Time-dependent Ginzburg-Landau equations for interacting neutron-proton superfluid in neutron stars

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Devoted to 80th anniversary of professor

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Superfluidity of nuclear matter

A new stage in the investigation of the internal structure of neutron stars and atomic nuclei began with the advent of the microscopic theory of the superconductivity:

J. Bardeen, L.N. Cooper, J.R. Schrieffer, Phys. Rev., 108, 1175, 1957.

A. Bohr, B. Mottelson, D.Pines considered superfluid states in atomic nuclei:

A. Bohr, B. Mottelson, D.Pines, Phys. Rev., 110, 936, 1958.

The basic methods of the microscopic theory of the superconductivity were then used to analyze the internal structure of neutron stars. A.B. Migdal Investigated equation of state of the neutron fluid and was led to possibilityof superfluidity in neutron stars V.L Ginzburg D.A. Kirzhnits used analogy withrotating He II to suggest possibility of configuration of neutron lines in a rotating neutron superfluid. They estimated also the neutron energy gap in the one S zero state was of the order of a few MeV

Introduction. Systems with two types of syperfluid condensates: solution of He^3 atoms in liquid He^4 below point of phase transition of He^3 to the superfluid state, hadronic phase of neutron stars, consisting of neutrons, protons and electrons.

Neutrons in crust pair due to interaction in $^1S_0$ channel.

Neutrons due to strong interaction in $^3P_2$ channel form superfluid condensate and create due to rotation quantized vortex lines :

- A.B. Migdal, , Sov. Phys. JETP 10, 176, 1960.
- V.L. Ginzburg, D.A. Kirzhnits, Sov. Phys. JETP 20, 1346, 1965.
- M. Hoffberg,A. Glassgold, R. Richardson, M. Ruderman, Phys. Rev. Lett. 24, 775, 1970.
- Protons due to interaction in $^1S_0$ channel form superconducting condensate, it supports magnetic fields by forming quantized magnetic flux tubes with quantum of magnetic flux $\Phi_0 = \pi\hbar c / e$:
- R.A. Wolf, Astrophys. J., 145, 166, 1965.
- G. Baym, C. Pethick, D. Pines, Nature, 224, 674, 1969.

The magnetic field f pulsars.
D.M. Sedrakyan, K.M. Shahabasyan, Astrophysics, 8, 326, 1972.

- It is shown that if the proton fluid is assumed superconducting and the electron fluid normal, a magnetic moment arises as a result of rotation and it is parallel to theaxis of rotation

Time-independent GL equations for interacting neutron-proton superfluid neutron stars

Equations for condensate wave functions and supercurrents

$$\frac{1}{4m'_p}\left[-i\hbar\nabla-\frac{2e}{c}\left(\vec{A}+\vec{A}'\right)\right]^2\Psi_1+\left(\alpha_1+\alpha'_1\left|\Psi_2\right|^2\right)\Psi_1+\beta_1\left|\Psi_1\right|^2\Psi_1=0\ ,$$

$$\vec{j}_p=\frac{ie\hbar}{2m'_p}\left(\Psi_1\nabla\Psi_1^*-\Psi_1^*\nabla\Psi_1\right)-\frac{2e^2}{cm'_p}\left(\vec{A}+\vec{A}'\right)\left|\Psi_1\right|^2,$$

$$\frac{1}{4m'_n}\left[-i\hbar\nabla+\vec{A}_p\right]^2\Psi_2+\left(\alpha_2+\alpha'_1\left|\Psi_1\right|^2\right)\Psi_2+\beta_2\left|\Psi_2\right|^2\Psi_2=0,$$

$$\vec{j}_n=\frac{i\hbar m_n}{2m'_n}\left(\Psi_2\nabla\Psi_2^*-\Psi_2^*\nabla\Psi_2\right)+\frac{m_n}{m'_n}\vec{A}_p\left|\Psi_1\right|^2,$$

Coefficients of GL equations

Entrainment vector potential

$$\vec{A}' = -\frac{i\hbar c m'_p N_p}{8 e m_p C_p C_n}\left(\Psi_2 \nabla \Psi_2^* - \Psi_2^* \nabla \Psi_2\right).$$

$$\alpha_1 = \frac{6\pi^2 k^2 T_{c1}\left(T - T_{c1}\right)}{7\zeta(3)\mu_p}, \beta_1 = \frac{6\pi^2 k^2 T_{c1}^2}{7\zeta(3)\mu_p n_p}, \alpha'_1 = -\frac{0,29 g_3^2 m_p^2 m_n^2 \left(kT_{c1}\right)^3}{n_n n_p^{2/3} \hbar^8},$$

Potential due to entrainment of superfluid neutrons by protons

$$\vec{A}_p = \frac{i\hbar m'_n N_n}{4 m_p C_p C_n}\left(\Psi_1 \nabla \Psi_1^* - \Psi_1^* \nabla \Psi_1 + \frac{4ie}{\hbar c}\vec{A}\left|\Psi_1\right|^2\right).$$

BCS type short range neutron-proton nuclear interaction Hamiltonian

It describes interactions which create proton and neutron pairs but there is no neutron- proton pairs due to difference of neutron and proton Fermi levels

$$\hat{H} = \int \Bigg(-\hat{\Psi}^{+}_{\alpha}(\vec{r}) \frac{\nabla^{2}}{2m_{p}} \hat{\Psi}_{\alpha}(\vec{r}) + \frac{g_{1}}{2} \hat{\Psi}^{+}_{\beta}(\vec{r}) \hat{\Psi}^{+}_{\alpha}(\vec{r}) \Psi_{\alpha}(\vec{r}) \Psi_{\beta}(\vec{r}) - \hat{\Phi}^{+}_{\alpha}(\vec{r}) \frac{\nabla^{2}}{2m_{n}} \Phi_{\alpha}(\vec{r})$$
$$+ \frac{g_{2}}{2} \hat{\Phi}^{+}_{\beta}(\vec{r}) \hat{\Phi}^{+}_{\alpha}(\vec{r}) \hat{\Phi}_{\alpha}(\vec{r}) \hat{\Phi}_{\beta}(\vec{r}) + g_{3} \hat{\Psi}^{+}_{\beta}(\vec{r}) \hat{\Phi}^{+}_{\alpha}(\vec{r}) \hat{\Phi}_{\alpha}(\vec{r}) \Psi_{\beta}(r) \Bigg) dV$$

where $g_{1} = g_{2} = g_{3}$ are negative.Formalism of anomalous Green functions is used and the solution of equations for the temperature Green functions of protons and neutrons are obtained near the lower critical temperature.

References

L.P. Gorkov, Sov. Phys. JETP 7, 505, 1958.

L.P. Gorkov, Sov. Phys. JETP 9,1364, 1959.

D.M. Sedrakian, K.M. Shahabasyan, G.A. Vardanyan, Uch. Zapiski EGU No2, 72, 19.

D.M. Sedrakian, K.M. Shahabasyan, Uch. Zapiski EGU No1, 46, 1980.

D.M. Sedrakian, K.M. Shahabasyan, Astrophysics, 16, 417, 1980.

D.M. Sedrakian, K.M. Shahabasyan, Sov. Phys. Usp. 34(7), 555, 1991.

Coefficients of entrainments potentials

Condensate wave functions of protons and neutrons and their superfluid velocieties

$$N_p = \frac{0.318 g_3^2 m_p^2 m_n n_p}{\pi^4 \hbar^6 \left(kT_{c1}\right)^3}, L_p = \frac{0.768 g_3^2 m_p m_n^2 n_p}{\pi^4 \hbar^6 \left(kT_{c1}\right)^3},$$

$$C_p = \frac{7\zeta(3) n_p}{16\pi^2 \left(kT_{c1}\right)^2}, C_n = \frac{7\zeta(3) n_n}{16\pi^2 \left(kT_{c1}\right)^2}, m_p' = \frac{m_p}{1 + \left(L_p \left|\Psi_2\right|^2\right) / 2C_p C_n}.$$

$$\Psi_1 = \sqrt{\left(n_{ps} / 2\right)} e^{i\varphi_1}, \Psi_2 = \sqrt{\left(n_{ns} / 2\right)} e^{i\varphi_2},$$

$$v_1 = \frac{\hbar}{2m_p} \nabla \varphi_1 - \frac{e}{m_p c} \vec{A}, v_2 = \frac{\hbar}{2m_n} \nabla \varphi_2.$$

Proton super-current , proton and neutron mass currents

Coefficients of proton and neutron mass currents

$$\vec{j}_p = en_{ps} K_p \vec{\mathrm{v}}_1 + en_{ps} R_p \vec{\mathrm{v}}_2 ,$$

$$\vec{j}_{pm} = m_p n_{ps} K_p \vec{\mathrm{v}}_1 + m_p n_{ps} R_p \vec{\mathrm{v}}_2 ,$$

$$\vec{j}_n = m_n n_{ns} K_n \vec{\mathrm{v}}_2 + m_n n_{ns} R_n \vec{\mathrm{v}}_1 ,$$

$$K_p = 1 + \frac{0.69 g_3^2 m_p m_n^2 n_{ns} kT_{c1}}{\pi^2 \hbar^6 n_n}, \mathrm{R}_p = \frac{1.15 g_3^2 m_p m_n^2 n_{ns} kT_{c1}}{\hbar^6 n_n}.$$

$$K_n = 1 + \frac{0.69 g_3^2 m_p^2 m_n n_{ps} kT_{c1}}{\pi^2 \hbar^6 n_n^{1/3} n_p^{2/3}}, \mathrm{R}_n = \frac{1.15 g_3^2 m_p^2 m_n n_{ps} kT_{c1}}{\hbar^6 n_n}.$$

Kinetic energy density

Proton and neutron muss currents can be written in the form

$$j_{pm} = \frac{m_p}{e} j_p = \rho_{11} \mathrm{v}_1 + \rho_{12} \mathrm{v}_2 ,$$

$$j_n = \rho_{21} \mathrm{v}_1 + \rho_{22} \mathrm{v}_2$$

some superfluid protons travel with the velocity superfluid neutrons this produces electric entrainment current. Here

ρ_{12} -is mass density of entrained protons and ρ_{21} is of entrained neutrons. ρ_{11} and ρ_{22} - mass densities of unentrained protons and neutrons .

The kinetic energy assumes the form

$$T_k = \frac{1}{2}\left(\rho_{11} \mathrm{v}_1^2 + 2\rho_{12} \mathrm{v}_1 \mathrm{v}_2 + \rho_{22} \mathrm{v}_2^2\right)$$

Entrainment of superconducting protons by superfluid neutrons at temperature T=0

Interaction ^3He condensate with moving ^4He condensate :

$$\varepsilon(p) = \frac{p^2}{2m^*} + \frac{\delta m}{m^*} p \mathrm{v}_2 - \frac{\delta m}{m^*} \frac{m\mathrm{v}_2^2}{2}$$

J. Bardeen, G. Baym, D. Pines, Phys. Rev.,1967, 156, 1967.

Protons energy spectrum becomes

$$\varepsilon(p) = \frac{1}{2m_p^*}\left(p - \frac{e}{c}A'\right)^2 - \frac{1}{2}\delta m_p \mathrm{v}_n^2$$

$$A' = A - \frac{c\delta m_p}{e}\mathrm{v}_n^2$$

D.M. Sedrakyan, K.M. Shahabasyan, Astrophysics, 16, 417, 1980.

$$j_p = -\frac{n_p e^2}{m_p^* c} A' = -\frac{e^2 n_p}{m_p^* c} A + \frac{e n_p \delta m_p}{m_p^*} \mathrm{v}_n$$

Magnetic field generation in pulsars.

D.M. Sedrakyan, Astrophysics, 18, 253, 1982.

D.M. Sedrakyan, K.M. Shahabasyan, A.G. Movsisyan, Astrophysics, 19, 175, 1983.

London equation for the hadronic phase of neutron star.

Magnetic fields generated by the entrainment currents is

$$curl\vec{H} = \frac{4\pi}{c}\vec{j}_{12}$$

Due to presence non-entrainmented superfluid protons $\vec{H}$ is different from magnetic induction $\vec{B}$ which is determined by

$$curl\vec{B} = \frac{4\pi}{c}\left(\vec{j}_{11} + \vec{j}_{12}\right)$$

Substituting $\vec{j}_p$ into the equation of magnetic induction and recalling that

$$curl\vec{\mathrm{v}}_1 = -\frac{e}{m_1 c}\vec{B} + \frac{\pi\hbar}{m_p}\vec{i}_1\sum_i \delta\left(r - r_i\right)$$

$$curl\vec{\mathrm{v}}_2 = \frac{\pi\hbar}{m_n}\vec{i}_2\sum_j \delta\left(\vec{r} - \vec{r}_j\right),$$

we obtain

$$\vec{B} + \lambda^2 curlcurl\vec{B} = \Phi_0\vec{i}_1\sum_i \delta\left(\vec{r} - \vec{r}_i\right) + \Phi_1\vec{i}_2\sum_j \delta\left(\vec{r} - \vec{r}_j\right)$$

Free energy for a two-component superfluid

For the identification of the vortex structures formed in the system we shall write free energy for the system

$$F = \frac{1}{2}\int\left(\rho_{11}\mathrm{v}_1^2 + 2\rho_{12}\mathrm{v}_1\mathrm{v}_2 + \rho_{22}\mathrm{v}_2^2 + \rho^{(n)}\vec{\mathrm{v}}_n^2\right)dV + \frac{1}{8\pi}\int \vec{B}^2 dV - \vec{M}\vec{\Omega}$$

Angular momentum $\vec{M}$ of the superfluid is given by

$$\vec{M} = \int\left[\vec{r}\times\left(\vec{j}_{pm} + \vec{j}_n + \rho^{(n)}\vec{\mathrm{v}}^{(n)}\right)\right]dV$$

where $\vec{\mathrm{v}}_1$ is determined from Maxwell equation. Substituting for $\vec{\mathrm{v}}_1$ in free energy, we obtain

$$F = \frac{1}{8\pi}\int\left[\vec{B}^2 + \lambda^2\left(curl\vec{B}\right)^2\right]dV + \frac{1}{2}\int\rho'_{22}\left(\vec{\mathrm{v}}_2 - \left[\vec{\Omega}\times\vec{r}\right]\right)^2 dV - \frac{1}{2}\int\rho\left[\vec{\Omega}\times\vec{r}\right]^2 dV$$

where

$$\rho'_{22} = \rho_{22} - \frac{\rho_{12}^2}{\rho_{11}}$$

Gibbs free energy for a two-component superfluid

Correct thermodynamic potential in presence of given entrainment currents:

$$G = F - \frac{1}{c}\int \vec{j}_{12}\vec{A}dV = F - \frac{1}{4\pi}\int \vec{H}\vec{B}dV$$

Magnetic structure of neutron vortex is given by a distribution density of proton vortices n_p . To find it we must minimize Gibbs potential for an individual neutron vortex: $G_p = G + \int n_p \varepsilon_p dV$

Energy of proton vortex ε_p and entrainment current j_{12} are:

$$\varepsilon_p = \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \ln\frac{\lambda}{\xi_1}, j_{12} = \rho_{12}\frac{e}{m_p}\frac{\kappa_2}{2\pi}\frac{1}{r}.$$

Magnetic field strength by entrainment currents has the form;

$$H(r) = \frac{\Phi_1}{2\pi\lambda^2}\ln\left(\frac{b}{r}\right), \Phi_1 = \Phi_0\frac{\delta m_p}{m_n}, \Phi_0 = \frac{\pi\hbar c}{e}, H(b) = 0$$

Cluster of proton vortices in a Neutron vortex

Following expression for n_p had been found:

$$n_p = \frac{H(r) - H_{c1}}{\Phi_0}, H_{c1} = \frac{\Phi_0}{4\pi\lambda^2} \ln\frac{\lambda}{\xi_p}, \quad \bar{B} =$$

where H_{c1} i s critical field for the formation of proton vortex. From $H(r) = H_{c1}$

$$r_1 = \left(\frac{\lambda}{\xi_p}\right)^{-1/2|k|}.$$

Average induction $\bar{B}$ of a neutron vortex:

$$\bar{B} = \frac{1}{\pi b^2}\int_0^{r_1} \Phi_0 n_p 2\pi r dr = \frac{k\Phi_0}{4\pi\lambda^2}\left(\frac{\xi_p}{\lambda}\right)^{1/|k|}, \mathrm{k} = \frac{\delta \mathrm{m}_p}{m_n}$$

$$\Omega = 200 s^{-1}, n_p = 10^{36} cm^{-3}, R = 10 km, b = 1.25 \cdot 10^{-3} cm, \lambda = 2.2 \cdot 10^{-11} cm$$

Mean magnetic induction and magnetic moment of star

Magnetic moment M

$$M = \frac{4}{3}\pi R^3 \frac{3}{8\pi}\bar{B} = \frac{\pi R^3}{2}\bar{B}$$

Coefficient $3/8\pi$ appears because magnetization of a uniformally magnetized sphere is $\mu = 3\bar{B}/8\pi$. Substituting usual values $\xi_p = 10^{-12}\,cm$ and $\lambda = 10^{-11}\,cm$ we obtain $\bar{B} = 10^{12}\,G$ in a neutron star . The magnetic moments are of the order of $M = 10^{30}\,G \cdot cm^3$.

Superfluid Core Rotation inPulsars. I Vortex Cluster Dynamics. A.D. Sedrakian, D.M. Sedrakian, Ap. J. 447, 305, 1995.

Magnetohydrodynamic theory for rotating neutron-proton superfluid mixture in neutron star cores is formulated. Theory incorporates effects of energy dissipation and mutual friction. Equation of motion of uncoupled neutron and proton vortices in the bulk and boundaries of superfluid core are derived. As a result of entraimant of proton supercurrents by superfluid neutron vortex circulation, rotation induced supercurrents and magnetic fields are generated. Magnetic field enters the vicinity of each vortex line by forming triangular two-dimensional lattice (vortex cluster) confined around neutron vortex line within macroscopic length scale 10-5 cm. Net number of proton vortex lines in cluster is 1012-1013, producing mean induction 1014 G. Acissymmetric magnetic field induction averaged over coreis of the order 1011-1012 G. Generated component of neutron star magnetic field, in contrast to a possible magnetic field is indipendent of its magnetic history prior to the nucleation superconducting phase.

Superfluid Core Rotation inPulsars.I I Postjump Relaxation. A.D. Sedrakian, D.M. Sedrakian, J.M. Cordes, Y. terzian Ap. J. 447, 324, 1995.

Theory of nonstationary dynamics of neutron superfluid core rotation, based on the dynamics of proton vortex clusters is presented. Exact solutions describing the postjump relaxation of the superfluid component of the star are given with allowance for the spatial dependence of viscous friction. In this theory the core is coupled on timescales of hours to years, rather than few seconds coupling times in models where vortex clusters are ignored. The postjump relaxation of vela pulsar can be understood in terms of dynamics of the superfluid core. It is predicted that millisecond pulsars will not showtiming irregularities on timescales larger than a few days.

Time-dependent GL equations

Time-dependent GL equations for proton condensate:

$$-\Gamma_1\left(\frac{\partial}{\partial t}+\frac{2ie}{\hbar}(\phi+\phi_r)\right)\Psi_1=\left(\alpha_1+\alpha_1'\left|\Psi_2\right|^2\right)\Psi_1+\beta_1\left|\Psi_1\right|^2\Psi_1+\frac{1}{4m_p'}\left[-i\hbar\nabla-\frac{2e}{c}\left(\vec{A}+\vec{A}'+\vec{A}_r\right)\right]^2\Psi_1,$$

$$\vec{j}_1=\frac{ie\hbar}{2m_p'}\left(\Psi_1\nabla\Psi_1^*-\Psi_1^*\nabla\Psi_1\right)-\frac{2e^2}{cm_p'}\left(\vec{A}+\vec{A}'+\vec{A}_r\right)\left|\Psi_1\right|^2+\sigma_n\left[-\frac{1}{c}\frac{\partial\left(\vec{A}+\vec{A}'+\vec{A}_r\right)}{\partial t}-\nabla\phi+\vec{E}_r\right],(13)$$

$$\varphi_r=\frac{1}{2}\frac{m_p\omega^2r^2}{e},\vec{E}_r=-\frac{m_p\omega^2\vec{r}}{e},\vec{A}_r=\frac{m_pc}{e}\left[\vec{\omega}\vec{r}\right].$$

$$\Gamma_1=\frac{3\pi^3\hbar kT_{c1}}{28\zeta(3)\mu_p}.\qquad\qquad \tau_1=\frac{\Gamma_1}{\left|\alpha_1\right|}=\frac{\pi\hbar}{8k\left(T_{c1}-T\right)}$$

$$\tau_j=\frac{\sigma_n m_p'\beta_1}{2e^2\left|\alpha_1\right|}=\frac{3\sqrt{3}c^2\hbar^4n_p^{1/2}m_p'}{\pi^{3/2}m_p^{3/2}e^3k^2T_{c1}^2}\left(1-\frac{T}{T_{c1}}\right)^{-1}$$

Time-dependent GL equations , method:
A. Schmid, Phys. Kondens. Materie, 5, 302, 1966.

- Time-dependent GL equations for neutron condensate

$$-\Gamma_2 \frac{\partial \Psi_2}{\partial t} = \left(\alpha_2 + \alpha_1' \left|\Psi_1\right|^2\right)\Psi_2 + \beta_2 \left|\Psi_2\right|^2 \Psi_2 + \frac{1}{4m_n'}\left[-i\hbar\nabla + \vec{A}_p\right]^2 \Psi_2,$$

$$\vec{j}_n = \frac{i\hbar m_n}{2m_n'}\left(\Psi_2 \nabla \Psi_2^* - \Psi_2^* \nabla \Psi_2\right) + \frac{m_n}{m_n'}\vec{A}_p \left|\Psi_2\right|^2,$$

$$\Gamma_2 = \frac{3\pi^3 \hbar k T_{c2}}{28\zeta(3)\mu_n}.$$

$$\tau_2 = \frac{\Gamma_2}{\left|\alpha_2\right|} = \frac{\pi\hbar}{8k\left(T_{c2} - T\right)}.$$

Energy balance

Free energy of electromagnetic field

$$F_{em} = \frac{1}{8\pi}\int\left(E^2 + H^2\right)dV,$$

Free energy of superconducting state

$$F_{sn} = \int\Big\{\alpha_1\left|\Psi_1\right|^2 + \alpha_1'\left|\Psi_1\right|^2\left|\Psi_2\right|^2 + \frac{1}{2}\beta_1\left|\Psi_1\right|^4 + \alpha_2\left|\Psi_2\right|^2 + \frac{1}{2}\beta_2\left|\Psi_2\right|^2 +$$

$$+\frac{1}{4m_p'}\left|\left[-i\hbar\nabla - \frac{2e}{c}\left(\vec{A} + \vec{A}' + \vec{A}_r\right)\right]\Psi_1\right|^2 + \frac{1}{4m_n'}\left|\left[-i\hbar\nabla\right]\Psi_2\right|^2 \Big\}dV.$$