More on Complexity in Finite Cut Off

By:

S. Sedigheh Hashemi

In collaboration with : Ghadir Jafari, Ali Naseh and Hamed Zolfi

Institute for Reserach in Fundamental Sciences (IPM), Tehran, Iran

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Yerevan State University

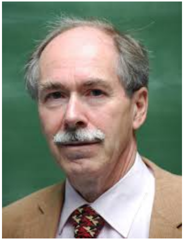
Outline

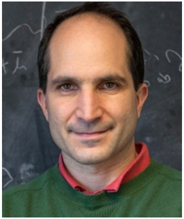
The following topics will be covered:

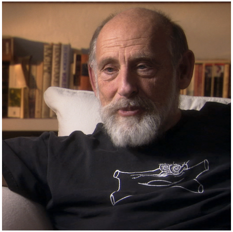
1. Holography
2. Complexity=Volume, Complexity = Action.
3. Einstein-Hilbert-Maxwell Theory at Finite Cut Off
4. Gauss-Bonnet-Maxwell Theory at Finite Cut Off
5. Conclusion.

Holography

- Quantum gravity in $(d + 1)$ dimensions must be described by a non gravitational theory in d dimensions.
  * Transition between both theories should be possible: DICTIONARY.
  * Not so easy to find precise example of holographic theories.
- Successful example: AdS/CFT.
  * Finding the entries of the dictionary is manageable.







- The leading example of such relations is: Ryu-Takayanagi: Provides a geometrical realization of entanglement entropy in dual CFT.
- Susskind proposal: Quantum computational complexity of a boundary state is dualized to a portion of spacetime. This proposal can be used to understand the rich geometrical structures that exist behind the horizon.

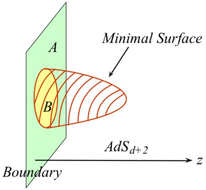


Figure 3: The holographic calculation of entanglement entropy via AdS/CFT.

Holographic Proposals:

To describe the quantum complexity of state in boundary QFT, two holographic proposals have been developed:
Complexity=Volume and Complexity=Area.

$$\mathcal{C}_A = \frac{I_{WdW}}{\pi \hbar} \tag{1}$$

I_{WdW} is the on-shell gravitational action evaluated on a certain subregion of spacetime known as WdW patch.
Lloyd's bound says that the upper bound on rate of complexity growth is twice the enrgy of system:

$$\frac{d\mathcal{C}_A}{dt} = 2M_{Bh} \tag{2}$$

Question:

What is a general structure of an effective QFT for which the UV behavior is not described by a CFT and we can holographically probe this field theory?

* This question was answered by Smirnov and Zamalodchikov. They discovered a general class of exactly solvable irrelevant deformation of two dimensional CFTs.

* The holographic dual of these authors is a 3 dimensional Einstein-Hilbert gravity at finite radial cut off.

We find the relation between this new cut off and boundary finite cut off in charged black hole solutions of Einstein Hilbert theory and Gauss-Bonnet theory.

Einstein-Hilbert-Maxwell Theory at Finite Cut Off

Metric

$$ds^2 = -f(r)dt^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{d}\Sigma^2_{\kappa,d-1}, \tag{3}$$

where the blackening factor $f(r)$ is given by

$$f(r) = \kappa + \frac{r^2}{L^2} - \frac{\omega^{d-2}}{r^{d-2}} + \frac{q^2}{r^{2(d-2)}}. \tag{4}$$

The thermodynamic quantities describing the black hole (3) are

$$M = \frac{(d-1)\Omega_{k,d-1}}{16\pi G}\,\omega^{d-2}, \qquad S = \frac{\Omega_{k,d-1}}{4G} r_+^{d-1}, \qquad T = \frac{1}{4\pi}\frac{\partial f}{\partial r}\Big|_{r=r_+}. \tag{5}$$

The charge and the Maxwell potential also respectively are

$$Q = \oint *F = \frac{q\ \Omega_{k,d-1}\sqrt{(d-1)(d-2)}}{g\sqrt{8\pi G}},$$

$$A_t(r) = \frac{g}{\sqrt{8\pi G}}\sqrt{\frac{d-1}{d-2}}\ \left(\frac{q}{r_+^{d-2}} - \frac{q}{r^{d-2}}\right). \quad (6)$$

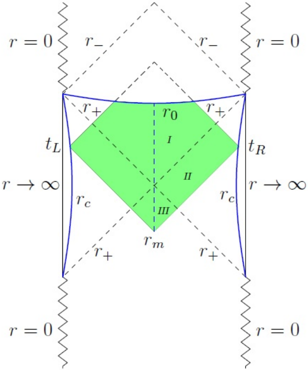


Figure: Penrose diagram of the eternal charged AdS black hole. The theory is defined at a radial finite cut off r_c and the actual WdW patch is shown in green color.

In order to consider the null sheets bounding the WdW patch the tortoise coordinate is defined as

$$r^*(r) = -\int_r^\infty \frac{\mathrm{d}r}{f(r)}, \tag{7}$$

which by using Eddington-Finkelstein coordinates, u and v, describing out- and in-going null rays, such that

$$v = t + r^*(r), \qquad u = t - r^*(r). \tag{8}$$

It is also useful to fix the notation for null vectors respectively associated with constant v and u surfaces

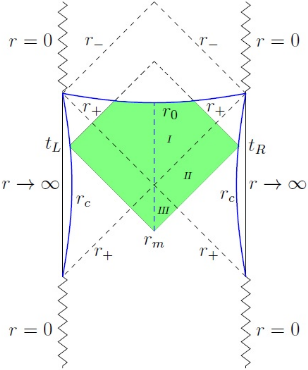
$$k_1 = \alpha\left(\partial_t + \frac{1}{f(r)}\partial_r\right), \qquad k_2 = \alpha\left(\partial_t - \frac{1}{f(r)}\partial_r\right), \tag{9}$$

The Einstein-Hilbert-Maxwell gravitational action can be written as

$$\begin{aligned}I = &\frac{1}{16\pi G}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left(R + \frac{d(d-1)}{L^2} - \frac{1}{4g^2}F^2\right)\\ &+ \frac{1}{8\pi G}\int_{\Sigma_t^d} K_t\, d\Sigma_t \pm \frac{1}{8\pi G}\int_{\Sigma_s^d} K_s\, d\Sigma_s \pm \frac{1}{8\pi G}\int_{\Sigma_n^d} K_n\, dSd\lambda\\ &\pm \frac{1}{8\pi G}\int_{J^{d-1}} a\, dS,\end{aligned} \tag{10}$$

For our case we have 3 terms: bulk integration, GHY contribution for the cut off surface r_0 and joint term at r_m,

$$I_{\text{tot}} = I_{\text{bulk}} + I_{\text{GHY}} + I_{\text{joint}}. \tag{11}$$



To calculate the contribution from the bulk action, we divide the WdW patch into three regions:
I, the region between r_0 and the outer horizon r_+;
II, the region outside the outer horizon r_+;
III, the region behind the outer horizon.

$$I_{\text{bulk}} = \int_{\text{WDW}} \mathrm{d}^{d+1}x\sqrt{-g}\,\mathcal{L} = \int_{\text{WDW}} \mathrm{d}t\,\mathrm{d}r\,I(r), \tag{12}$$

in which $I(r) = \mathrm{d}\mathcal{I}(r)/\mathrm{d}r$, and

$$\mathcal{I}(r) = \frac{\Omega_{k,d-1}}{16\pi G}\left[-\frac{2(d-1)q^2}{r^{d-2}} + (d-1)\omega^{d-2} - r^{d-1}f'(r)\right]. \tag{13}$$

The bulk contributions are given by

$$
\begin{aligned}
I^{\rm I}_{\rm bulk} &= 2\int_{r_0}^{r_+} dr\, I(r)\left(\frac{t}{2} - r^*(r)\right),\\
I^{\rm II}_{\rm bulk} &= 4\int_{r_+}^{r_{\rm max}} dr\, I(r)\left(-r^*(r)\right),\\
I^{\rm III}_{\rm bulk} &= 2\int_{r_m}^{r_+} dr\, I(r)\left(-\frac{t}{2} - r^*(r)\right),
\end{aligned}
\tag{14}
$$

Adding the above contributions and take a time derivative we arrive to

$$
\frac{\mathrm{d}I_{\rm bulk}}{\mathrm{d}t} = \int_{r_0}^{r_m} I(r)\,\mathrm{d}r = \mathcal{I}(r)\Big|_{r_0}^{r_m}. \tag{15}
$$

Now, substituting (13) in the above equation yields

$$
\frac{\mathrm{d}I_{\rm bulk}}{\mathrm{d}t} = \frac{\Omega_{k,d-1}}{8\pi G}\left[q^2\left(\frac{1}{r_0^{d-2}} - \frac{1}{r_m^{d-2}}\right) + \frac{1}{L^2}\left(r_0^d - r_m^d\right)\right]. \tag{16}
$$

The GHY action at the cut off surface $r = r_0$ is

$$I_{\text{GHY}} = -\frac{2}{8\pi G}\int_{r=r_0} \mathrm{d}t\, \mathrm{d}^{d-1}x\sqrt{-h}\,K_s, \tag{17}$$

extrinsic curvature K:

$$K = \frac{n_r}{2}\left(\partial_r f(r) + \frac{2(d-1)}{r} f(r)\right), \tag{18}$$

n_r the normal vector to this cut off surface. $\Rightarrow$

$$I_{\text{GHY}} = -\frac{\Omega_{k,d-1}}{8\pi G} r^{d-1}\left(\partial_r f(r) + \frac{2(d-1)f(r)}{r}\right)\left(\frac{t}{2} - r^*(r)\right)\Bigg|_{r=r_0}. \tag{19}$$

Taking the time derivative of I_{GHY} leads to

$$\frac{\mathrm{d}I_{\text{GHY}}}{dt}\Bigg|_{r=r_0} = -\frac{\Omega_{k,d-1}}{16\pi G}\left[\frac{2q^2}{r_0^{d-2}} + \frac{2(d-1)\kappa}{r_0^{2-d}} + \frac{2d}{L^2} r_0^d - d\,\omega^{d-2}\right]. \tag{20}$$

The time dependent joint term contribution can be calculated:

$$I_{\text{joint}} = -\frac{\Omega_{k,d-1}}{8\pi G} \; r_m^{d-1} \log\left(\frac{|f(r_m)|}{\alpha^2}\right), \tag{21}$$

Using the identity $\frac{\mathrm{d}r_m}{\mathrm{d}t} = -\frac{f(r_m)}{2}$,

$$\frac{\mathrm{d}I_{\text{joint}}}{\mathrm{d}t} = \frac{\Omega_{k,d-1}}{16\pi G}\Bigg(\frac{2r_m^d}{L^2} - \frac{2q^2(d-2)}{r_m^{d-2}} + (d-2)\omega^{d-2} + \log\frac{|f(r_m)|}{\alpha^2} f(r_m)(d-1)r_m^{d-2}\Bigg). \tag{22}$$

Now by summing (16), (20), (22)

$$\dot{\mathcal{C}} = \frac{\mathrm{d}I_{\text{tot}}}{\mathrm{d}t} = \frac{(d-1)\Omega_{k,d-1}}{8\pi G}\left[-\frac{r_0^d}{L^2} + \omega^{d-2} - \kappa\, r_0^{d-2} - \frac{q^2}{r_+^{d-2}}\right]. \tag{23}$$

For small charged black holes, $r_- < r_0 < r_+ \ll L$, with spherical horizon one has

$$\dot{\mathcal{C}} = 2(E - \mu Q), \tag{24}$$

at late times.

The gravitational quasi-local energy is

$$E_{\rm gr} = \frac{(d-1)\Omega_{k,d-1}\ r_c^{d-1}}{8\pi G}\Bigg(\frac{1}{L} + \frac{\kappa\, L}{2r_c^2} - \frac{L^3}{8r_c^4} - \sqrt{\frac{1}{L^2} - \frac{\omega^{d-2}}{r_c^d} + \frac{q^2}{r_c^{2d-2}} + \frac{\kappa}{r_c^2}}\ \Bigg), \tag{25}$$

and its relation with E is given by

$$E_{\rm gr} = \frac{L}{r_c} E. \tag{26}$$

therefore:

$$2\left((E-E_{global})-\mu Q\right)=2\frac{(d-1)\Omega_{k,d-1}}{8\pi G}\left(\frac{r_c^d}{L}(\sqrt{\frac{1}{L^2}+\frac{\kappa}{r_c^2}})\right.$$
$$\left.-\sqrt{\frac{1}{L^2}-\frac{\omega^{d-2}}{r_c^d}+\frac{q^2}{r_c^{2d-2}}+\frac{\kappa}{r_c^2}}-q^2(\frac{1}{r_+^{d-2}}-\frac{1}{r_c^{d-2}})\right) \tag{27}$$

for small black holes $q^2=r_-^{d-2}r_+^{d-2}$, we then have

$$r_0=r_-\left(1+\frac{L^2}{2(d-2)r_c^2}(1+\frac{r_+^{d-2}}{r_-^{d-2}})\right) \tag{28}$$

Gauss-Bonnet metric is given by:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Sigma_{\kappa,d-1}, \tag{29}$$

with

$$f(r) = \kappa + \frac{r^2}{2\tilde{\alpha}_{\rm GB}}\left[1 - \sqrt{1 + 4\tilde{\alpha}_{\rm GB}\left(\frac{\omega^{d-2}}{r^d} - \frac{1}{L^2} - \frac{q^2}{r^{2(d-1)}}\right)}\ \right]$$
$$\tilde{\alpha}_{\rm GB} = \alpha_{\rm GB}(d-2)(d-3) \tag{30}$$

The bulk action of Gauss-Bonnet-Maxwell theory is

$$I_{\text{bulk}} = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-g}\,\Bigg(R + \frac{d(d-1)}{L^2} - \frac{1}{4g^2}F_{\mu\nu}^2 + \alpha_{\text{GB}}\left(R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2\right)\Bigg), \tag{31}$$

the proper GHY term is

$$I_{\text{GHY}} = \frac{1}{16\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^{d}x\sqrt{-h}\,\Bigg(2K + \alpha_{\text{GB}}\left[8\,\mathcal{G}_{ij}K^{ij} - \tfrac{8}{3}K_i^{\,j}K^{ik}K_{kj} + 4KK_{ij}K^{ij} - \tfrac{4}{3}K^3\right]\Bigg), \tag{32}$$

and the proper counterterm, just for small value of $\alpha_{\rm GB}$, becomes

$$I_{\rm ct}=-\frac{1}{16\pi G}\int_{\partial\mathcal{M}}{\rm d}^d x\sqrt{-h}\left(I_{\rm ct}^{(0)}+I_{\rm ct}^{(2)}\;\mathcal{R}+I_{\rm ct}^{(4)}\;(\mathcal{R}_{ij}^2-\frac{d}{4(d-1)}\mathcal{R}^2)+...\right) \tag{33}$$

with

$$I_{\rm ct}^{(0)}=\frac{2(d-1)}{L}\left(1-\frac{\alpha_{\rm GB}(d-2)(d-3)}{6L^2}\right),$$

$$I_{\rm ct}^{(2)}=\frac{L}{(d-2)}\left(1+\frac{3\alpha_{\rm GB}(d-2)(d-3)}{2L^2}\right),$$

$$I_{\rm ct}^{(4)}=\frac{L^3}{(d-2)^2(d-4)}\left(1-\frac{15\alpha_{\rm GB}(d-2)(d-3)}{2L^2}\right).$$

The $\mathcal{G}_{ij}$, K_{ij} and $\mathcal{R}, \mathcal{R}_{ij}$ are respectively Einstein tensor, extrinsic curvature tensor and intrinsic curvature tensors of induced metric h_{ij}.

the quasi local energy (**??**) at finite cut off $r = r_c$ becomes

$$E_{\rm gr} = \frac{(d-1)\Omega_{\kappa,d-1}\, r_c^{d-1}}{16\pi G}\Big[\frac{2}{L} + \frac{\kappa L}{r_c^2} - \frac{L^3}{4r_c^4} - \frac{2}{r_c}\sqrt{f(r_c)} - \frac{\tilde{\alpha}_{\rm GB}}{L^2}(\frac{1}{3L} - \frac{3\kappa L}{2r_c^2} - \frac{15L^3}{8r_c^4}) - 4\tilde{\alpha}_{\rm GB}\frac{\sqrt{f(r_c)}}{r_c^3}(\kappa - \frac{1}{3}f(r_c))\Big] \tag{34}$$

Complexity in Gauss-Bonnet-Maxwell theory at finite cut off

$$\dot{\mathcal{C}} = \frac{\mathrm{d}}{\mathrm{d}t}(I_{\text{tot}} + I_{\text{GH}} + I_{\text{joint}} + I_{\text{amb}})$$

$$= \frac{2(d-1)\Omega_{k,d-1}}{16\pi G}\left(\frac{q^2}{r_0^{d-2}} - \frac{q^2}{r_m^{d-2}} - f(r_0)r_0^{d-2} - 2\tilde{\alpha}_{\text{GB}}\, f(r_0)r_0^{d-4}\left(\kappa - \frac{1}{3}f(r_0)\right)\right). \quad (35)$$

hence:

$$r_0 = r_-\left(1 + \frac{L^2}{2(d-2)r_c^2}\left(1 + \frac{r_+^{d-2}}{r_-^{d-2}}\right)\right) \quad (36)$$

Conlusion

- We have studied holographic complexity for charged AdS black holes at finite cut off in Einstein-Hilbert- Maxwell theory and Gauss-Bonnet-Maxwell theory.
- This proposal implies that a behind the horizon cut off exists in addition to the boundary cut off.
- We see that in order to have a late time behavior consistent with generalized Lloyd's bound one is forced to have a cut off behind the outer horizon and in front of inner horizon whose value is fixed by the inner horizon and boundary cut off.

Thanks for your attention :)