Vacuum currents in braneworlds with compact dimensions

Stefano Bellucci

INFN, Laboratori Nazionali di Frascati

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Outline

- Motivation
- Background geometry: AdS spacetime with compact dimensions
- Vacuum currents in AdS spacetime
- Vacuum currents in the presence of a brane
- Conclusions

QFT effects in models with non-trivial topology

- Quantum field theory plays an important role in models with non-trivial topology
- Fields propagating in the bulk are subject to boundary conditions along compact dimensions
- Imposition of boundary conditions on the field leads to the change of the spectrum for vacuum (zero-point) fluctuations
- As a result the vacuum expectation values of physical observables are changed (topological Casimir effect)
- Vacuum energy depends on the size of compact space

Stabilization mechanism for compact dimensions

Aim

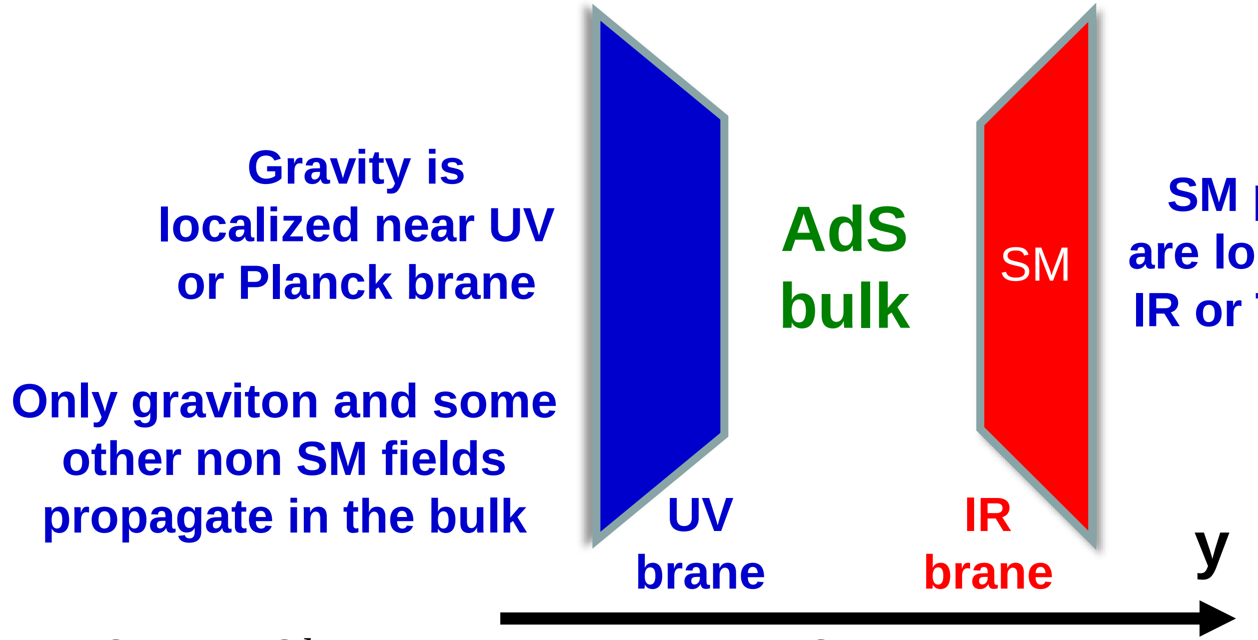
- We aim to consider combined effects of topology and gravity on the properties of quantum vacuum
- Gravitational field is considered as a classical curved background
- Back-reaction of quantum effects is described by Einstein equations with the expectation value of the energy-momentum tensor for quantum fields in the right-hand side
- This hybrid but very useful scheme is an important intermediate step to the development of quantum gravity
- Among the most interesting effects in this field are the particle production and the vacuum polarization by strong gravitational fields
- As a background geometry we consider AdS spacetime

Importance of AdS in QFT on curved backgorunds

- Because of the high symmetry, numerous problems are exactly solvable on AdS bulk and this may shed light on the influence of a classical gravitational field on the quantum matter in more general geometries
- Questions of principal nature related to the quantization of fields propagating on curved backgrounds
- AdS spacetime generically arises as a ground state in extended supergravity and in string theories
- AdS/Conformal Field Theory correspondence: Relates string theories or supergravity in the bulk of AdS with a conformal field theory living on its boundary
- Braneworld models: Provide a solution to the hierarchy problem between the gravitational and electroweak scales
- Braneworlds naturally appear in string/M-theory context and provide a novel setting for discussing phenomenological and cosmological issues related to extra dimensions

Randall-Sundrum-type braneworlds

- Original Randall-Sundrum model (RS1) offers a solution to the hierarchy problem by postulating 5D AdS spacetime bounded by two (3+1)-dimensional branes



$$ds^2 = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

- Hierarchy problem between the gravitational and electroweak scales is solved for

k·distance between branes = 40

Geometry

- (D+1)-dimensional AdS spacetime

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{-2y/a}\eta_{ik}dx^i dx^k - dy^2$$

$$\eta_{ik} = \mathrm{diag}(1, -1, \ldots, -1)$$

- New coordinate $z = ae^{y/a},\ 0 \leqslant z < \infty$

$$ds^2 = (a/z)^2(\eta_{ik}dx^i dx^k - dz^2)$$

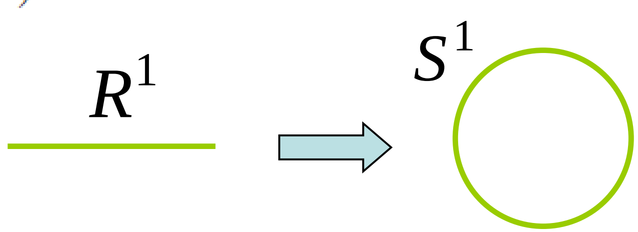
- Topology $R^p \times (S^1)^q,\ q + p = D - 1$

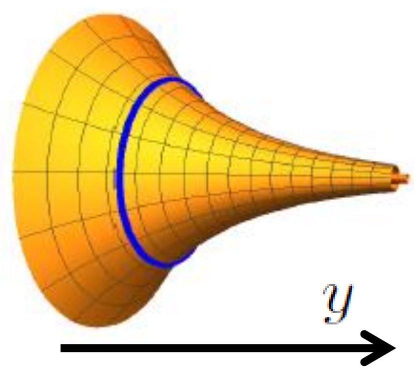
  q-dimensional torus

- Cartesian coordinates along uncompactified and compactified dimensions

$$\mathbf{x}_q = (x^{p+1}, \ldots, x^{D-1}) \qquad \mathbf{x}_p = (x^1, \ldots, x^p)$$

Compactification





$L_l \Rightarrow$ length of the *l*-th compact dimension

Field

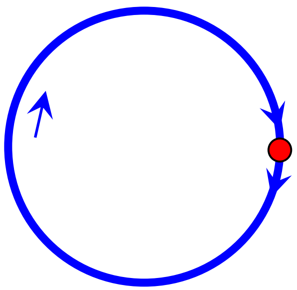
- Charged scalar field with general curvature coupling

$$\left(g^{\mu\nu}D_\mu D_\nu + m^2 + \xi R\right)\varphi(x) = 0,\ D_\mu = \nabla_\mu + ieA_\mu$$

External classical gauge field

- In models with nontrivial topology one need also to specify the periodicity conditions obeyed by the field operator along compact dimensions

$$\varphi(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\alpha_l}\varphi(t, \mathbf{x}_p, \mathbf{x}_q)$$



- Special cases:
  - Untwisted fields $\alpha_l = 0$
  - Twisted fields $\alpha_l = \pi$

- We assume that the gauge field is constant: $A_\mu = \mathrm{const}$

- Though the corresponding field strength vanishes, the nontrivial topology gives rise to Aharaonov-Bohm-like effects

Current density

- We are interested in the effects of non-trivial topology and gravity on the vacuum expectation value (VEV) of the current

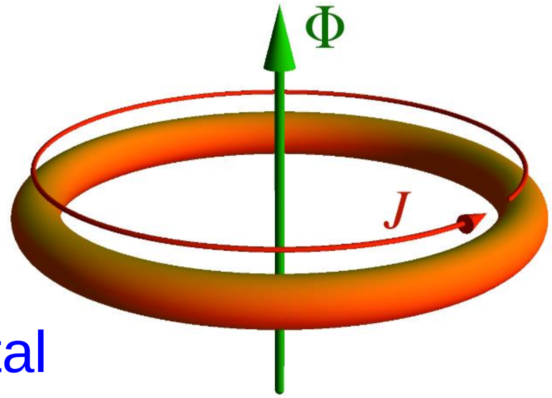
$$j_\mu(x) = ie[\varphi^+(x)D_\mu\varphi(x) - (D_\mu\varphi^+(x))\varphi(x)]$$

- This VEV is among the most important quantities that characterize the properties of the quantum vacuum

- Although the corresponding operator is local, due to the global nature of the vacuum, the VEV carries important information about the global properties of the background space-time

- Current acts as the source in the Maxwell equations and therefore plays an important role in modeling a self-consistent dynamics involving the electromagnetic field

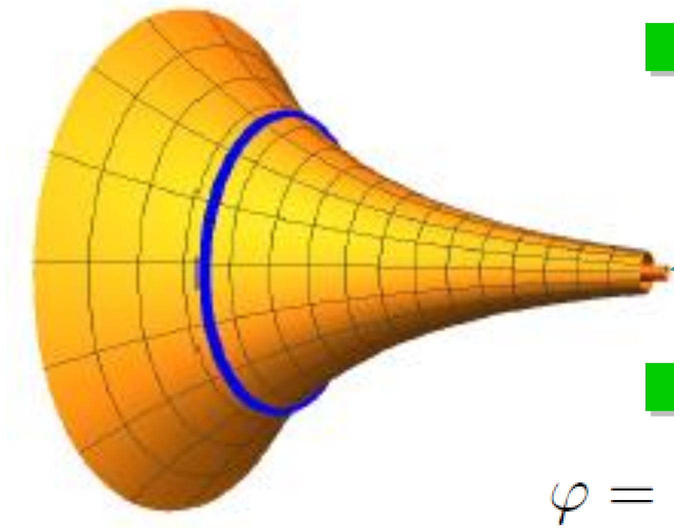
Analog from condensed matter physics: Persistent currents



- Persistent currents in metallic rings are predicted in *M. Büttiker, Y. Imry, R. Landauer, Phys. Lett. A* ***96****, 7 (1983)*.
- Existence of persistent currents in normal metal rings is a signature of phase coherence in mesoscopic systems and an example of the Aharonov-Bohm effect
- Temperature must be sufficiently low to reduce the probability of inelastic scattering and the circumference of the ring short enough that the phase coherence of the electronic wave functions is preserved around the loop
- Measurements of persistent currents in nanoscale gold and aluminum rings: A.C. Bleszynski-Jayich et. al., *Science* **326** (2009); H. Bluhm et. al., *Phys. Rev. Lett.* **102** (2009).

Induced currents in models with compact dimensions

- In the problem under consideration the presence of a constant gauge field is equivalent to the magnetic flux enclosed by the compact dimension



- Flux of the field strength which threads the *l*-th compact dimension

$$A_l L_l = \oint dx^l A_l$$

- By the gauge transformation

$$\varphi = \varphi' e^{-ie\chi},\ A_\mu = A'_\mu + \partial_\mu \chi \quad \chi = A_\mu x^\mu$$

the problem with a constant gauge field is reduced to the problem in the absence of the gauge field with the shifted phases in the periodicity conditions:

$$\varphi'(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\tilde{\alpha}_l} \varphi'(t, \mathbf{x}_p, \mathbf{x}_q),$$

$$\tilde{\alpha}_l = \alpha_l + eA_l L_l$$

Evaluation procedure

- VEV of the current density can be expressed in terms of the Hadamard function

$$G^{(1)}(x,x') = \langle 0|\varphi(x)\varphi^{+}(x') + \varphi^{+}(x')\varphi(x)|0\rangle$$

Vacuum state

- The corresponding relation

$$\langle 0|j_\mu(x)|0\rangle = \langle j_\mu\rangle = \frac{i}{2}e\lim_{x'\to x}(\partial_\mu - \partial'_\mu)G^{(1)}(x,x')$$

- Mode sum for the Hadamard function

$$G^{(1)}(x,x') = \sum_\sigma \sum_{s=\pm} \varphi_\sigma^{(s)}(x)\varphi_\sigma^{(s)*}(x')$$

$\{\varphi_\sigma^{(+)}(x), \varphi_\sigma^{(-)}(x)\}$ complete set of normalized positive- and negative-energy solutions to the field equation obeying the periodicity conditions

Vacuum current density

- Charge density vanishes
- Components of the current density along uncompact dimensions vanish
- Current density along *l*-th compact dimension

$$\langle j^l \rangle = \frac{4ea^{-1-D}L_l}{(2\pi)^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1}) q_{\nu-1/2}^{(D+1)/2}(1 + g_{\mathbf{n}_q}^2/(2z^2))$$

$$\mathbf{n}_{q-1} = (n_{p+1}, \ldots, n_{l-1}, n_{l+1}, \ldots, n_{D-1}), \quad -\infty < n_i < +\infty, \; i \neq l$$

$$\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1} = \sum_{i=1, \neq l}^{D-1} \tilde{\alpha}_i n_i, \;\; g_{\mathbf{n}_q} = \left(\sum_{i=p+1}^{D-1} n_i^2 L_i^2 \right)^{1/2}$$

$$q_\alpha^\mu(x) = \frac{e^{-i\pi\mu} Q_\alpha^\mu(x)}{(x^2-1)^{\mu/2}}$$

← Associated Legendre function of the second kind

$$\tilde{\alpha}_l = \alpha_l + eA_l L_l$$

Current density: Properties

- Current density along the *l*-th compact dimension is an odd periodic function of the phase $\tilde{\alpha}_l$ and an even periodic function of the phases $\tilde{\alpha}_i,\ i \neq l$, with the period 2π
- In particular, the current density is a periodic function of the magnetic fluxes with the period equal to the flux quantum $2\pi/|e|$
- In the absence of the gauge field, the current density along the *l*-th compact dimension vanishes for untwisted and twisted fields along that direction
- Charge flux through (*D -1*)-dimensional hypersurface $x^l = \mathrm{const}$

  $n_l \langle j^l \rangle,\ n_l = a/z$ ⬅ Normal to the hypersurface

- Charge flux depends on the coordinate lengths of the compact dimensions in the form of

  L_i/z = proper length of the compact dimension, measured in units of the AdS curvature radius

  $L_{(p)i} = aL_i/z$ ⬅ proper length of the compact dimension

Limiting cases

- Large values of the curvature radius $a \to \infty$

$$\langle j^l \rangle \approx \frac{4eL_l m^{(D+1)/2}}{(2\pi)^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1}) \frac{K_{(D+1)/2}(m g_{\mathbf{n}_q})}{g_{\mathbf{n}_q}^{(D+1)/2}}$$

$K_{(D+1)/2}(x)$ ⬅ MacDonald function

Leading term coincides with the current in Minkowski spacetime with toroidally compactified dimensions

- Large values of the proper length compared with the AdS curvature radius: $L_l/z \gg 1$
  - $\tilde{\alpha}_i = 0,\ i = p+1, \dots, D-1,\ i \neq l,$

$$\langle j^l \rangle \approx \frac{4e\Gamma(p/2+\nu+2)}{\pi^{p/2+1}\Gamma(\nu+1)a^{D+1}V_q} \frac{z^{D+2\nu+2}}{L_l^{p+2\nu+2}} \sum_{n_l=1}^{\infty} \frac{\sin(\tilde{\alpha}_l n_l)}{n_l^{p+2\nu+3}}$$

  - At least one of the phases $\tilde{\alpha}_i,\ i \neq l,$ is not equal to zero

$$\langle j^l \rangle \approx \frac{2ea^{-1-D}}{\pi^{(p+1)/2}} \frac{\sin(\tilde{\alpha}_l) z^{D+2\nu+2}}{\Gamma(\nu+1)V_q e^{\beta_{q-1}L_l}} \frac{\beta_{q-1}^{(p+3)/2+\nu}}{(2L_l)^{(p+1)/2+\nu}},\quad \beta_{q-1} = \left(\sum_{i=p+1,\neq l}^{D-1} \tilde{\alpha}_i^2/L_i^2\right)^{1/2}$$

Limiting cases

- Small values of the proper length compared with the AdS curvature radius: $L_l/z \ll 1$

$$\langle j^l \rangle \approx \frac{2e\Gamma((D+1)/2)}{\pi^{(D+1)/2}(a/z)^{D+1}L_l^D} \sum_{n_l=1}^{\infty} \frac{\sin(\tilde{\alpha}_l n_l)}{n_l^D}$$

Coincides with the VEV of the current density for a massless scalar field in (*D* + 1)-dimensional Minkowski spacetime compactified along the direction x^l

- Near the AdS boundary, $z \to 0$

$$\langle j^l \rangle \approx \frac{4eL_l\Gamma(\nu + D/2 + 1)}{\pi^{D/2}\Gamma(\nu+1)a^{D+1}} z^{D+2\nu+2} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \frac{\cos(\tilde{\alpha}_{q-1}\cdot \mathbf{n}_{q-1})}{g_{\mathbf{n}_q}^{D+2\nu+2}}$$

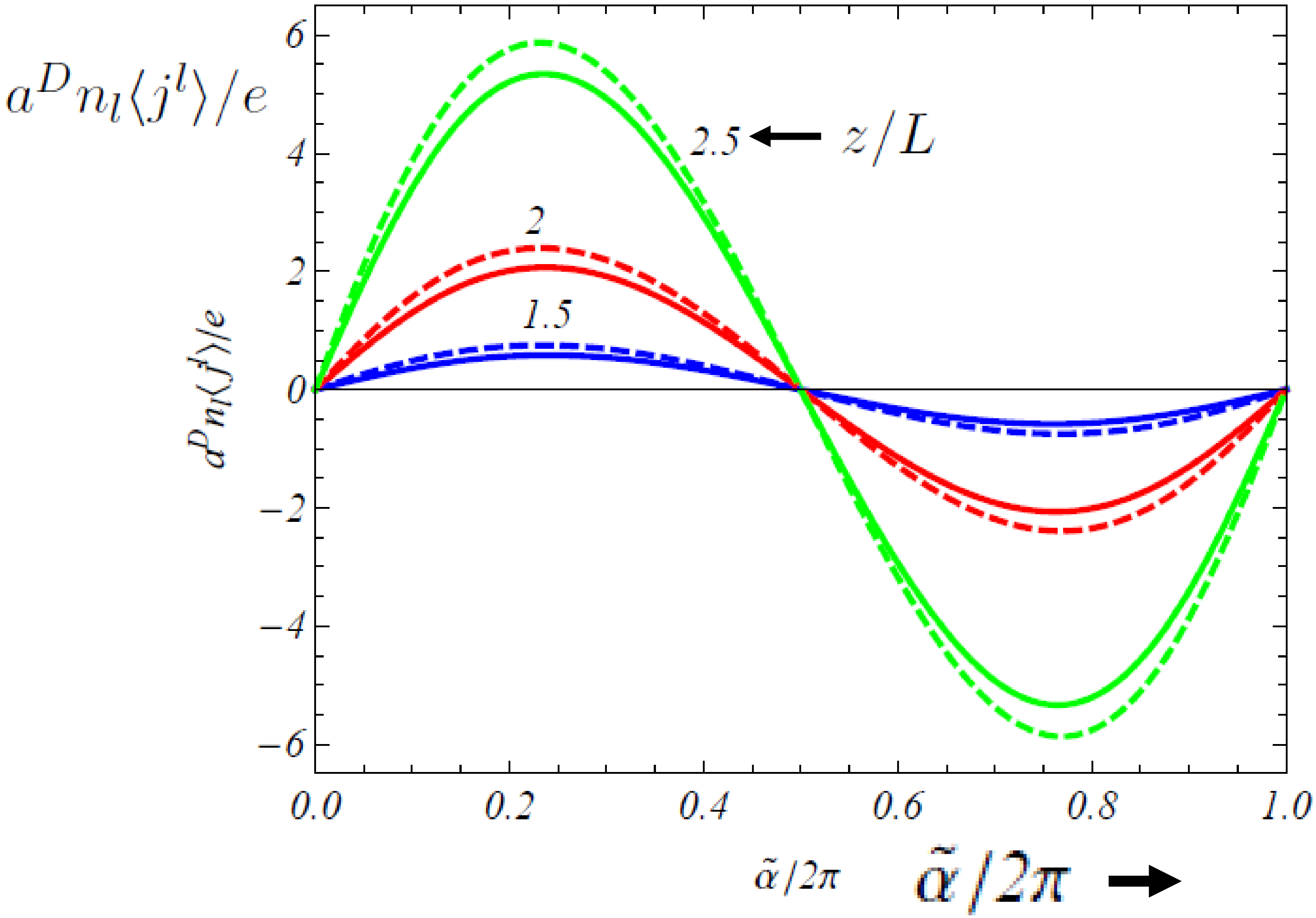
- Near the AdS horizon, $z \to \infty$: $\langle j^l \rangle \approx (z/a)^{D+1} \langle j^l \rangle_M,$

$$\langle j^l \rangle_M = 2eL_l \frac{\Gamma((D+1)/2)}{\pi^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \frac{\cos(\tilde{\alpha}_{q-1}\cdot \mathbf{n}_{q-1})}{g_{\mathbf{n}_q}^{D+1}}$$

Current density in Minkowski spacetime for a massless scalar field

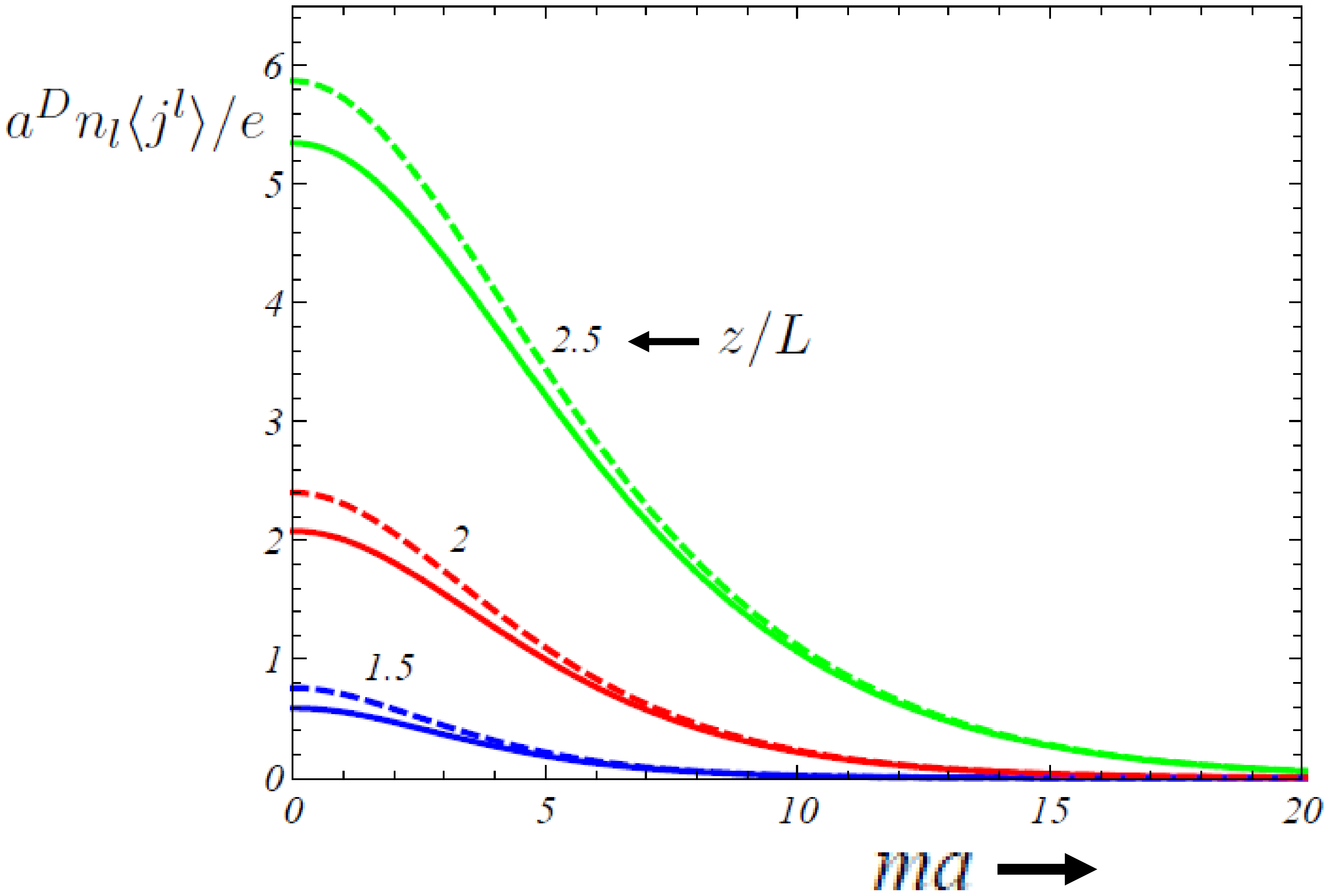
Numerical example

- D=4 minimally (full curves) and conformally (dashed curves) coupled fields
- Single compact dimension

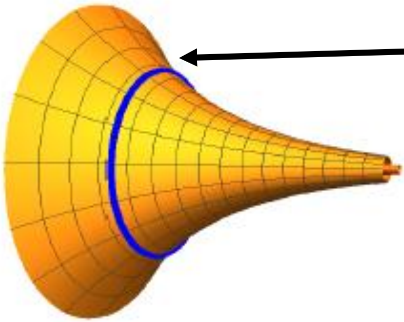


Numerical example

$$\tilde{\alpha} = \pi/2$$



Geometry with a brane



Brane at $y = y_0$

- Boundary condition on the brane

$$(1 + \beta n^\mu D_\mu)\varphi(x) = 0, \quad y = y_0$$

Constant   Normal to the brane

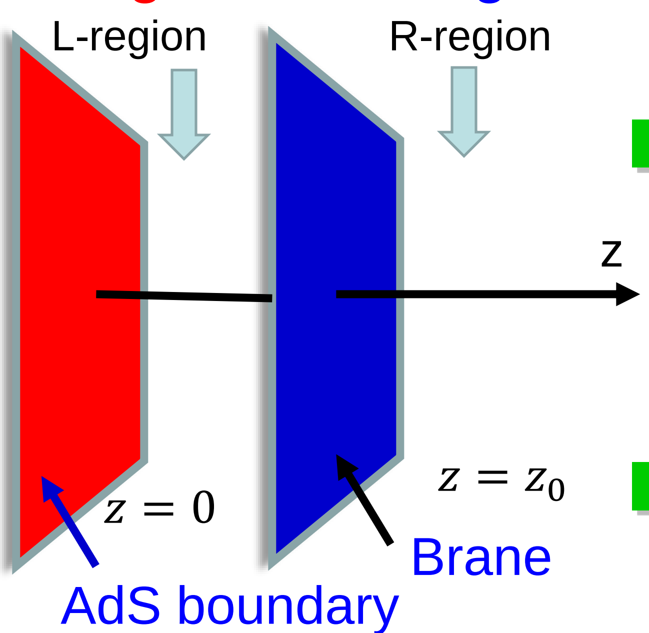
- Robin boundary condition
- Special cases: Dirichlet ($\beta = 0$) and Neumann ($\beta = \infty$)
- There is a region in the space of the parameter β in which the vacuum becomes unstable
- Critical value for the Robin coefficient depends on the lengths of the compact dimensions, on the phases in periodicity conditions and on the mass of the field

Geometry with a brane

- Properties of the vacuum are different in L- and R-regions

L-region ➡ Region between the brane and AdS boundary

R-region ➡ Region between the brane and horizon



- For both L- and R-regions the Hadamard function is decomposed into pure AdS and brane-induced contributions

- Current density along the *l*-th compact dimension

$$\langle j^l \rangle = \langle j^l \rangle_0 + \langle j^l \rangle_b, \; l = p+1, \ldots, D-1,$$

Pure AdS (in the absence of the brane)

Brane-induced

Brane-induced current density

- Brane-induced current density in the R-region

$$\langle j^l \rangle_b = -\frac{eC_p z^{D+2}}{2^{p-1} a^{D+1} V_q} \sum_{\mathbf{n}_q} k_l \int_{k_{(q)}}^{\infty} dx\, x (x^2 - k_{(q)}^2)^{\frac{p-1}{2}} \frac{\bar{I}_\nu(z_0 x)}{\bar{K}_\nu(z_0 x)} K_\nu^2(zx),$$

I_ν, K_ν modified Bessel functions $\quad C_p = \dfrac{\pi^{-(p+1)/2}}{\Gamma((p+1)/2)}$

$$k_{(q)}^2 = \sum_{l=p+1}^{D-1} k_l^2 = \sum_{l=p+1}^{D-1} (2\pi n_l + \tilde{\alpha}_l)^2 / L_l^2,$$

Barred notation for a given function *F(x)*

$$\bar{F}(x) = A_0 F(x) + B_0 x F'(x), \ A_0 = 1 + \frac{D}{2}\delta_y \beta / a, \quad B_0 = \delta_y \beta / a$$

$\delta_y = 1$ in the R-region and $\delta_y = -1$ in the L-region

- Brane-induced current density in the L-region by the replacements $\quad I_\nu \to K_\nu,\ K_\nu \to I_\nu$

Asymptotics of the brane-induced current density in R-region

- At large distances from the brane compared with the AdS curvature radius $z \gg z_0$

$$\langle j^l \rangle_b \approx -\frac{e z^{D-(p-1)/2} \tilde{\alpha}_l k_{(q)}^{(0)(p-1)/2}}{2^{p+1}\pi^{(p-1)/2} a^{D+1} V_q L_l} \frac{\bar{I}_\nu(z_0 k_{(q)}^{(0)})}{\bar{K}_\nu(z_0 k_{(q)}^{(0)})} e^{-2zk_{(q)}^{(0)}}$$

Near the horizon the boundary-free part dominates

$$k_{(q)}^{(0)2} = \sum_{i=p+1}^{D-1} \tilde{\alpha}_i^2 / L_i^2$$

- When the location of the brane tends to the AdS boundary, $z_0 \to 0$, the VEV vanishes as $z_0^{2\nu}$

- An important result is that the VEV of the current density is finite on the brane

- For Dirichlet boundary condition both the current density and its normal derivative vanish on the brane

Finiteness of the current density on the brane

- Finiteness of the current density is in clear contrast to the behavior of the VEVs for the field squared and the energy-momentum tensor which suffer surface divergences

- Feature that the VEV of the current density is finite on the brane could be argued on the base of general arguments

- In quantum field theory the ultraviolet divergences in the VEVs of physical observables bilinear in the field are determined by the local geometrical characteristics of the bulk and boundary

- On the background of standard AdS geometry with non-compact dimensions the VEV of the current density in the problem under consideration vanishes by the symmetry

- Compactication of the part of spatial dimensions to torus does not change the local bulk and boundary geometries and does not add new divergences compared with the case of trivial topology

Asymptotics of the brane-induced current density in L-region

- On the AdS boundary the brane-induced contribution vanishes as $z^{D+2\nu+2}$, $z \to 0$
- Near the AdS boundary the boundary-free part in the VEV of the current density behaves in a similar way

On the AdS boundary the ratio of the brane-induced and boundary-free contributions tend to a finite limiting value

- For a fixed value of *z*, when the brane location tends to the AdS horizon, the brane-induced contribution is exponentially suppressed

$$\langle j^l \rangle_b \approx \frac{(1-2\delta_{0B_0})e\tilde{\alpha}_l z^{D+2} e^{-2z_0 k_{(q)}^{(0)}}}{2^p \pi^{(p-1)/2} a^{D+1} V_q L_l z_0^{(p+1)/2}} k_{(q)}^{(0)(p+1)/2} I_\nu^2(z k_{(q)}^{(0)}), \; z_0 \to \infty$$

Applications to Randall-Sundrum 1-brane model

- From the results for the R-region one can obtain the current density in Z_2- symmetric braneworld models of the Randall-Sundrum type with a single brane

- In the original RS 1-brane model the universe is realized as a Z_2-symmetric positive tension brane in 5D AdS and the only contribution to the curvature comes from the negative cosmological constant in the bulk

- Most scenarios motivated from string theories predict the presence of other bulk fields, such as scalar fields

- In addition, string theories also predict small compact dimensions originating from 10D string backgrounds

Generalized RS 1-brane model with compact dimensions

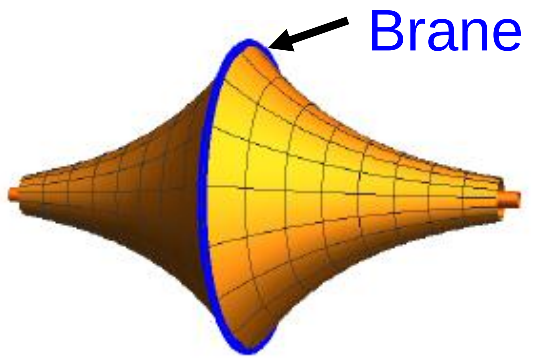
- Background geometry: $ds^2 = e^{-2|y-y_0|/a}\eta_{ik}dx^i dx^k - dy^2$

  Topology $R^p \times T^q$

- Background geometry contains two patches $y > y_0$ of the AdS glued by the brane and related by the Z_2-symmetry identification

$$y - y_0 \longleftrightarrow y_0 - y$$

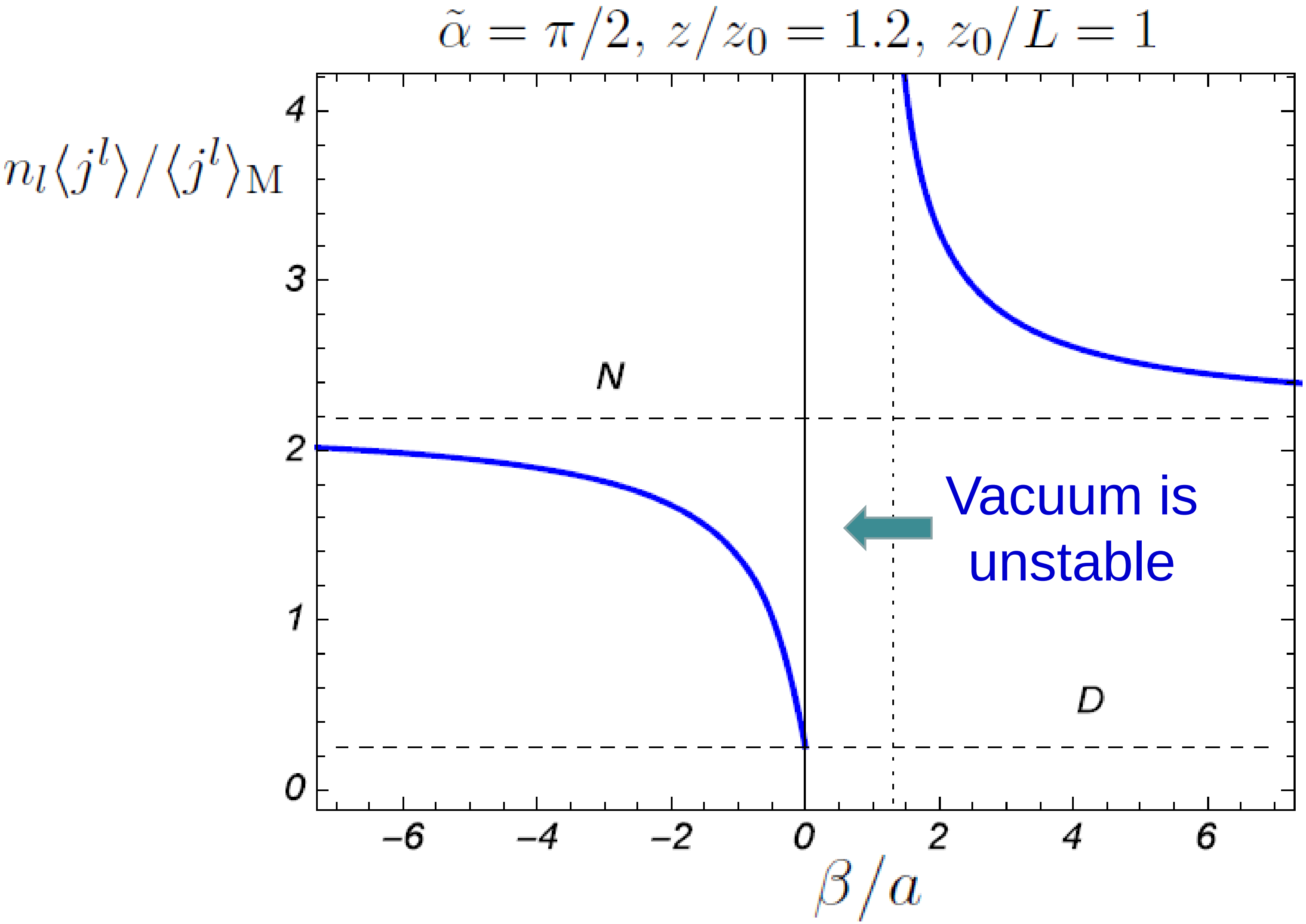
- Spatial geometry in the case $D = 2$, embedded into a 3D Euclidean space



- For fields even under the reflection with respect to the brane (untwisted scalar field) the boundary condition is of the Robin type with $\beta = -1/(c + 2D\xi/a)$, with c being the brane mass term
- For fields odd with respect to the reflection (twisted fields) the boundary condition is reduced to the Dirichlet one

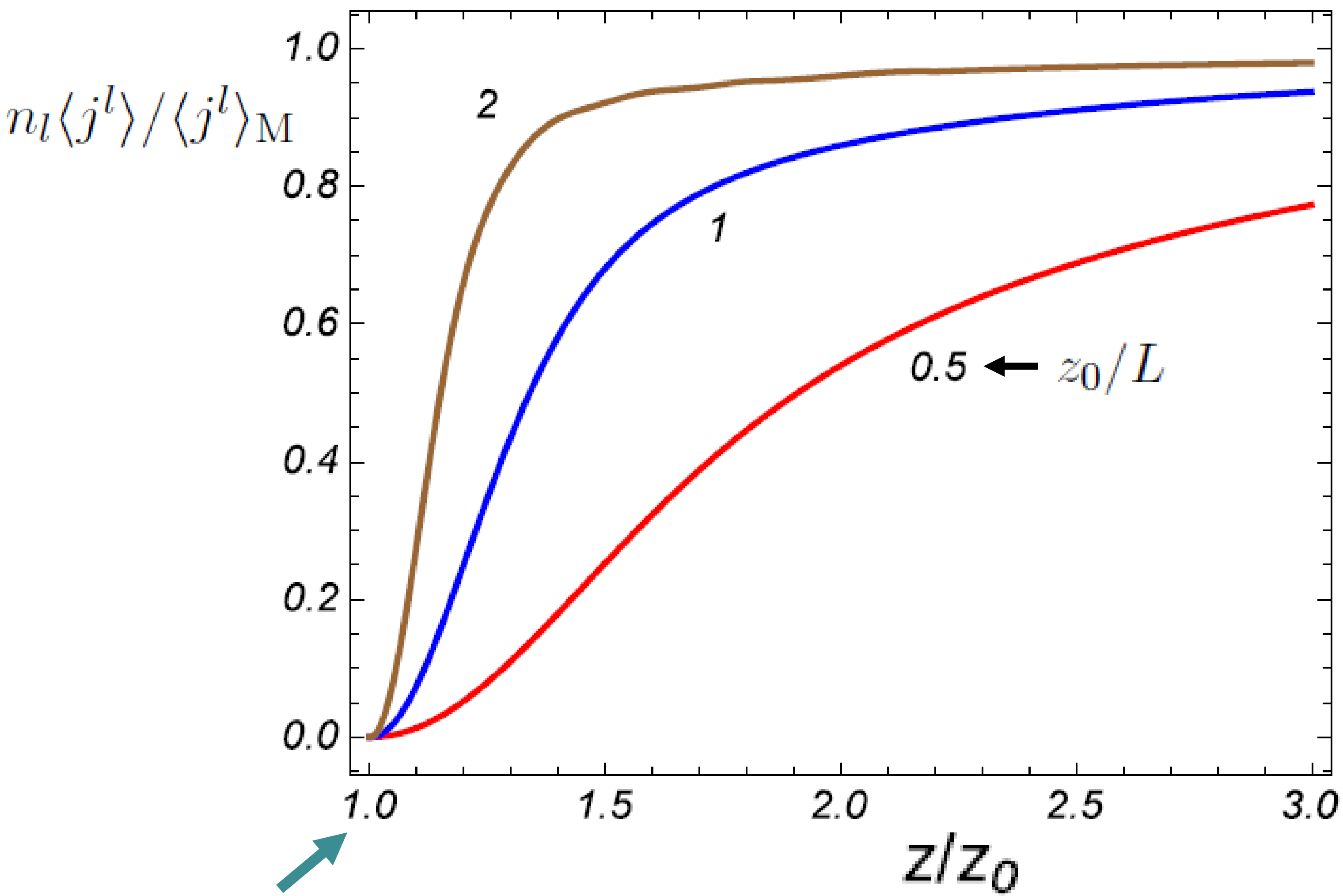
Vacuum current as a function of the Robin coefficient

D = 4 AdS space with a single compact dimension



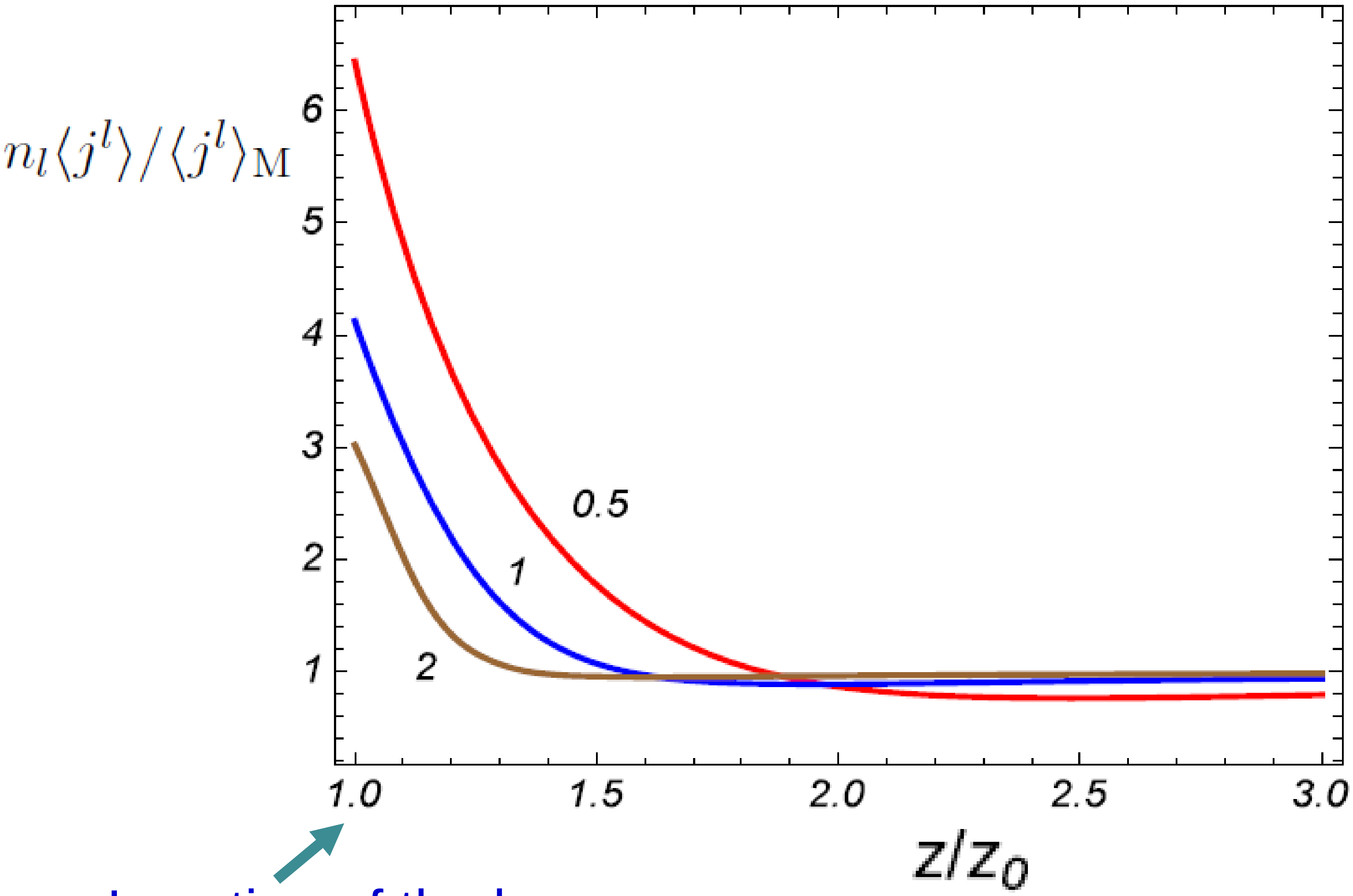
Vacuum current: Dirichlet BC, R-region

$\tilde{\alpha} = \pi/2$



Location of the brane

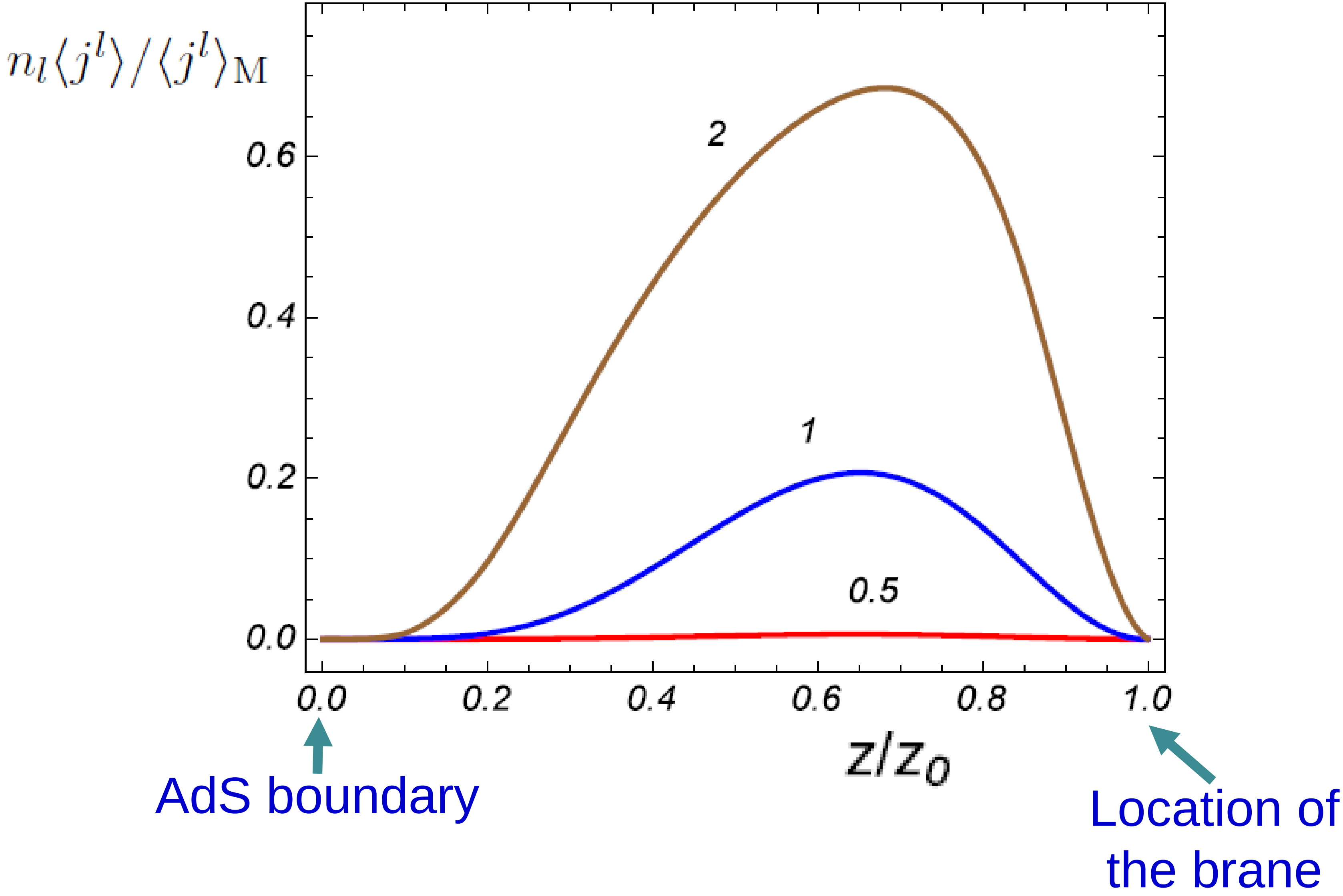
Vacuum current: Neumann BC, R-region



Location of the brane

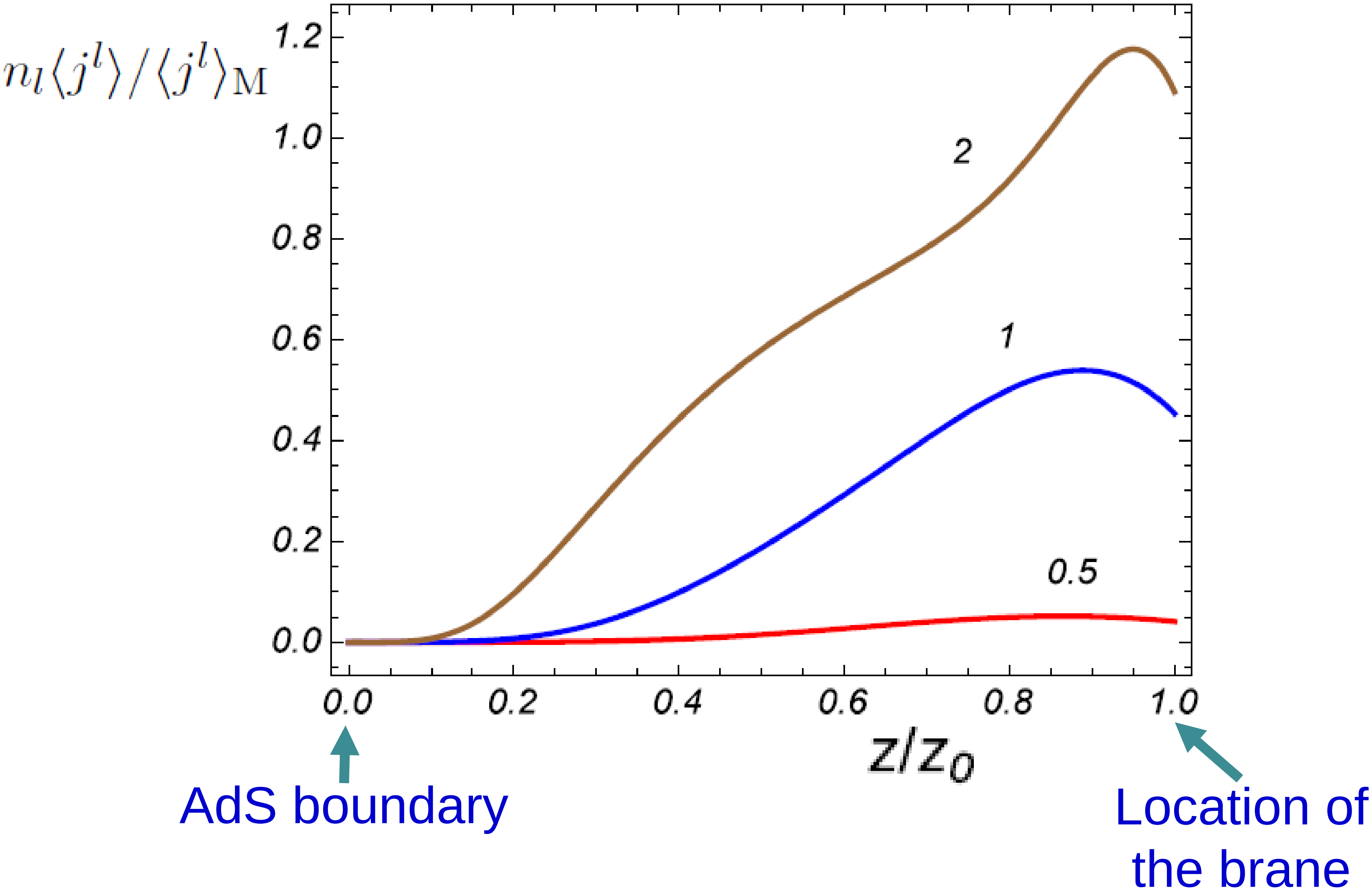
Vacuum current: Dirichlet BC, L-region

Region between the AdS boundary and the brane



Vacuum current: Neumann BC, L-region

Region between the AdS boundary and the brane



Fermionic currents

- Charged Dirac field $i\gamma^{\mu} D_{\mu}\psi - m\psi = 0,\ D_{\mu} = \partial_{\mu} + \Gamma_{\mu} + ieA_{\mu}$

  Spin connection

- Periodicity condition along *l*-th compact dimension

$$\psi(t, x^1, \dots, x^l + L_l, \dots, x^D) = e^{i\alpha_l}\psi(t, x^1, \dots, x^l, \dots, x^D)$$

- Current density along *l*-th compact dimension: AdS spacetime

$$\langle j^l \rangle = -\frac{eNa^{-D-1}L_l}{(2\pi)^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\tilde{\boldsymbol{\alpha}}_{q-1} \cdot \mathbf{n}_{q-1}) \sum_{j=0,1} q_{ma-j}^{(D+1)/2}(b_{\mathbf{n}_q})$$

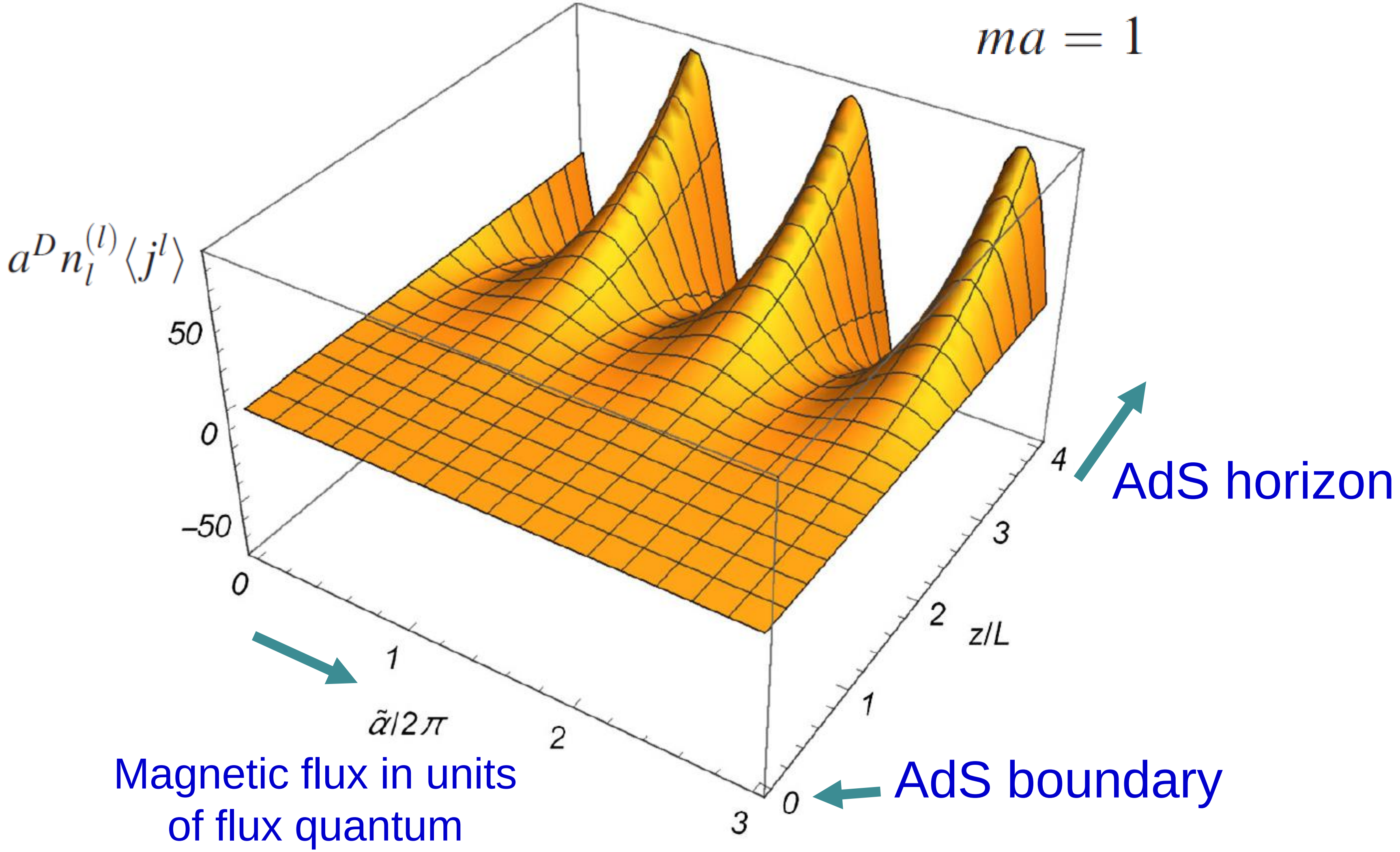
- Current density: Minkowski spacetime

$$\langle j^l \rangle_{\mathrm{M}} = -\frac{2eNL_l m^{(D+1)/2}}{(2\pi)^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1}) \frac{K_{(D+1)/2}(mg_{\mathbf{n}_q})}{g_{\mathbf{n}_q}^{(D+1)/2}}$$

$$b_{\mathbf{n}_q} = 1 + \frac{g_{\mathbf{n}_q}^2}{2z^2}, \qquad g_{\mathbf{n}_q}^2 = \sum_{i=p+1}^{D-1} n_i^2 L_i^2.$$

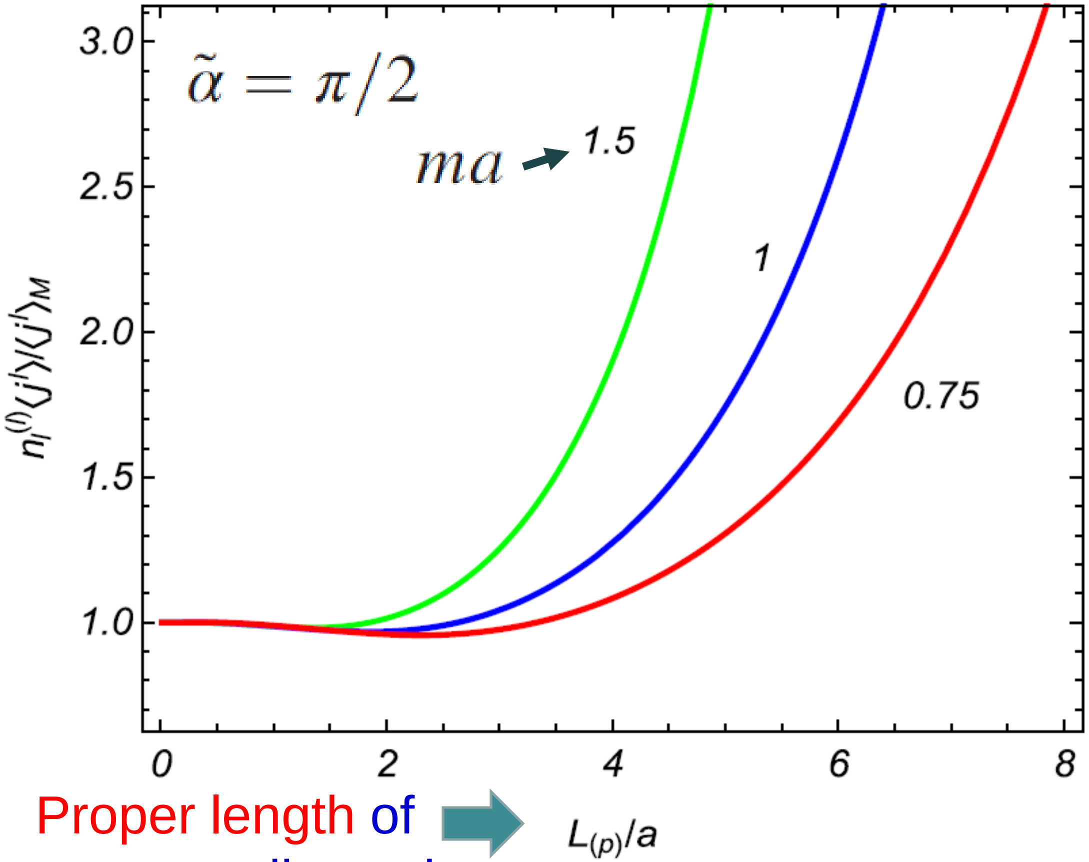
Fermionic current: Example

$D = 4$ model with a single compact dimension



Fermionic current: Example

Ratio of the current densities in locally AdS and Minkowski spacetimes



Proper length of compact dimension

For large values of the proper length the charge flux in the AdS bulk is essentially larger than that for the Minkowski case.

In AdS spacetime the decay of the current density for large values of the proper length goes like a power law

In Minkowski background and for a massive field the decay is exponential

Fermionic currents in models with branes

- Fermionic currents in models with a single brane parallel to AdS boundary

  S. Bellucci, A. A. Saharian, D. H. Simonyan, V. V. Vardanyan, Phys. Rev. D 98, 085020 (2018).

- Scalar and fermionic currents in models with a two branes (generalized Randall-Sundrum model)

  S. Bellucci, A. A. Saharian, V. V. Vardanyan, Phys. Rev. D 93, 084011 (2016).

  S. Bellucci, A. A. Saharian, H. G. Sargsyan, V. V. Vardanyan, arXiv:1907.13379.

- The results for $D=2$ model are applied to deformed graphene nanotubes described by the Dirac model in the long-wavelength approximation

- Current densities along compact dimensions are sources of magnetic fields along uncompact dimensions

Conclusions

- VEV of the current density for a massive scalar field is investigated in the background of AdS spacetime with spatial topology $R^p \times (S^1)^q$
- Charge density and the components along the uncompactified dimensions vanish
- Current density along compactified dimensions is a periodic function of the magnetic flux with the period of the flux quantum
- Current density vanishes on the AdS boundary
- Near the horizon the effects induced by the background curvature are small
- In Kaluza-Klein-type models the current with the components along compact dimensions is a source of cosmological magnetic fields

Thank you