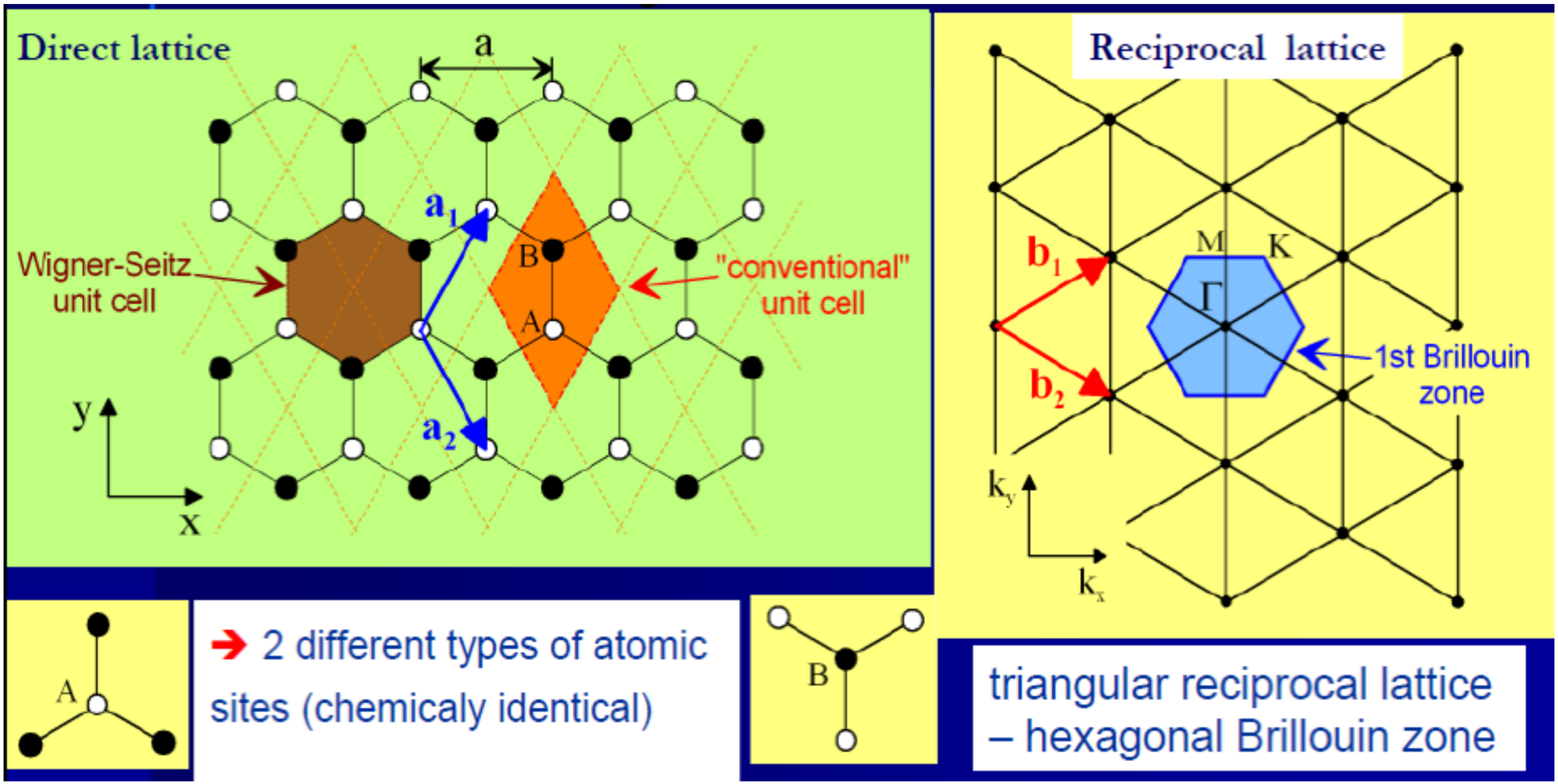
- Graphene is a single 2D sheet of carbon atoms in a honeycomb lattice



- Graphene is the basis for the understanding of the electronic properties in other allotropes
- Graphite can be viewed as a stack of graphene layers
- Carbon nanotubes are rolled-up cylinders of graphene
- Fullerenes (C_{60}) are molecules consisting of wrapped graphene by the introduction of pentagons on the hexagonal lattice



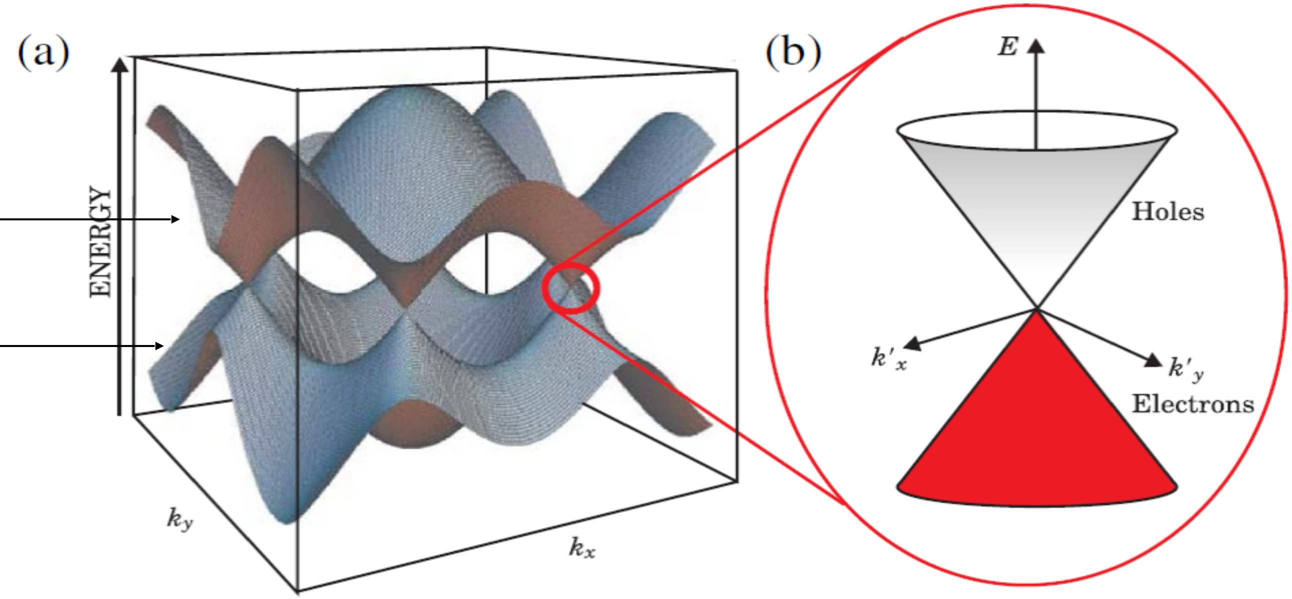
Honeycomb lattice of graphene



Graphene band structure

Conduction band

Valence band



- Dirac points are two inequivalent corners of the Brillouin zone

$$\mathbf{K} = (2\pi/(3a), 2\pi/(3\sqrt{3}a)) \qquad a \approx 0.142 \text{ nm}$$

$$\mathbf{K}' = (2\pi/(3a), -2\pi/(3\sqrt{3}a))$$

- The most important aspect of graphene's energy dispersion is its linear energy–momentum relationship with the conduction and valence bands intersecting, with no energy gap

- Graphene is a zero band-gap semiconductor with a linear long-wavelength energy dispersion

Dirac-like theory

- Graphene electronic band structure close to the Dirac points shows a conical dispersion

$$E(\mathbf{k}) = v_{\mathrm{F}}|\mathbf{k}| \qquad v_F \approx 10^8 \text{ cm/s}$$

Fermi velocity

momentum measured relatively to Dirac points

- The linear long-wavelength Dirac dispersion is the most distinguishing feature of graphene in addition to its 2D nature

- The low-energy excitations can be described by a pair of two-component massless spinors, residing on the two different sublattices of the graphene lattice

- In the presence of an external magnetic field an effective mass term is generated for the fermionic excitations

Dirac model for long-wavelength excitations

- Low-energy excitations of the electronic subsystem are described by a pair of two-component spinors, ψ_A and ψ_B, corresponding to the two different triangular sublattices of the honeycomb lattice of graphene
- Dirac equation for these spinors has the form

$$(iv_F^{-1}\gamma^0 D_0 + i\gamma^l D_l - m)\psi_J = 0,\ J = A, B,$$

- For a plane graphene sheet the topology for the Dirac-like theory is trivial $\Longrightarrow$ **R^2**

VEV of fermionic current

- Decomposed form of the VEV: $\langle j^{\mu} \rangle = \langle j^{\mu} \rangle_{0,\mathrm{ren}} + \langle j^{\mu} \rangle_{\mathrm{b}}$

  Boundary-free VEV

  Induced by circular boundary

- Both parts are periodic function of the parameter

$$\alpha = e\phi_0 A/2\pi = -e\Phi/2\pi$$ Period = 1

Periodic functions of the flux with the period equal to the flux quantum

- If we present: $\alpha = \alpha_0 + n_0, \; |\alpha_0| < 1/2$

  VEVs are (odd) functions of α_0 alone

- Radial component of the current density vanishes

Boundary-free part

Boundary-free parts in the charge density and azimuthal current

$$\begin{aligned}
\langle j^0 \rangle_{0,\mathrm{ren}} &= \frac{em}{2\pi r}\Big[\sum_{l=1}^{p}(-1)^l \sin(2\pi l\alpha_0)e^{-2mr\sin(\pi l/q)} \\
&\quad -\frac{q}{2\pi}\int_0^\infty dy \frac{e^{-2mr\cosh y}}{\cosh y}\frac{\sum_{\delta=\pm1}\delta f(q,\delta\alpha_0,2y)}{\cosh(2qy)-\cos(q\pi)}\Big].
\end{aligned}$$

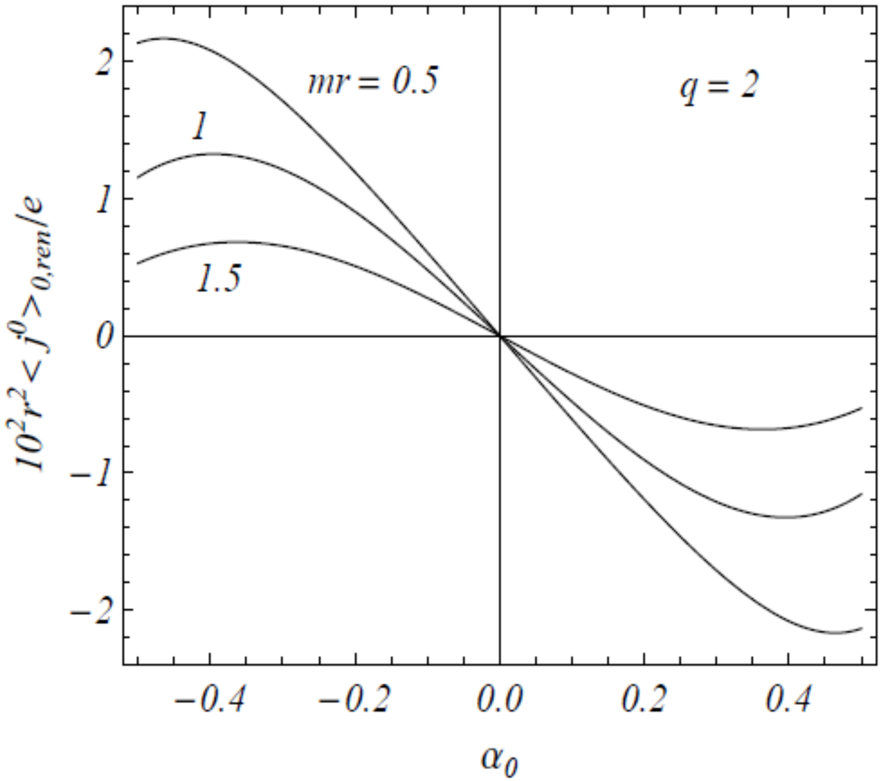
$$\begin{aligned}
\langle j^2 \rangle_{0,\mathrm{ren}} &= \frac{e}{4\pi r^3}\Big[\sum_{l=1}^{p}\frac{(-1)^l \sin(2\pi l\alpha_0)}{\sin^2(\pi l/q)}\frac{1+2mr\sin(\pi l/q)}{e^{2mr\sin(\pi l/q)}} \\
&\quad -\frac{q}{4\pi}\int_0^\infty dy \frac{\sum_{\delta=\pm1}\delta f(q,\delta\alpha_0,2y)}{\cosh(2qy)-\cos(q\pi)}\frac{1+2mr\cosh y}{e^{2mr\cosh y}\cosh^3 y}\Big]
\end{aligned}$$

$2p \leqslant q < 2p+2$, $q \equiv 2\pi/\phi_0$,

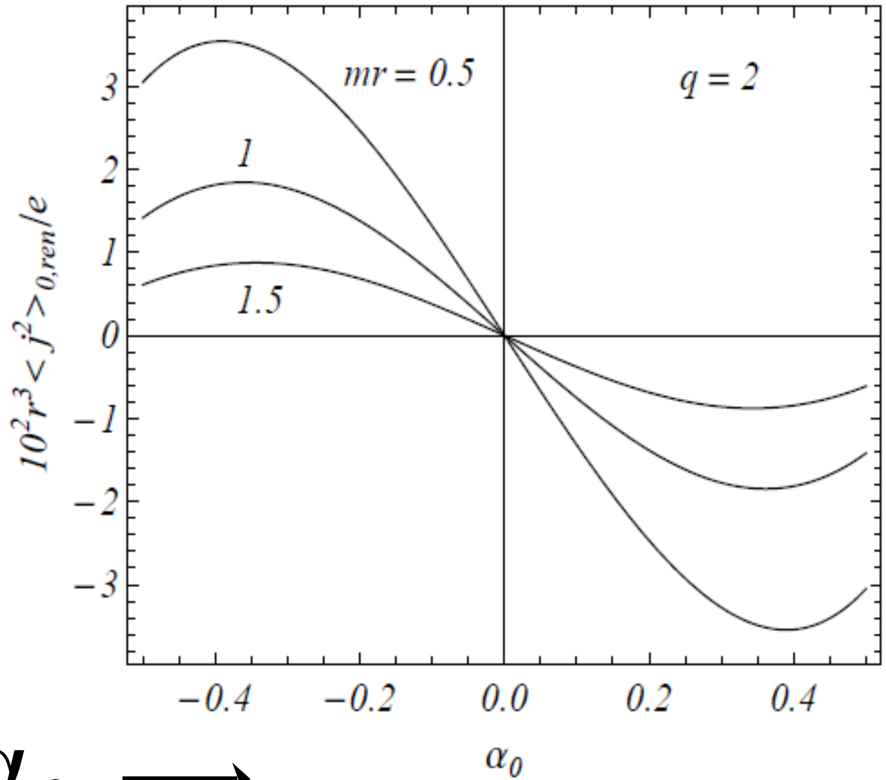
$$\begin{aligned}
f(q,\alpha_0,2y) &= \cos\left[q\pi\left(1/2-\alpha_0\right)\right]\cosh\left[\left(2q\alpha_0+q-1\right)y\right] \\
&\quad -\cos\left[q\pi\left(1/2+\alpha_0\right)\right]\cosh\left[\left(2q\alpha_0-q-1\right)y\right]
\end{aligned}$$

Boundary-free charge and current densities

Charge density

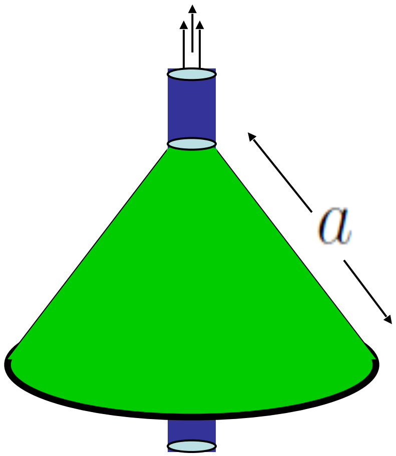


Azimuthal current



$\alpha_0 \longrightarrow$

Current density inside a circular boundary



- Boundary-induced part in the charge density for a massless field

$$\langle j^0\rangle_{\mathrm{b}} = \frac{eq}{2\pi^2 a^2}\sum_j \int_0^\infty dx\, \frac{I^2_{\beta_j}(xr/a) - I^2_{\beta_j+\epsilon_j}(xr/a)}{I^2_{\beta_j}(x) + I^2_{\beta_j+\epsilon_j}(x)}$$

$$j = \pm 1/2, \pm 3/2, \ldots$$

Modified Bessel function

$$\beta_j = q|j+\alpha| - \epsilon_j/2,\ \epsilon_j = \begin{cases} 1, & j > -\alpha \\ -1, & j < -\alpha \end{cases}$$

- Azimuthal current

$$\langle j^2\rangle_{\mathrm{b}} = \frac{eq}{\pi^2 r a^2}\sum_j \int_0^\infty dx\, \frac{I_{\beta_j}(xr/a) I_{\beta_j+\epsilon_j}(xr/a)}{I^2_{\beta_j}(x) + I^2_{\beta_j+\epsilon_j}(x)} W_{\beta_j,\beta_j+\epsilon_j}(x)$$

$$W_{\nu,\sigma}(x) = x\left[I_\nu(x)K_\nu(x) - I_\sigma(x)K_\sigma(x)\right]$$

Macdonald function

- Near the cone apex, $r \to 0$, VEVs behave as

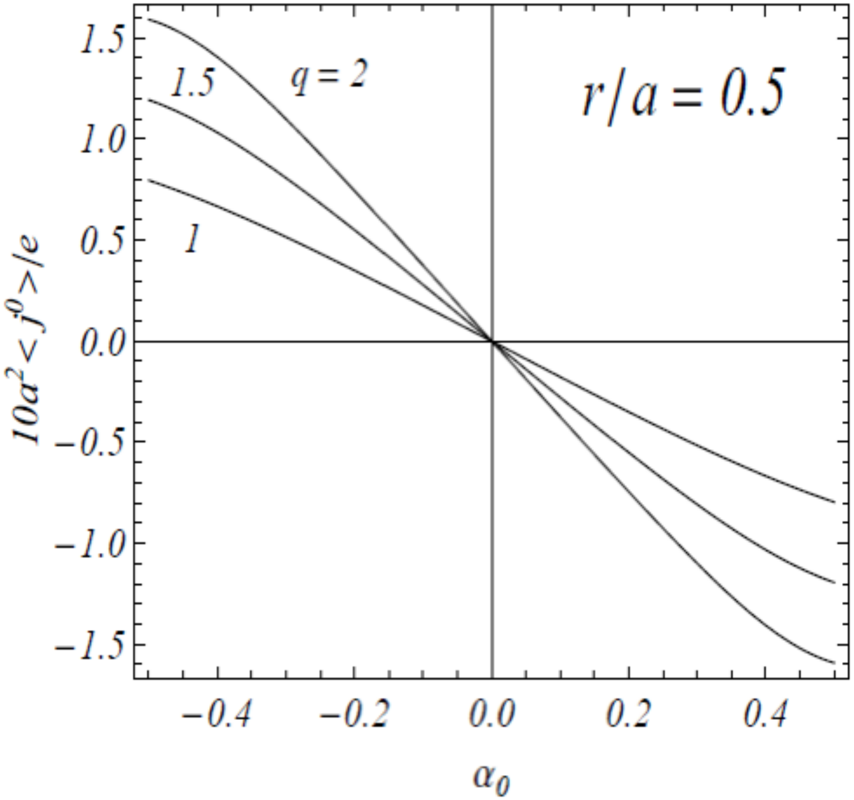
$$(r/a)^{2q_\alpha - 1},\quad q_\alpha = q(1/2 - |\alpha_0|)$$

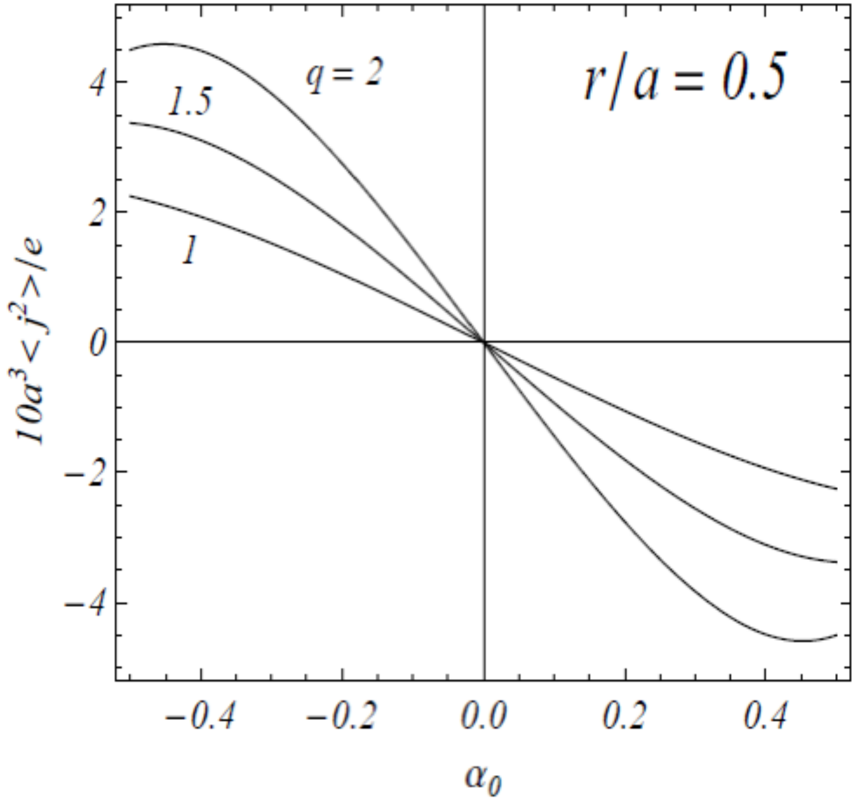
Vanish for $|\alpha_0| < 1/2 - 1/(2q)$ and diverge for $|\alpha_0| > 1/2 - 1/(2q)$

Charge and current densities inside a circle

Charge density

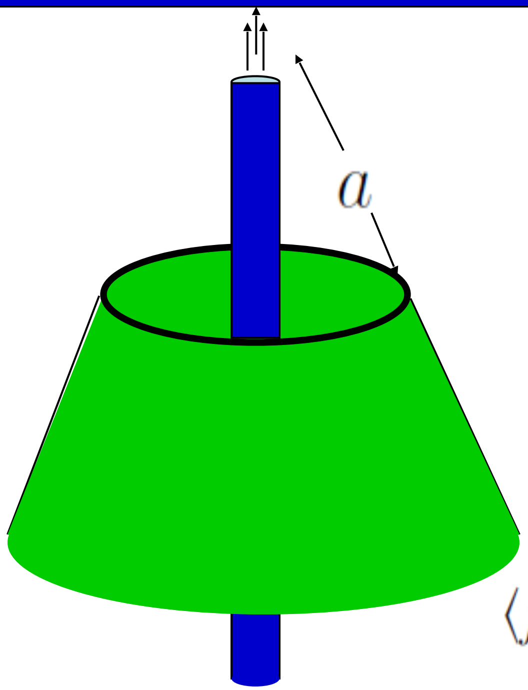
Azimuthal current





α_0 →

Current density outside a circular boundary



- Boundary-induced part in the charge density for a massless field

$$\langle j^0(x)\rangle_{\rm b} = \frac{eq}{2\pi^2 a^2}\sum_j \int_0^\infty dz \frac{K^2_{\beta_j}(zr/a) - K^2_{\beta_j+\epsilon_j}(zr/a)}{K^2_{\beta_j}(z) + K^2_{\beta_j+\epsilon_j}(z)}$$

- Azimuthal current

$$\langle j^2(x)\rangle_{\rm b} = -\frac{eq}{\pi^2 a^2 r}\sum_j \int_0^\infty dz \frac{K_{\beta_j}(zr/a)K_{\beta_j+\epsilon_j}(zr/a)}{K^2_{\beta_j}(z) + K^2_{\beta_j+\epsilon_j}(z)} W_{\beta_j,\beta_j+\epsilon_j}(z)$$

- Large distances from the circle

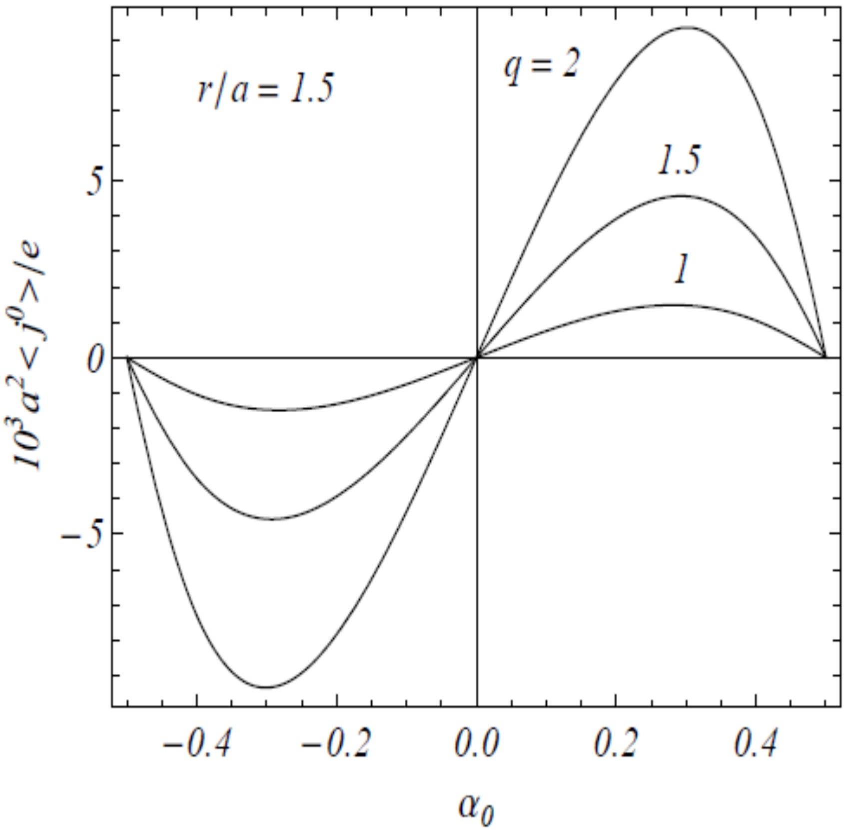
$$\langle j^0\rangle_{\rm b} \propto r^{-2q_\alpha}, \quad q_\alpha = q(1/2 - |\alpha_0|)$$

$$\langle j^2\rangle_{\rm b} \propto r^{-2q_\alpha - 1} \text{ for } q_\alpha > 1/2$$

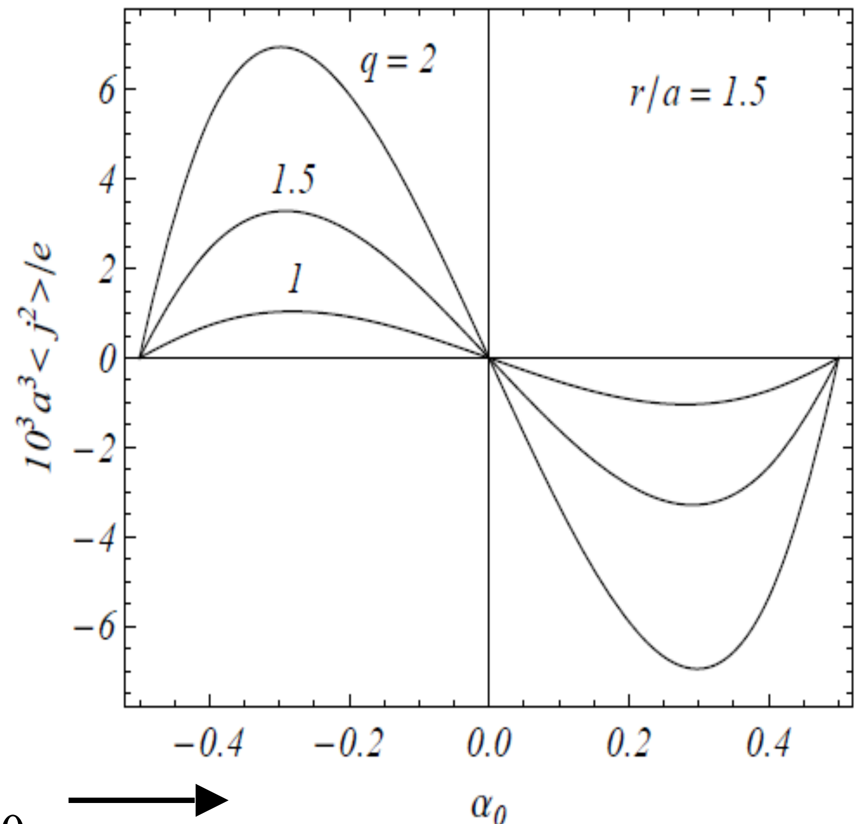
$$\langle j^2\rangle_{\rm b} \propto r^{-4q_\alpha} \text{ for } q_\alpha < 1/2$$

Charge and current densities outside a circle

Charge density



Azimuthal current



α_0 →

Thank you!