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Some Quantum Effects in de Sitter Spacetime with Compact Dimensions

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Introduction

- The Casimir effect is among the most striking macroscopic manifestations of the nontrivial properties of the quantum vacuum, common to all systems characterized by fluctuating quantities on which external boundary conditions are imposed.
- The boundary conditions imposed on a quantum field lead to the modification of the spectrum for zero-point fluctuations.
- As a result, the expectation values for physical quantities bilinear in the field are shifted. In particular, the confinement of quantum fluctuations causes forces that act on constraining boundaries.
- These forces depend on the nature of the quantum field, on the bulk and boundary geometries, and on the specific boundary conditions on the field operator.

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- K. A. Milton, The Casimir Effect: Physical Manifestation of Zero-Point Energy (World Scientific, Singapore, 2002)
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Introduction

- We investigate the effects of background curvature, of nontrivial topology and of a planar boundary on the properties of the vacuum state for a charged scalar field
  - Background geometry is locally dS
  - Nontrivial topology is induced by compactification of a part of spatial dimensions to a torus
  - Planar boundary is perpendicular to one of infinite dimension and on it the field obeys the Robin boundary condition
  - Along compact dimensions general quasiperiodicity conditions are imposed and, in addition, the presence of a constant gauge field is assumed. The latter induces Aharonov-Bohm-type effect on the vacuum expectation values (VEVs) of physical observables.

- The high symmetry of the background geometry allows to obtain closed expressions for the VEVs of field squared and of the energy-momentum tensor. They are decomposed into the VEVs for the locally de Sitter spacetime with compact dimensions in the absence of a boundary and the boundary induced part.

Introduction. Models with compact dimensions

- In recent years much attention has been paid to the possibility for the Universe to have non-trivial topology. In particular, a number of fundamental physical theories are formulated in spacetimes with compact extra dimensions. The idea of compactified dimensions has been extensively used in supergravity and superstring theories.
- The models of a compact Universe with non-trivial topology may play an important role by providing proper initial conditions for inflation.

[1] *A. Linde, JCAP 0410, 004 (2004)*

The quantum creation of the Universe having toroidal spatial topology is discussed in [2] and in Refs. [3] within the framework of supergravity theories.

[2] *Y.B. Zeldovich and A.A. Starobinsky, Sov. Astron. Lett. 10, 135 (1984).*

[3] *Yu.P. Goncharov and A.A. Bytsenko, Phys. Lett. B 160, 385 (1985);*
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Yu.P. Goncharov and A.A. Bytsenko, Class. Quant. Grav. 4, 555 (1987).

Introduction. Models with compact dimensions

- Models with the compact dimensions are popular in different physical problems
  - Kaluza-Klein type theories:
    Additional compact dimensions are used in order to unify different physical interactions.
  - Condensed matter physics:
    Examples: Cylindrical and Toroidal graphene nanotubes
  - Shift in the vacuum energy as a source for dark energy

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2. K.A. Milton, Grav. Cosmol. 9, 66 (2003); A.A. Saharian, Phys. Rev. D 70, 064026 (2004);
3. E. Elizalde, J. Phys. A 39, 6299 (2006); A.A. Saharian, Phys. Rev. D 74, 124009 (2006);
4. B. Green and J. Levin, JHEP 0711, 096 (2007); P. Burikham, A. Chatrabhuti, P. Patcharamaneepakorn, and K. Pimsamarn, JHEP 0807, 013 (2008);
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The $D>3$ models both Kaluza-Klein and braneworld type, are widely used for the unification of physical interaction.
The field theoretical models with $D<3$ appear in effective theories of a number of condensed matter systems, like graphene and topological insulators.

Introduction. Topological Casimir Effect

Periodicity conditions imposed on fields along compact dimensions give rise to the modification of the spectrum for normal modes and, related to this, the expectation values of physical observables are changed. A well known effect of this type, demonstrating the connection between quantum phenomena and global properties of space-time, is the topological Casimir effect.

1. *V.M. Mostepanenko and N.N. Trunov, The Casimir Effect and Its Applications (Clarendon, Oxford, 1997); M. Bordag, G.L. Klimchitskaya, U. Mohideen, and V.M. Mostepanenko, Advances in the Casimir Effect (Oxford University Press, Oxford, 2009);*
2. *K.A. Milton, The Casimir Effect: Physical Manifestation of Zero-Point Energy (World Scientific, Singapore, 2002);*
3. *A.A. Bytsenko, G. Cognola, L. Vanzo, and S. Zerbini, Phys. Rep. 266, (1996); A.A. Bytsenko, G. Cognola, E. Elizalde, V. Moretti, and S. Zerbini, Analytic Aspects of Quantum Fields (World Scientific, Singapore, 2003);*
4. *E. Elizalde, Ten Physical Applications of Spectral Zeta Functions (Springer Verlag, 2012).*

Introduction. Curved backgrounds – de Sitter

- The investigation of quantum effects in fixed gravitational backgrounds is among the most interesting topics in quantum field theory

- The corresponding problems are exactly solvable for highly symmetric background geometries only

- dS spacetime is considered in different inflationary models in order to solve the problems of standard cosmological model

- Nowadays, the Universe is expanding with acceleration and its evolution can be well approximated by a model with positive cosmological constant

- Different physical problems can be solved exactly in de-Sitter spacetime due to its high symmetry

Introduction. Topological Casimir Effect

The vacuum expectation values (VEVs) for the field squared and the energy-momentum tensor are considered for untwisted and twisted fields. These quantities are among the most important local characteristics of the quantum vacuum and are closely related with the structure of spacetime through the theory of gravitation.

For charged fields another important bilinear characteristic is the VEV of the current density. In addition to describing the physical structure of the quantum field at a given point, the current acts as the source in the Maxwell equations and plays an important role in modeling a self-consistent dynamics involving the electromagnetic field.

Geometry of the problem

- We consider combined effects of topology, of the gravitational field and of a planar boundary on the properties of the vacuum state for a charged scalar field
- Along compact dimensions generic quasiperiodic boundary conditions are imposed with arbitrary phases.
- The presence of a classical constant gauge field is assumed.

Spatial topology ⇒ $R^{p+1} \times (S^1)^q$

$$\mathbf{x}_p = (x^1, \ldots, x^p) \qquad -\infty < x^l < \infty,\ l = 1, \ldots, p$$

$$\mathbf{x}_q = (x^{p+1}, \ldots, x^D) \qquad q = D - p$$

are compactified to circles with the lengths

$$L_l\colon\ 0 \le x^l \le L_l,\ l = p+1, \ldots, D.$$

Geometry of the problem

- (D + 1)-dimensional locally dS spacetime is described by the line element in inflationary coordinates:

$$ds^2 = dt^2 - e^{2t/\alpha}\sum_{l=1}^{D}(dx^l)^2$$

$$R^{p+1} \times (S^1)^q$$

the parameter α is related to the positive cosmological constant Λ by the formula

$$\alpha^2 = D(D-1)/(2\Lambda)$$

conformal time defined as $\tau = -\alpha e^{-t/\alpha}, \; -\infty < \tau < 0$

In terms of this coordinate the metric tensor takes a conformally flat form:

$$g_{ik} = (\alpha/\tau)^2 \mathrm{diag}(1, -1, ..., -1)$$

Complex scalar field

- We consider a complex scalar field $\varphi(x)$ with a curvature coupling parameter ξ, in the presence of a classical abelian gauge field A_μ. The corresponding field equation has the form

$$\left(D_\mu D^\mu + m^2 + \xi R\right)\varphi(x) = 0$$

- In the most important special cases:

minimally and conformally coupled scalar fields

$\xi = 0$ $\qquad$ $\xi = (D-1)/(4D)$

- $D_\mu = \nabla_\mu + ieA_\mu$ gauge-covariant derivative
- e coupling between the scalar and gauge fields
- $R = D(D+1)/\alpha^2$ scalar curvature

1 / m is the length of the Compton wavelength (h = c = 1 unit).

Complex scalar field

- Background space is multiply-connected and in addition to the field equation one should specify the periodicity conditions along compact dimensions. We will assume generic quasiperiodic boundary conditions

$$\varphi(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\alpha_l} \varphi(t, \mathbf{x}_p, \mathbf{x}_q),$$

$\alpha_l = const,$ $\boldsymbol{e}_l$ unit vector along the dimension $l = p+1, \ldots, D$

- The special cases of untwisted and twisted scalar fields (periodic and antiperiodic boundary conditions), most frequently discussed in the literature, correspond to

$\alpha_l = 0$ untwisted

$\alpha_l = \pi$ twisted

Gauge field configuration

- We consider the simplest configuration of the gauge field with $A_\mu = \text{const}$
The gauge field can be excluded from the field equation by the gauge transformation

$$A'_\mu = A_\mu + \partial_\mu \omega, \ \varphi'(x) = e^{-ie\omega}\varphi(x), \ \omega = -A_\mu x^\mu$$

- In the new gauge one has $A'_\mu = 0$ and $D'_\mu = \nabla_\mu$

- Quasi-periodicity condition after gauge transformation

$$\varphi'(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\tilde{\alpha}_l}\varphi'(t, \mathbf{x}_p, \mathbf{x}_q)$$

- The phases in the periodicity conditions and the value of the gauge field are related to each other through a gauge transformation.

$$\tilde{\alpha}_l = \alpha_l + eA_l L_l$$

We work in terms of the gauge transformed field $\varphi'(\mathbf{x})$ omitting the prime.

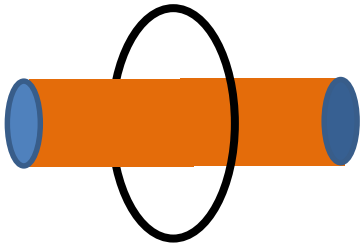
Field content

- Though the field strength vanishes, a constant gauge field shifts the phases in the periodicity conditions along compact dimensions

- This leads to the Aharonov-Bohm-like effects on the current density for charged fields

- The shift due to the gauge field may be written in the form,

$eA_l L_l = -eA_l L_l = -2\pi\Phi_l / \Phi_0$, where $\Phi_0 = 2\pi / e$ is the flux quantum and Φ_l is the flux enclosed by the circle corresponding to the l-th compact dimension



The Hadamard function

- In the problem under consideration the properties of the vacuum state are encoded in two-point functions. In particular, from the point of view of the evaluation of the VEV the Hadamard function is of special interest. The latter is defined as the expectation value

$$G^{(1)}(x,x') = \langle 0 | \varphi(x)\varphi^{+}(x') + \varphi^{+}(x')\varphi(x) | 0 \rangle$$

where $|0\rangle$ stands for the vacuum state and the dagger corresponds to the hermitian conjugate.

- Mode-sum for the Hadamard function

the mode functions are specified by the set of quantum numbers

$$\sigma = (\mathbf{k}_p, \mathbf{n}_q)$$

$$\mathbf{k}_p = (\mathbf{k}_{p-1}, k_p)$$

$$\mathbf{n}_q = (n_{p+1}, \ldots, n_D)$$

$$G^{(1)}(x,x') = \sum_{\sigma} \sum_{m=\pm} \varphi_{\sigma}^{(m)}(x) \varphi_{\sigma}^{(m)*}(x')$$

Complete set of solutions to the field equation

Here $\sum_{\sigma}$ includes the summation over the discrete components of the collective index σ and the integration over the continuous ones

These two classes are required in order to have a complete set of modes.

Complete set of solutions to the field equation

- We assume that the scalar field is prepared in the Bunch-Davies vacuum

$$\varphi_\sigma^{(+)}(x) = C_\sigma^{(+)}\eta^{D/2}H_\nu^{(1)}(k\eta)e^{i\mathbf{k}_p\cdot\mathbf{x}_p+i\mathbf{k}_q\cdot\mathbf{x}_q},$$
$$\varphi_\sigma^{(-)}(x) = C_\sigma^{(-)}\eta^{D/2}H_{\nu^*}^{(2)}(k\eta)e^{i\mathbf{k}_p\cdot\mathbf{x}_p+i\mathbf{k}_q\cdot\mathbf{x}_q}.$$

$$\eta = |\tau| = \alpha e^{-t/\alpha}$$

The module of the conform time

$$\nu = \left[D^2/4 - D(D+1)\xi - m^2\alpha^2\right]^{1/2}$$

The order of the Hankel functions can be either real or imaginary. Depending on that the behavior of the VEVs is monotonic or oscillatory

Momentum along uncompactified dimensions

$$-\infty < k_l < +\infty,\ l = 1, \ldots, p,$$

Along compactified dimensions

$$k_l = (2\pi n_l + \tilde{\alpha}_l)/L_l, \quad n_l = 0, \pm1, \pm2, \ldots, \quad l = p+1, \ldots, D$$

Current Density

- Current density operator for a charged scalar field is given by the standard expression:

$$j_\mu(x) = ie[\varphi^+(x)D_\mu\varphi(x) - (D_\mu\varphi^+(x))\varphi(x)]$$

in the gauge under consideration $D_\mu\varphi = \nabla_\mu\varphi = \partial_\mu\varphi$

- The VEV of the current density

$$\langle j_\mu(x)\rangle = \frac{i}{2}e\lim_{x'\to x}(\partial_\mu - \partial'_\mu)G^{(1)}(x,x')$$

Current Density

Having the Hadamard function, we can evaluate the expectation value for the current density It can be seen that the VEVs of the charge density ($\mu = 0$) and of the components of the current density along uncompactified dimensions vanish:

$$\langle j_\mu \rangle = 0, \ \mu = 0, 1, \ldots, p.$$

- VEV of the current density along the r-th compact dimension

$$\langle j^r \rangle = \frac{8e\alpha(\eta/\alpha)^{D+2}}{(2\pi)^{(p+3)/2} V_q L_r^{p-1}} \int_0^\infty dz\, z \left[I_{-\nu}(\eta z) + I_\nu(\eta z)\right] K_\nu(\eta z)$$
$$\times \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^p} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{(p+1)/2}(nL_r\sqrt{z^2+u_r^2}).$$

$$f_\mu(z) = z^\mu K_\mu(z)$$

- For single compact dimension

$$\langle j^r \rangle = \frac{8e\alpha(\eta/\alpha)^{D+2}}{(2\pi)^{D/2+1} L_r^{D-1}} \int_0^\infty dz\ z \left[I_{-\nu}(\eta z) + I_\nu(\eta z)\right] K_\nu(\eta z) \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{D-1}} f_{D/2}(nL_r z)$$

- For a conformally coupled massless scalar field $m = 0, \xi = (D-1)/(4D)$

$$\langle j^r \rangle = \frac{4e(\eta/\alpha)^{D+1}}{(2\pi)^{p/2+1} V_q L_r^p} \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{p+1}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{p/2+1}(nL_r u_r) = (\eta/\alpha)^{D+1} \langle j^r \rangle^{(\mathrm{M})}$$

with compact dimensions of the lengths $(L_{p+1}, \ldots, L_D)$

Limiting Cases

- In the case of small values of the length for compactified dimension

$$\langle j^r \rangle \approx (\eta/\alpha)^{D+1} \langle j^r \rangle^{(\mathrm{M})}, \quad L_r/\eta \ll 1$$

$$f_\mu(z) = z^\mu K_\mu(z)$$

- For large values of the length of compact dimension and for real ν

Monotonic decay

$$\langle j^r \rangle \approx e \frac{2^{\nu-(p-1)/2}(\eta/L_r)^{D-2\nu+2}\Gamma(\nu)}{\pi^{(p+3)/2}\alpha^{D+1}V_{q-1}L_r^{p-D}} \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{p-2\nu+2}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{(p+3)/2-\nu}(nL_r u_r) \quad , \eta/L_r \ll 1$$

- For large values of the length of compact dimension and for purely imaginary ν

Oscillatory decay

$$\langle j^r \rangle \approx \frac{8e\alpha B_s e^{-(D+2)t/\alpha}}{(2\pi)^{(p+3)/2}V_q L_r^{p+1}} \cos[2|\nu|t/\alpha + 2|\nu|\ln(L_r/\alpha) + \phi_s]$$

$$B_s e^{i\phi_s} = 2^{i|\nu|}\Gamma(i|\nu|) \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{p+2-2i|\nu|}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{(p+3)/2-i|\nu|}(nL_r u_r)$$

ratio L_l/η is the proper length of the lth compact dimension measured in the units of the dS curvature scale

Asymptotic at early and late times

- At early stages of cosmological expansion $t \to -\infty$

  VEV of current density increases

$$\langle j^r \rangle \sim \exp[-(D+1)t/\alpha]$$

- At late stages of cosmological expansion $t \to +\infty$

For real values of ν VEV of current density decays exponentially

$$\langle j^r \rangle \sim \exp[-(D-2\nu+2)t/\alpha]$$

Asymptotic behavior for large masses

- For a fixed value of the time coordinate t, the condition $\eta / L_r << 1$ corresponds to the length of the rth compact dimension much larger than the dS curvature scale $L_r >> \alpha$. Now for the values of the curvature scale $\alpha \geq m^{-1}$ one also has $L_r >> m^{-1}$ and the limit we have discussed corresponds to the length of the compact dimension much larger than the Compton wavelength of the field quanta. The decay of the current density is a power-law as a function of parameter mL_r

Power-law suppression

$$\langle j^r \rangle \approx \frac{2e\Gamma(\nu)\Gamma(D/2+1-\nu)\eta}{\pi^{D/2+1}\alpha^{D+1}(L_r/\eta)^{D-2\nu+1}} \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{D-2\nu+1}}.$$

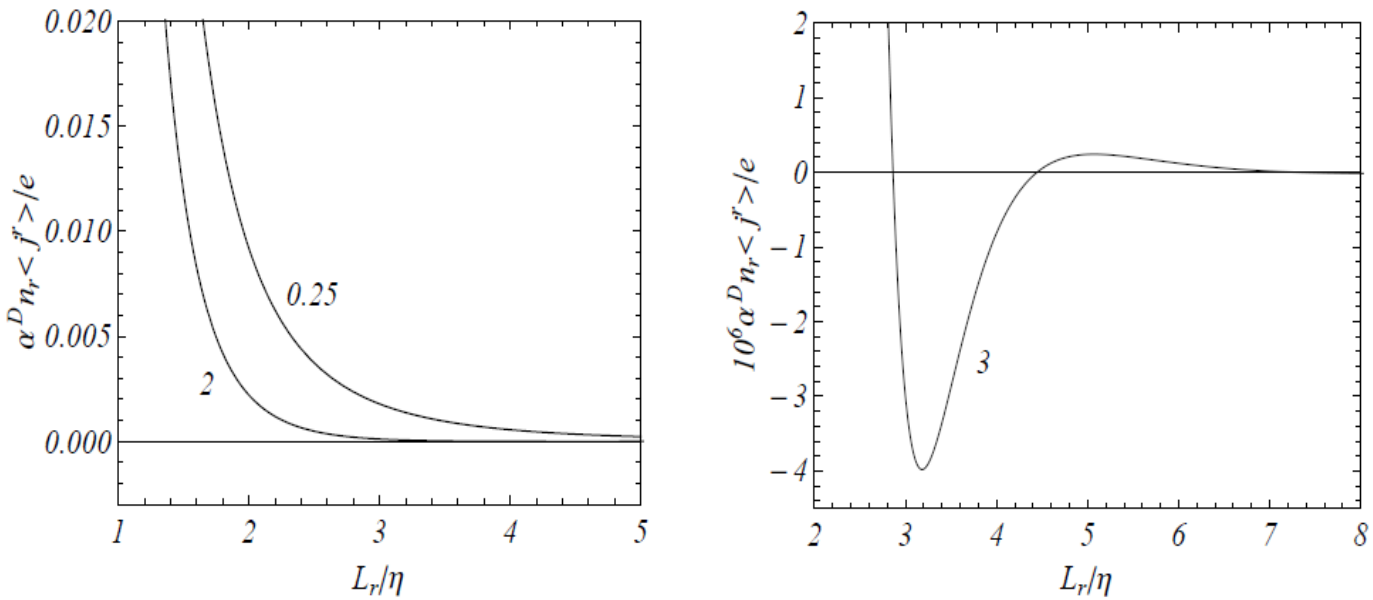
- In the Minkovski spacetime under the same conditions, in the case of a single compact dimension

Exponential suppression

$$\langle j^r \rangle^{(\mathrm{M})} \approx \frac{2em^D \sin(\tilde{\alpha}_r)}{(2\pi)^{D/2}(mL_r)^{D/2}} e^{-mL_r}$$

Numerical Results

$$\nu = \left[D^2/4 - D(D+1)\xi - m^2\alpha^2\right]^{1/2}$$



For the same model $D = 4$, $p = 3$, in figure 2, we have plotted the quantity $\alpha^D n_r \langle j^r \rangle$ for a conformally coupled scalar field versus the ratio L_r/η. The latter is the proper length of the compact dimension measured in units of the dS curvature scale α. In numerical evaluations we have taken $\tilde{\alpha}_r = \pi/2$ and the numbers near the curves correspond to the values of $m\alpha$. For small values of L_r/η, from the asymptotic analysis given above it follows that $n_r \langle j^r \rangle \propto (L_r/\eta)^{-D}$. As it has been explained before, depending on the parameter ν, for large values of L_r/η two different regimes arise with monotonic (for positive values of ν) and oscillatory (for imaginary values of ν) damping of the current density. Note that, for a conformally coupled field, ν is real for $m\alpha = 0.25$ and imaginary for $m\alpha = 2, 3$. In order to display the oscillatory behavior of the damping, on the right panel of figure 2 we have plotted the graph for $m\alpha = 3$. The value of the ratio L_r/η corresponding to the first zero of the current density decreases with increasing the value of $m\alpha$. For the first two zeros of the current density one has $L_r/\eta = 3.88, 8.29$ and $L_r/\eta = 2.87, 4.45$ in the cases $m\alpha = 2$ and $m\alpha = 3$, respectively.

VEV of the field squared

- Given the Hadamard function, the vacuum expectation value of the field squared is evaluated as the coincidence limit

$$\langle\varphi\varphi^\dagger\rangle = \frac{1}{2}\lim_{x'\to x} G^{(1)}(x,x')$$

- Mean field squared is presented in the form

$$\langle\varphi\varphi^\dagger\rangle_{p,q} = \langle\varphi\varphi^\dagger\rangle_{dS} + \langle\varphi\varphi^\dagger\rangle_c$$

Induced by the compactification (topological part)

VEV in the uncompactied dS spacetime

- In the model with a single compact dimension x^D $(q=1,\ p=D-1)$

$$\langle\varphi\varphi^\dagger\rangle_c = \frac{4\alpha^{1-D}\eta^D}{(2\pi)^{D/2+1}}\int_0^\infty dz\, z g_\nu(\eta z)\sum_{n=1}^{\infty}\frac{\cos(n\tilde{\alpha}_D)}{(nL_D)^{D-2}} f_{D/2-1}(nL_D z)$$

- Renormalization is needed for $\langle\varphi\varphi^\dagger\rangle_{dS}$

$$g_\nu(x) = K_\nu(x)[I_{-\nu}(x)+I_\nu(x)], \qquad f_\mu(x) = x^\mu K_\mu(x)$$

Energy-momentum tensor

- Vacuum expectation value of the energy-momentum tensor

$$\langle T_{ik}\rangle_{p,q} = \frac{1}{2}\lim_{x'\to x}\partial_i\partial'_k G_{p,q}(x,x') + \frac{1}{2}\left[\left(\xi - \frac{1}{4}\right)g_{ik}\nabla_l\nabla^l - \xi\nabla_i\nabla_k - D\xi g_{ik}/\alpha^2\right]G_{p,q}(x,x)$$

- By using the Hadamard function, the VEV is decomposed as

$$\langle T_i^k\rangle_{p,q} = \langle T_i^k\rangle_{dS} + \langle T_i^k\rangle_c$$

Induced by the compactification (topological part)

VEV in the uncompactied dS spacetime

- From the maximal symmetry of Bunch-Davies vacuum state

$$\langle T_i^k\rangle_{dS} = const\cdot\delta_i^k$$

- Off-diagonal components of the topological part vanish

Topological part in the energy-momentum tensor

- In the model with a single compact dimension x^D $(q=1,\ p=D-1)$

$$\langle T_i^i \rangle_c = \frac{4\alpha^{-1-D}}{(2\pi)^{D/2+1}} \sum_{n=1}^{\infty} \frac{\cos(n\tilde{\alpha}_D)}{\omega_n^D} \int_0^{\infty} dx x g_\nu(x/\omega_n) P^{(i)}(x),\ \omega_n = nL/\eta,$$

$$P^{(0)}(x) = \left(m^2\alpha^2 + D^2\xi + \frac{x^2}{4} - \frac{x^2}{\omega_n^2}\right) f_{D/2-1}(x) - \left(\frac{1}{2} + D\xi\right) f_{D/2}(x),$$

$$P^{(l)}(x) = \left[\left(\xi - \frac{1}{4}\right) x^2 - D\xi\right] f_{D/2-1}(x) + \left(\frac{1}{2} - \xi - \frac{1}{\omega_n^2}\right) f_{D/2}(x),$$

$$P^{(D)}(x) = P^{(1)}(x) + \frac{1}{\omega_n^2} f_{D/2+1}(x), \quad l = 1, \ldots, D-1.$$

- Separate components: $i = 0$ → energy density

$-\langle T_i^i \rangle, \quad i = 1, 2, \ldots, D+1$ → vacuum pressure along the corresponding dimension

Topological part in the energy-momentum tensor

- Vacuum pressure is anisotropic

the lengths of compact dimentions are different

the Lorentz invariance in that direction

- Unlike to Minkwoskian geometry the vacuum pressure along uncompact dimensions is not equal to - energy density

Asymptotic behavior

- For $L_r/\eta \ll 1$ (L_r/η is the physical length of compact dimensions measured in units of the curvature radius 1/α)

$$\Delta_r\langle\varphi\varphi^\dagger\rangle_{p,q} \approx \left(\frac{\eta}{\alpha}\right)^{D-1}\Delta_r\langle\varphi^2\rangle^{(M)}_{p,q} \qquad \Delta_r\langle T^i_i\rangle_{p,q} \approx (\eta/\alpha)^{D+1}\Delta_r\langle T^i_i\rangle^{(M)}_{p,q}$$

- For fixed values of L_r this limit corresponds to the early stages of the cosmological expansion, $t \to -\infty$

  Effects of gravity on topological contributions are weak

- For $L_r/\eta \gg 1$: Physical length of comapct dimension is large This corresponds to late stages of cosmological expansion $t \to +\infty$

  Topological contributions tend to zero

- Monotonically for $\nu^2 = D^2/4 - D(D+1)\xi - m^2\alpha^2 \geq 0$

$$\exp[-(D-2\nu)t/\alpha]$$

- Oscillatory for $\nu^2 < 0$ $\quad e^{-Dt/\alpha}\cos[2|\nu|t/\alpha + 2|\nu|\ln(L_r/\alpha) + \phi_s]$

Geometry of the problem

- In addition, we assume planar boundary at $x^p = 0$ on which the field obeys the Robin boundary condition

$$(1+\beta D_p)\varphi = 0, \quad x^p = 0$$

Special cases:

Dirichlet $\beta = 0$

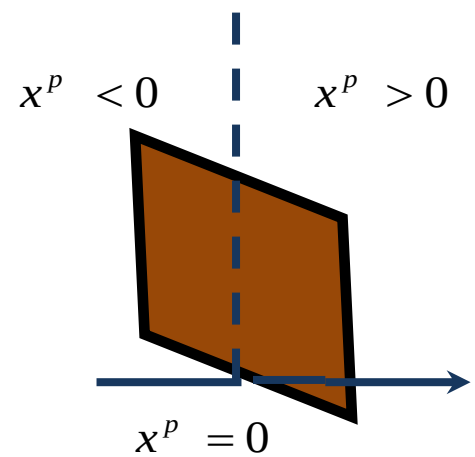
Neumann conditions $\beta = \infty$

- The Hadamard function decomposed into two parts

$$G^{(1)}(x,x') = G_0^{(1)}(x,x') + G_b^{(1)}(x,x')$$

without boundary

boundary-induced part



- We investigate boundary-induced effects

Complete set of solutions to the field equation

Complete set of solutions to the field equation

$$\varphi_\sigma^{(+)}(x) = C_\sigma^{(+)} \eta^{D/2} H_\nu^{(1)}(K\eta) \cos(k_p x^p + \alpha(k_p)) e^{i\mathbf{k}_{p-1}\mathbf{x}_{p-1} + i\mathbf{k}_q \mathbf{x}_q}$$

$$\varphi_\sigma^{(-)}(x) = \varphi_\sigma^{(+)*}(x)$$

$$e^{2i\alpha(k_p)} = \frac{i\beta k_p - 1}{i\beta k_p + 1}$$

$$K = \sqrt{\mathbf{k}_{p-1}^2 + k_p^2 + \mathbf{k}_q^2}$$

Field squared

- $\langle\varphi\varphi^+\rangle = \langle\varphi\varphi^+\rangle_0 + \langle\varphi\varphi^+\rangle_b$

without boundary

boundary-induced part

$$\langle\varphi\varphi^+\rangle_b = \frac{2^{1-p}\pi^{-(p+1)/2}\eta^D}{\Gamma((p-1)/2)V_q\alpha^{D-1}}\sum_{\mathbf{n}_q}\int_0^\infty dx\, x^p f(\sqrt{x^2+k^2_{\mathbf{n}_q}}, x^p)h_\nu(\eta x)$$

$$h_\nu(u) = \int_0^1 dz\, z(1-z^2)^{(p-3)/2}g_\nu(uz) \qquad k^2_{\mathbf{n}_q} = \sum_{l=p+1}^{D}(2\pi n_l + \tilde{\alpha}_l)^2/L_l^2$$

$$g_\nu(x) = K_\nu(x)[I_{-\nu}(x) + I_\nu(x)], \qquad f(u, x^p) = \frac{e^{-2ux^p}}{u}\frac{\beta u + 1}{\beta u - 1}$$

- For a conformally coupled massless scalar field

$$\langle\varphi\varphi^+\rangle_b = \frac{(\eta/\alpha)^{D-1}}{(4\pi)^{p/2}\Gamma(p/2)V_q}\sum_{\mathbf{n}_q}\int_{k_{\mathbf{n}_q}}^\infty dx\,(x^2 - k^2_{\mathbf{n}_q})^{p/2-1}e^{-2xx^p}\frac{\beta x + 1}{\beta x - 1}$$

Conformal factor

Conformally related to the result in Minkwoski bulk

- For Dirichlet and Neumann boundary conditions

$$\langle\varphi\varphi^+\rangle_b = \mp\frac{2\eta^D(2x^p)^{2-p}}{(2\pi)^{p/2+1}V_q\alpha^{D-1}}\int_0^\infty dy\, y g_\nu(\eta y)\sum_{\mathbf{n}_q} f_{p/2-1}(2x^p\sqrt{y^2+k^2_{\mathbf{n}_q}})$$

Energy-momentum tensor

- $\langle T_{ik} \rangle = \langle T_{ik} \rangle_0 + \langle T_{ik} \rangle_b$

without boundary

boundary-induced part

$$\langle T_i^k \rangle_b = \frac{A_p}{V_q \alpha^{D+1}} \sum_{\mathbf{n}_q} \int_0^\infty dx\, x^{p-D} f(\sqrt{x^2 + k_{\mathbf{n}_q}^2}, x^p) \times \left[\hat{F}_i^k(\eta) h_\nu^{(0)}(\eta x) + \eta^2 x^2 F_i^k h_\nu^{(1)}(\eta x) \right],$$

- The only nonzero off-diagonal componen$\langle T_0^p \rangle$

- Boundary-induced part obeys the covariant continuity equation

$$\nabla_k \langle T_i^k \rangle_b = 0$$

Energy-momentum tensor

- Off-diagonal componen $\langle T_0^p \rangle_b$ appears (energy flux)

$$\langle T_0^p \rangle_b = \frac{\pi^{-(p+1)/2}(\eta/\alpha)^{D+1}}{2^{p-2}\Gamma((p-1)/2)V_q} \sum_{\mathbf{n}_q} \int_0^\infty dy\, y F_\nu(\eta y)$$

$$\times \int_0^\infty dw\, w^{p-2} \frac{\beta u + 1}{\beta u - 1} e^{-2x^p u}\Big|_{u=\sqrt{y^2+w^2+k_{\mathbf{n}_q}^2}},$$

where $F_\nu(x) = \left[\left(\xi - \frac{1}{4}\right) x\partial_x + (D+1)\xi - \frac{D}{4}\right] g_\nu(x)$

- In special cases of Dirichlet and Neumann boundary conditions

$$\langle T_0^p \rangle = \mp \frac{4(\eta/\alpha)^{D+1}}{(2\pi)^{p/2+1} V_q (2x^p)^{p-1}} \sum_{\mathbf{n}_q} \int_0^\infty dy\, y f_{p/2}(2x^p \sqrt{y^2 + \mathbf{k}_q^2}) F_\nu(\eta y)$$

- For Dirichlet and Neumann boundary conditions the energy flux has opposite signs

Asymptotic behavior

- For $x^p/\eta \ll 1$ (x^p/η is the proper distance from the boundary in units of the curvature radius $1/\alpha$)
  the influence of the gravitational field is weak

- For $x^p \gg L_l$
  - $\nu^2 > 0$

$$\langle\varphi\varphi^+\rangle_b \approx \mp\frac{2^{2\nu-p}\Gamma(\nu)\Gamma(p/2-\nu)\eta^{D-2\nu}}{(2\pi)^{p/2+1}V_q\alpha^{D-1}(x^p)^{3p/2-1-2\nu}}$$

For $\tilde{\alpha}_l = 0,\ l = p+1,\ldots,D$

Power-law decay

$$\langle\varphi\varphi^+\rangle_b \approx \mp\frac{2^{2\nu-p}\Gamma(\nu)\eta^{D-2\nu}k_{(0)}^{(p-1)/2-\nu}e^{-2x^pk_{(0)}}}{(2\pi)^{(p+1)/2}V_q\alpha^{D-1}(x^p)^{p-\nu-1/2}}$$

If, at least one $\tilde{\alpha}_l \neq 0$

Exponential decay

  - $\nu^2 < 0$

For $\tilde{\alpha}_l = 0,\ l = p+1,\ldots,D$

$$\langle\varphi\varphi^+\rangle_b \approx \mp\frac{\pi^{-p/2-1}A_p\eta^D}{2^{p+1}V_q\alpha^{D-1}(x^p)^p}\cos\left[2|\nu|\ln(2x^p/\eta)+\phi_1\right]$$

If, at least one $\tilde{\alpha}_l \neq 0$

$$\langle\varphi\varphi^+\rangle_b \approx \mp\frac{\pi^{-p/2}\eta^Dk_{(0)}^{(p-1)/2}e^{-2x^pk_{(0)}}}{2^{p+1}V_q\alpha^{D-1}(x^p)^{(p+1)/2}}\frac{\cos\left[|\nu|\ln(4x^p/\eta^2k_0)+\phi_2\right]}{\sqrt{|\nu|\sinh(\pi|\nu|)}}$$

Asymptotic behavior

- At early stages of the expansion $(t \to -\infty)$

Increases as the power function

$$\langle T_0^p \rangle \approx \mp \frac{2D\,(\xi - \xi_D)\,(\eta/\alpha)^D}{(2\pi)^{(p+1)/2}\,\alpha V_q (2x^p)^p} \sum_{\mathbf{n}_q} f_{(p+1)/2}(2x^p k_{\mathbf{n}_q})$$

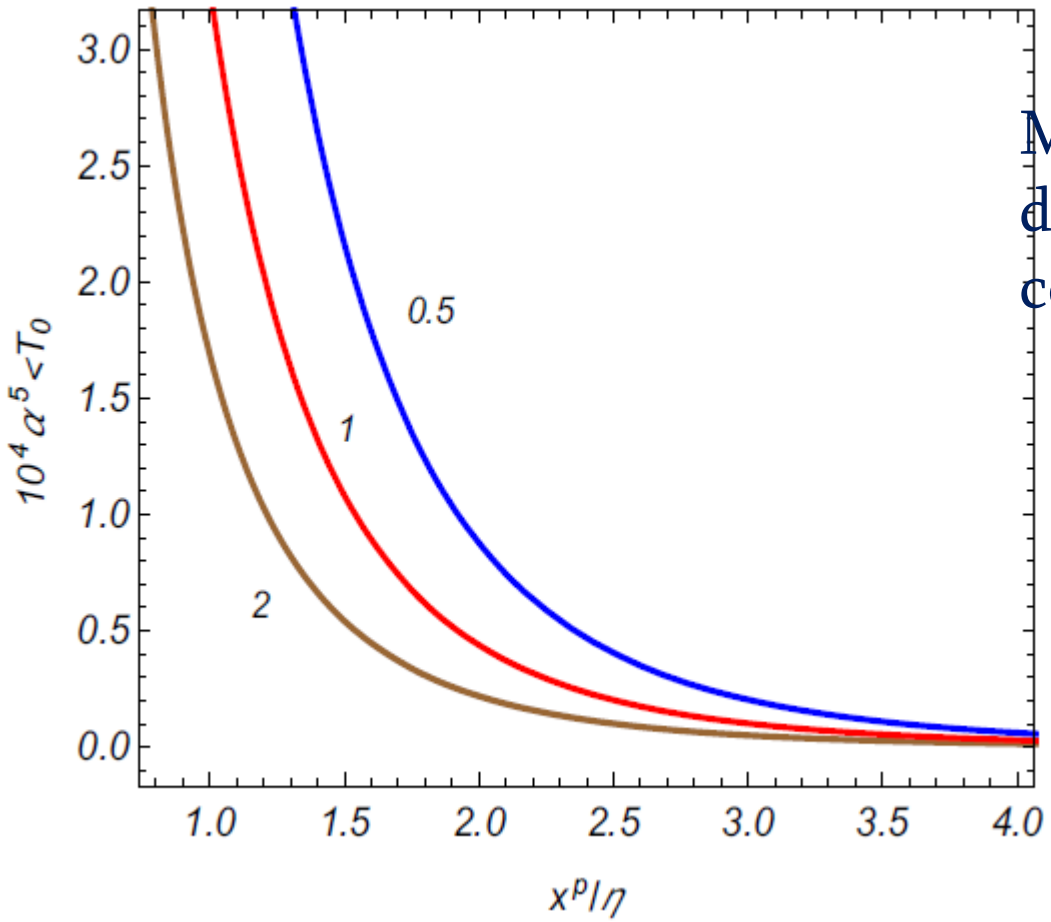
- At late stages of the expansion $(t \to +\infty)$
  - $\nu^2 > 0$ Decay is monotonically

$$\langle T_0^p \rangle \approx \mp 2^{p/2+\nu+2}\Gamma(\nu)(\eta/\alpha)^{D+1-2\nu} \frac{2\nu\,(1/4-\xi) + (D+1)\xi - D/4}{(4\pi)^{p/2+1}\,\alpha^{2\nu} V_q (2x^p)^{p+1-2\nu}} \sum_{\mathbf{n}_q} g_{p/2+1-\nu}(2x^p k_{\mathbf{n}_q})$$

  - $\nu^2 < 0$ Decay is oscillatory

$$\langle T_0^p \rangle \approx -\frac{(\eta/\alpha)^{D+1}}{2^{p-1}\pi^{(p+1)/2} V_q} \sum_{\mathbf{n}_q} A(x^p, k_{\mathbf{n}_q}) \sin[2|\nu| \ln(\eta/2) - \phi(x^p, k_{\mathbf{n}_q})]$$

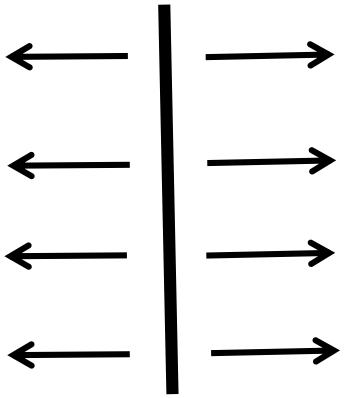
Energy flux



$$D = 4$$

Model with a single compact dimension Dirichlet boundary condition

$$m\alpha = 1, \ \tilde{\alpha} = \pi/4$$



Energy flux is directed from the boundary (to the boundary for Neumann b.c.)

The appearance of the off-diagonal component is related to that the problem is inhomogeneous with respect to time and with respect to the coordinate perpendicular to the boundary.

Conclusion

- We evaluate the Hadamard function for general values of the phases in the periodicity conditions obeyed by the field operator along compact dimensions. Additionally, the presence of a constant gauge field is assumed. The latter may be excluded by a gauge transformation that leads to the shift in the phases of periodicity conditions.
- The shift is expressed in terms of the ratio of the magnetic flux enclosed by compact dimension to the flux quantum.
- Current density along the compact dimension is an odd periodic function of the magnetic flux enclosed by the dimension with the period equal to the flux quantum
- In the presence of an external constant gauge field, compact dimensions lead to Aharonov- Bohm like effect for current densities
- Because the toroidal compactification does not change the local geometry, the renormalization of the VEVs for physical quantities bilinear in the field is reduced to the one for the uncompactified geometry
- Effects of gravity are small, when the length of compact dimension is small compared with the curvature radius
- Effects of gravity are essential when the length of compact dimension is larger than the curvature scale of the background spacetime

Conclusion

- Vacuum energy-momentum tensor is decomposed into the pure dS and topological parts
- VEVs of the energy density and stresses are even periodic functions of the magnetic flux enclosed by the dimension with the period equal to the flux quantum
- Gravitational effects are small, when the length of compactified dimension is small (early stages of the cosmological expansion)
- Gravitational effects are significant, when the length of compactified dimension is larger than the curvature scale of the background spacetime (late stages of the cosmological expansion)

Conclusion

- The presence of a planar boundary give rise to the energy flux in the vacuum state along the direction normal to the plate

- Vacuum expectation values are decomposed into the boundary-free and boundary-induced parts

- The flux is an even periodic function of magnetic fluxes enclosed by compact dimensions with the period equal to flux quantum

- Near the boundary the inuence of the gravitational field on boundary-induced quantum effects is weak