Low-mass stars in modified gravity

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Motivation: Low-mass stars and modified gravity...?

So far, the modified gravity community is used to play mainly in the cosmological framework and/or astrophysical ones:

- neutron stars[1]
- black holes[2]
- exotic environment (wormholes, branes, cosmic strings, ...).

Growing data sets, especially on gravitational waves detections, allows to test and rule out[3] some of the gravitational theory proposal.

A long way to go in order to extract useful information about the internal structure, dynamics and composition of neutron stars:
a degeneracy in the mass-radius profiles (see the next slide).

[1] B.P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 119 (2017)161101;

[2] B. P. Abbottet al.[LIGO Scientific and Virgo Collabo-rations], Phys. Rev. Lett.116(2016) 061102; K. Akiyamaet al.[Event Horizon Telescope Collabora-tion], Astrophys. J.875(2019) L1

[3] J. Ezquiaga et al., Phys. Rev. Lett. 119, 251304 (2017); S. Boran et al., Phys. Rev. D 97, 041501 (2018); R. Sanders, Int.J.Mod.Phys.D, 27, 14, (2018) 1847027; B. P. Abbottet al.[LIGO Scientific and Virgo Collabo-rations], Phys. Rev. Lett. 123, 011102 (2019); S. Jana et al., Phys. Rev. D 99, 044056 (2019);

$f(R)$ gravity in metric formalism[4]

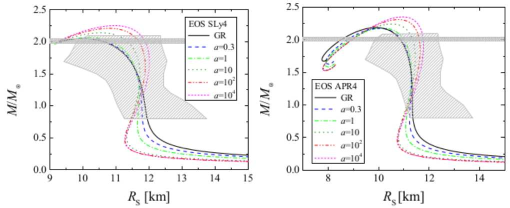


Figure 1. The mass of radius relation for EOS SLy4 (left panel) and APR4 (right panel). Different styles and colors of the curves correspond to different values of the parameter a. The current observational constrains are shown as shaded regions.

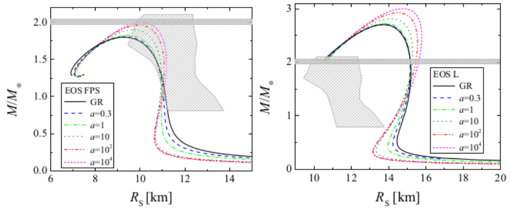


Figure 2. The mass of radius relation for EOS FPS (left panel) and L (right panel). Different styles and colors of the curves correspond to different values of the parameter a. The current observational constrains are shown as shaded regions.

[4] S. Yazadjiev, D. Doneva, K. Kokkotas, K. Staylov, 2014

Non-relativistic stars (NRS)

Assuming that $p << \rho$ together with $4\pi r^3 p << \mathcal{M}$ and $\frac{2G\mathcal{M}}{r} << 1$, the relativistic Tolman-Oppenheimer-Volkoff equation

$$\frac{dp}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2}\frac{\left(1+\frac{p(r)}{\rho(r)}\right)\left(1+\frac{4\pi r^3 p(r)}{\mathcal{M}(r)}\right)}{1-\frac{2G\mathcal{M}(r)}{r}}$$

is simplified to Newton's hydrostatic equilibrium

$$\frac{dp}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2}.$$

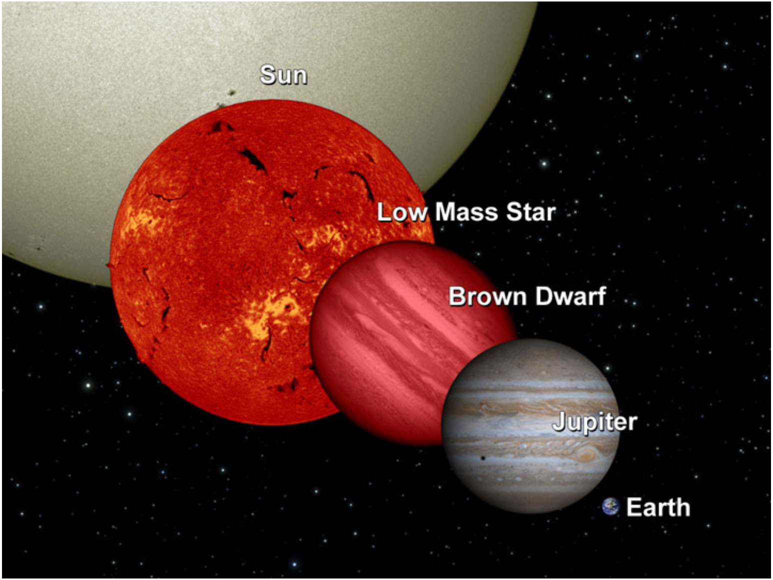
A simple model, together with the polytropic EoS to described the main-sequence stars.
However, the low mass objects such as red and **brown dwarf stars** are well described by it.

Why I am interested in low mass stars: Brown dwarfs[5]

- BD stars are well-described by the **polytropic models**:
  - chemically homogeneous except for the photosphere,
  - their observational properties are weak functions of their opacity and metallicity,
  - static: the lack of the chemical evolution apart from their very early phases,
  - **weakly dependent on non-gravitational physics thus perfect for testing gravitational theories which modify Lane-Emden equations.**
- A radius of BD almost independent of its mass over a large range of masses:
  - polytropic EoS in GR framework predict $0.1R_\odot$ - close to the observed value,
  - measuring the mass-radius relation of objects can place a new constraint on modified gravity.

[5] A. Burrows, J. Liebert, Rev. Mod. Phys. 65, 301 (1993)

Brown dwarf stars



Brown dwarf stars[6]

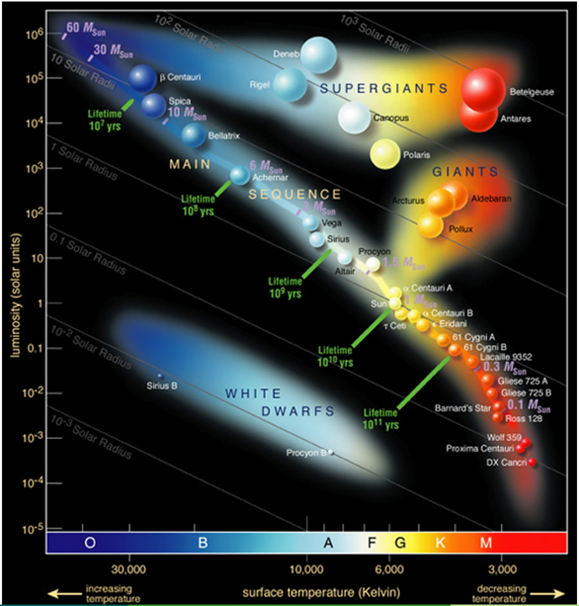
... with mass between a giant planet such as Jupiter $M_J \approx 0.001 M_\odot$ and main sequence stars $M \lesssim 0.08 M_\odot$.

"Aborted stars" or **"Failed stars"**

- Composed of molecular H and He in the liquid metallic phase.
- Near the surface - a thin layer with very low density. The fluid there exists as a weakly coupled plasma satisfying **the ideal gas law**.
- During a star formation it contracts under its own self-gravity $\rightarrow$ T_c and ρ_c grow.
- The contraction stops at the onset of **electron degeneracy** or thermonuclear fusion.
- Only object sufficiently heavy can reach central conditions capable of thermonuclear ignition before the fluid becomes degenerate.
- A star does not achieve thermonuclear ignition before degeneracy $\rightarrow$ **brown dwarf**.
- Pressure almost independent of the temperature, surface radiation does not lead to further contraction so **the brown dwarf cools over time**.
- Typically: $T_c \sim 10^6$K, $\rho_c \sim 10^3$g/cm^3.

[6] A. Burrows, J. Liebert, Rev. Mod. Phys. 65, 301 (1993)

Hertzsprung-Russel diagram



Non-relativistic stars: Lane-Emden equation

The **polytropic equation of state** (K constant depending on the composition of the fluid) with $n = \{1, \frac{3}{2}, 3\}$ (neutron stars, red giants & brown dwarfs, white dwarfs):

$$p = K\rho^{\Gamma}, \qquad \Gamma = 1 + \frac{1}{n}.$$

Newtonian regime of TOV eq's: $p << \rho$, $4\pi r^3 p << \mathcal{M}$, $2G\mathcal{M}/r << 1$:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}p\right) = -4\pi r^2 \rho.$$

Introducing dimensionless variables:

$$r = r_c \xi;\ \rho = \rho_c \theta^{\frac{1}{\Gamma-1}};\ r_c^2 = \frac{K\Gamma}{4\pi G(\Gamma-1)}\rho_c^{\Gamma-2}$$

allows to rewrite Newtonian TOV as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0.$$

with the boundary conditions at the center of the star $\theta(0) = 1$, $\theta'(0) = 0$.

The first zero $\theta(\xi_1) = 0$ ($R = r_c \xi_1$) defines the radius of the star.

Solutions of the Lane-Emden equation - some formulas

Three important dimensionless quantities are

$$\omega_n \equiv -\xi^2 \left. \frac{d\theta}{d\xi} \right|_{\xi=\xi_R},$$

$$\gamma_n \equiv (4\pi)^{\frac{1}{n-3}} (n+1)^{\frac{n}{3-n}} \omega_n^{\frac{n-1}{3-n}} \xi_R, \quad \text{and}$$

$$\delta_n \equiv -\frac{\xi_R}{3d\theta/d\xi|_{\xi=\xi_R}},$$

which appear in the formula for the mass

$$M = 4\pi r_c^3 \rho_c \omega_n,$$

the mass-radius relation

$$R = \gamma_n \left(\frac{K}{G}\right)^{\frac{n}{3-n}} M^{\frac{n-1}{n-3}},$$

and the central density

$$\rho_c = \delta_n \left[\frac{3M}{4\pi R^3}\right].$$

Brown dwarf stars and modified gravity

Theories of gravity which modify the Lane-Emden equation

- Scalar-tensor theories[7]
- Degenerate Higher-Order Scalar Tensor (DHOST)[8]
- **Palatini** $f(\mathcal{R})$ **gravity**[9]
- ... others not done yet!

https://news.virginia.edu/content/uva-astronomers-find-oasis-brown-dwarf-desert

[7] J. Sakstein, Phys. Rev. Lett. 115 (2015) 201101; Phys. Rev. D 92 (2015) 124045

[8] M. Crisostomi, M. Lewandowski and F. Vernizzi, Phys. Rev. D 100 no.2, 024025 (2019)

[9] G. Olmo, D. Rubiera-Garcia, A. Wojnar, Phys.Rev. D 100 no.4, 044020 (2019)

Palatini gravity in a nutshell

$$S = S_{\rm g} + S_{\rm m} = \frac{1}{2\kappa}\int \sqrt{-g} f(\hat{R}) d^4x + S_{\rm m}(g_{\mu\nu}, \psi_m),$$

where $\hat{R} = \hat{R}^{\mu\nu}(\hat{\Gamma})g_{\mu\nu}$. Modified field equations wrt $g_{\mu\nu}$ and $\hat{\Gamma}$ are

$$\begin{aligned} f'(\hat{R})\hat{R}_{\mu\nu} - \frac{1}{2}f(\hat{R})g_{\mu\nu} =& \kappa T_{\mu\nu}, \\ \hat{\nabla}_\beta(\sqrt{-g}f'(\hat{R})g^{\mu\nu}) =& 0 \quad \rightarrow \quad h_{\mu\nu} = f'(\hat{R})g_{\mu\nu}. \end{aligned}$$

The trace of (**??**) wrt $g_{\mu\nu}$ gives the structural equation

$$f'(\hat{R})\hat{R} - 2f(\hat{R}) = \kappa T,$$

where T is a trace of e-m tensor $T_{\mu\nu}$ wrt $g_{\mu\nu}$, provides $\hat{R} = \hat{R}(T)$.

- Non-linear system of a second order PDE.
- $f(\hat{R}) = \hat{R} - 2\Lambda$ is fully equivalent to the Einstein $R - 2\Lambda$.
- Any $f(\hat{R})$ vacuum solution ($T_{\mu\nu} = 0$) $\rightarrow$ Einstein vacuum solution with the cosmological constant.
- EPS interpretation[10].

[10] J. Ehlers, F.A.E. Pirani, A. Schild, in General Relativity, ed. L.O.'Raifeartaigh (Clarendon, Oxford, 1972)

$f(\hat{R})$ Palatini gravity in Einstein frame[12]

Let us recall that $h_{\mu\nu} = \Phi g_{\mu\nu}$, where $\Phi \equiv f'(\hat{R})$, together with the useful relations[11]

$$\hat{R}_{\mu\nu} = \bar{R}_{\mu\nu}, \quad \bar{R} = h^{\mu\nu}\bar{R}_{\mu\nu} = \Phi^{-1}\hat{R}, \quad h_{\mu\nu}\bar{R} = \ g_{\mu\nu}\hat{R}$$

allowing to rewrite the previous MFE as (Einstein frame):

$$\begin{aligned}\bar{R}_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{R} &= \kappa\bar{T}_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\bar{U}(\Phi),\\ \Phi\,\bar{U}'(\Phi) + \kappa\bar{T} &= 0\end{aligned}$$

where $\bar{U}(\Phi) = \frac{\hat{R}\Phi - f(\hat{R})}{\Phi^2}$ and $\bar{T}_{\mu\nu} = \Phi^{-1}T_{\mu\nu}$.

Scalar-tensor action for the metric $h_{\mu\nu}$ and (non-dynamical) scalar field Φ

$$S(h_{\mu\nu},\Phi) = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-h}\left(\bar{R} - \bar{U}(\Phi)\right) + S_m(\Phi^{-1}h_{\mu\nu},\psi).$$

[11] A. Stachowski, M. Szydłowski, A. Borowiec, EPJC 77, 406 (2017); EPJC 77, 603 (2017)

[12] V. I. Afonso, G. J. Olmo and D. Rubiera-Garcia, Phys. Rev. D 97 (2018) 021503; V. I. Afonso, G. J. Olmo, E. Orazi and D. Rubiera-Garcia, arXiv:1807.06385

Palatini in troubles? Well... no

Metric-affine theories (Palatini) seems to suffer serious **problems at the star's surface** when polytropic stars considered:

- For $3/2 < \Gamma < 2$ a singular behavior of the metric curvature invariants[13]
- For $\Gamma = 5/3$ (degenerate non-relativistic electron gas): infinite tidal forces[14]
- EiBI gravity in similar troubles[15]

How to deal with that? [16]

- Inappropriate EoS used.
- Understanding the structure of the theory.
- Unphysical values at the surface.
- Gravitational backreaction on the matter dynamics modifies the effective description.

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[13] E. Barausse, T.P. Sotiriou, J.C. Miller, CQG 25.6 (2008) 062001, EAS Publications Series 30 (2008) 189

[14] E. Barausse, Enrico, T.P. Sotiriou, J.C. Miller, CQG 25.10 (2008): 105008

[15] P. Pani and T. P. Sotiriou, Phys. Rev. Lett. 109 (2012) 251102

[16] F.A.T. Pannia et al., Gen.Rel.Grav. 49.2 (2017): 25; G.J. Olmo, PRD 78.10 (2008) 104026; H-Ch. Kim, Phys. Rev. D 89.6 (2014): 064001; A. Mana, L. Fatibene, M. Ferraris, JCAP 2015.10 (2015): 040

Relativistic stars in Palatini $f(\mathcal{R})$ gravity

The modified TOV equations are provided, together with the stability analysis which resembles the one in General Relativity[17]

$$\mathcal{M}(r) = \int_0^r 4\pi\tilde{r}^2 \frac{Q(\tilde{r})}{\phi(\tilde{r})^2} d\tilde{r},$$

$$\frac{d}{dr}\Big(\frac{\Pi(r)}{\phi^2(r)}\Big) = -\frac{\tilde{A}G\mathcal{M}}{r^2}\left(\frac{\Pi + Q}{\phi(r)^2}\right)\left(1 + 4\pi r^3 \frac{\Pi}{\phi(r)^2\mathcal{M}}\right)$$

where $\phi = f'(\hat{R})$ while $\Pi = \rho + \frac{U}{2\kappa^2 c^2}$ and $Q = \rho - \frac{U}{2\kappa^2 c^2}$ are generalized pressure and energy density, respectively.

Notice that the coordinate r is the one from the Einstein frame so before running your numerical analysis make the conformal transformation

$$r^2 \to \phi r^2$$

[17] A. Wojnar, EPJC 78 (2018) no.5, 421

Polytropic Palatini stars: generalized Lane-Emden equation

For the quadratic model (Starobinsky)

$$f(\mathcal{R}) = \mathcal{R} + \beta\mathcal{R}^2$$

we write the generalized Lane-Emden equation[18]

$$\xi^2\theta^n\Phi^{3/2} + \frac{1}{1+\frac{\xi\Phi_\xi}{2\Phi}}\frac{d}{d\xi}\left(\frac{\xi^2\Phi^{3/2}}{1+\frac{\xi\Phi_\xi}{2\Phi}}\frac{d\theta}{d\xi}\right) = 0 \; .$$

where $\Phi = 1 + 2\alpha\theta^n$, $\Phi_\xi = d\Phi/d\xi$, and

$$\alpha = -\kappa^2 c^2 \beta\rho_c$$

with ρ_c being the star's central density.

Two exact solutions[19] for $n = \{0, 1\}$ while numerical solutions - **no problems!**

[18] A. Wojnar, EPJ C 79 no.1, 51 (2019)

[19] A. Sergyeyev, A. Wojnar, arXiv:1901.10448

Crucial ingredients of Lane-Emden eq. in Palatini gravity

The star's mass, radius, central density, and temperature seems to be like in GR

$$M = 4\pi r_c^3 \rho_c \omega_n, \qquad \rho_c = \delta_n \left(\frac{3M}{4\pi R^3} \right)$$

$$R = \gamma_n \left(\frac{K}{G} \right)^{\frac{n}{3-n}} M^{\frac{n-1}{n-3}} \xi_R, \qquad T = \frac{K\mu}{k_B} \rho_c^{\frac{1}{n}} \theta_n,$$

where k_B is Boltzmann's constant and μ the mean molecular weight.

However, ω_n, γ_n, and δ_n depend now on the solutions of LE eq.[20] and $\Phi = 1 + 2\alpha\theta^n$

$$\omega_n = -\frac{\xi^2 \Phi^{\frac{3}{2}}}{1 + \frac{1}{2}\xi \frac{\Phi_\xi}{\Phi}} \frac{d\theta}{d\xi} \Big|_{\xi=\xi_R}, \qquad \delta_n = -\frac{\xi_R}{3 \frac{\Phi^{-\frac{1}{2}}}{1+\frac{1}{2}\xi\frac{\Phi_\xi}{\Phi}} \frac{d\theta}{d\xi} \Big|_{\xi=\xi_R}}$$

$$\gamma_n = (4\pi)^{\frac{1}{n-3}} (n+1)^{\frac{n}{3-n}} \omega_n^{\frac{n-1}{3-n}} \xi_R$$

[20] A. Sergyeyev, A. Wojnar, arXiv:1901.10448

Minimum main sequence mass[21]

... is the lowest mass where the rate-limiting reaction for hydrogen burning can be sustained stably.
It means that energy generated in the core is compasated by energy radiated from the surface, which corresponds to the mass where

$$L_{\text{hydrogen burning}} = L_{\text{photosphere}}.$$

The thermonuclear ignition is powered by the 3 main chain reactions:

$$\begin{aligned} p + p &\to d + e^+ + \nu_e, \\ p + e^- + p &\to d + \nu_e, \\ p + d &\to {}^3H_e + \gamma \end{aligned}$$

where the first one is a slow process acting as a bottle-neck behind the MMSM bound. The energy generation rate per unit mass if this process

$$\dot{\epsilon}_{pp} = \dot{\epsilon}_c \left(\frac{T}{T_c}\right)^s \left(\frac{\rho}{\rho_c}\right)^{u-1}, \quad s \approx 6.31, \quad u \approx 2.28.$$

21 A. Burrows and J. Liebert, Rev. Mod. Phys. **65** (1993) 301.

Luminosity from hydrogen burning

Integrate $\dot{\epsilon}_{pp}$ over the stellar volume with $\theta(\xi \approx 0) = \exp\left[-\frac{\xi^2}{6}\right]$ to get:

$$L_{HB} = 4\pi r_c^3 \rho_c \dot{\epsilon}_c \int_0^{\xi_R} \xi^2 \theta^{n(u+\frac{2}{3}s)} d\xi = \frac{3\sqrt{6\pi}}{2\omega_{3/2}(\frac{3}{2}u+s)^{\frac{3}{2}}} \dot{\epsilon}_c M \approx 0.079 \dot{\epsilon}_c M$$

where the constant $\dot{\epsilon}_c = \epsilon_0 T_c^s \rho_c^{u-1}$ and $\dot{\epsilon}_0 \approx 3.4 \times 10^{-9}$ ergs $\mathrm{g}^{-1}\mathrm{s}^{-1}$. Finally,

$$\boxed{L_{\mathrm{HB}} = 1.53 \times 10^7 L_\odot \frac{\delta_{3/2}^{5.487}}{\omega_{3/2}\gamma_{3/2}^{16.46}} M_{-1}^{11.977} \frac{\eta^{10.15}}{(\eta + \alpha_d)^{16.46}}}$$

where we have defined $M_{-1} = M/(0.1 M_\odot)$ and $L_\odot$ is the solar luminosity[22].

[22] $\Big($ $\alpha_d \approx 4.82$, a ratio of the number of baryons per electron and $\eta \equiv \frac{\mu_F}{\kappa_B T}$ measures the degree of the degeneracy electroc pressure of the star $\Big)$

Temperature at the photosphere

Photosphere: the radius when the optical depth $\tau(r)$ is equaled to $2/3$

$$\tau(r) = \int_r^\infty \kappa_{\rm R}\rho dr, \quad \kappa_{\rm R} \text{ is Rosseland mean opacity.}$$

Assumption: the surface gravity is a constant value $g \equiv \frac{GM(r)}{r^2} \approx \frac{GM(R)}{R^2}$. From the modified hydrostatic equilibrium

$$p'_{ph} = -g\rho\left(1 + 8\beta\frac{g}{c^2 r}\right) \quad \rightarrow \quad p_{ph} = \frac{2g\left(1 + 8\beta\frac{g}{c^2 R}\right)}{3\kappa_R}$$

Then, from the ideal gas law

$$\frac{\rho k_B T}{\mu m_H} = \frac{2g\left(1 + 8\beta\frac{g}{c^2 R}\right)}{3\kappa_R}.$$

we find out the photosphere temperature:

$$\frac{T_{ph}}{\text{K}} = 2.88 \times 10^4 \frac{M_{-1}^{0.49}}{\gamma_{3/2}^{0.59}\eta^{1.09}} \left(\frac{1 + 8\beta\frac{g}{c^2 R}}{\kappa_{-2}}\right)^{0.296} \left(1 + \frac{\alpha_d}{\eta}\right)^{-0.59}.$$

Luminosity at the photosphere and MMSM[23]

The stellar luminosity: $L_{ph} = 4\pi R^2 \sigma T_{ph}^4$ is found to be

$$L_{ph} = 28.18 L_{\odot} \frac{M_{-1}^{1.305}}{\gamma_{3/2}^{2.366}\eta^{4.351}} \left(1+\frac{\alpha_d}{\eta}\right)^{-0.366} \left(\frac{\delta_{3/2} - 1.31\alpha\left(1+\frac{\alpha_d}{\eta}\right)^4}{\delta_{3/2}\kappa_{-2}}\right)^{1.183}$$

Thus, the minimum main sequence mass in quadratic Palatini gravity:

$$\boxed{M_{-1}^{MMSM} = 0.290 \frac{\gamma_{3/2}^{1.32}\omega_{3/2}^{0.09}}{\delta_{3/2}^{0.51}} I(\eta,\alpha)}$$

where ($\alpha \equiv -\kappa^2 c^2 \beta \rho_c$) and the new function $I(\eta,\alpha)$:

$$I(\eta,\alpha) = \frac{(\alpha_d+\eta)^{1.509}}{\eta^{1.325}} \left(1 - 1.31\alpha \frac{\left(\frac{\alpha_d+\eta}{\eta}\right)^4}{\delta_{3/2}\kappa_{-2}}\right)^{0.111}$$

[23] G. Olmo, D. Rubiera-Garcia, A. Wojnar, Phys.Rev. D 100 no.4, 044020 (2019)

Testing and constraining the theory[26]

$$M_{-1}^{MMSM} = 0.290\frac{\gamma_{3/2}^{1.32}\omega_{3/2}^{0.09}}{\delta_{3/2}^{0.51}}\frac{(\alpha_d+\eta)^{1.509}}{\eta^{1.325}}\left(1-1.31\alpha\frac{\left(\frac{\alpha_d+\eta}{\eta}\right)^4}{\delta_{3/2}\kappa_{-2}}\right)^{0.111}$$

The observational bound[24]: M-dwarf star G1 866C with the mass $(0.0930 \pm 0.0008)M_\odot$
The GR theoretical prediction[25]: $\sim 0.08 - 0.09 M_\odot$

α	ξ_R	$\omega_{3/2}(\xi_R)$	$\gamma_{3/2}(\xi_R)$	$\delta_{3/2}(\xi_R)$	$M/M\odot$
-0.100	3.64	2.39	2.25	6.67	0.0810
-0.010	3.65	2.68	2.35	6.09	0.0910
0 (GR)	3.65	2.71	2.36	5.97	0.0922
0.006	3.66	2.73	2.36	5.95	0.0929
0.010	3.66	2.75	2.37	5.93	0.0933
0.015	3.66	2.77	2.46	5.89	0.0980

Table: Numerical values of ξ_R obtained from $\theta(\xi_R)=0$, and the associated values of the functions $\gamma_{3/2}$, $\omega_{3/2}$, and $\delta_{3/2}$ for different values of $\alpha = -\kappa^2 c^2 \beta \rho_c$. The last column provides the estimations for the normalized (in solar mass units) MMSM for each value of α, which must be compared with the observational bound $0.0930 \pm 0.0008 M_\odot$ of the M-dwarf star G1 866C.

[24] D. Segransan et al., Astron. Astrophys. 364 (2000) 665

[25] A. Burrows, J. Liebert, Rev. Mod. Phys. 65, 301 (1993)

[26] G. Olmo, D. Rubiera-Garcia, A. Wojnar, Phys.Rev. D 100 no.4, 044020 (2019)

Observational and toy-model issues

- Simplifications made on:
  - changing the opacity ($\propto \kappa_R^{-0.11}$) - a weak dependence,
  - the MMHB is weakly dependent on stellar rotation (increases MMHB),
  - little chemical evolution - (the values of the composition parameters),
  - the polytropic EoS and effects of mixing and transport processes.
- Taking into account the above effects into a full numerical modeling[27], the MMHB changes by less than $0.01 M_\odot$.
- The missing physics in non-gravitational.
- The empirical mass determination assumes GR?
  - no deviations from GR outside astrophysical bodies,
  - the masses obtained by the elipsing binary technique, or by measuring the motions of their satellites. The both methods rely on Newtonian mechanics and not the objects' intrinsic properties,
  - photometry for mass-luminosity relations used in the observation is empirical and has nothing to do with the relation we derived.

[27] S. S. Kumar, Astrophys. J. 137, 1121 (1963).

Of course life is not so easy - there are difficulties[28]

- Lane-Emden equation in modified gravity - sometimes not an easy task (for example the equation is not scale-invariant and thus the symmetry is not preserved by modified hydrostatic equilibrium)!
- The luminosity of the photosphere.
- The surface gravity.
- How the approximations of the above ones change our result.
- Do not use GR values in modified equations!

[28] Listen to Aneta's complains

Thanks!

Brown dwarf with asteroid ring

http://www.arcadiastreet.com